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LORENTZIAN PARA-SASAKIAN MANIFOLDS AND $*$-RICCI SOLITONS

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Abstract. We study the properties of Lorentzian para-Sasakian manifolds endowed with $*$-Ricci solitons and gradient $*$-Ricci solitons. Finally, the existence of $*$-Ricci soliton on a 4-dimensional Lorentzian para-Sasakian manifold is proved by constructing a non-trivial example.

1. Introduction

A Ricci soliton (g, F, λ) [12] on a semi-Riemannian manifold (M, g) is a generalization of Einstein metric such that

$$\frac{1}{2}\pounds_F g + S + \lambda g = 0,$$

where S is the Ricci tensor, $\pounds_F$ is the Lie derivative operator along the vector field F on M, g represents the semi-Riemannian metric of M and λ is a real number. The Ricci soliton is said to be shrinking, steady and expanding according to λ being less than 0, 0 and greater than 0, respectively.

In 1959, the notion of $*$-Ricci tensor on almost Hermitian manifolds was introduced by Tachibana [23] and further studied by Hamada [11] on real hypersurfaces of non-flat complex space forms. A semi-Riemannian metric g on a smooth manifold M is called a $*$-Ricci soliton [16] if there exists a smooth vector field F (called soliton vector field) and a real number λ, such that

$$\pounds_F g + 2S^* = -2\lambda g, \tag{1.1}$$

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where

$$S^*(U,V) = g(Q^*U,V) = \text{Trace}\,\{\phi \circ R(U,\phi V)\}\,,$$

for all vector fields U, V on M [6]. Here, ϕ is the $(1,1)$ tensor field and Q^* is the $(1,1)$ $*$-Ricci operator. If we choose λ as a smooth function in (1.1), then the soliton (g, F, λ) satisfying equation (1.1) is known as an almost $*$-Ricci soliton on M. In this connection, we recommend the papers [4, 10, 13, 15, 17, 21, 22, 24, 25] for more details about the study of Ricci solitons, η-Ricci solitons and $*$-Ricci solitons in the context of contact Riemannian geometry. As far as our knowledge goes, the study of $*$-Ricci solitons in the context of Lorentzian para-Sasakian manifolds is left. The main motive of this article is to fill this gap.

In 1989, K. Matsumoto [18] introduced the notion of $LP-$Sasakian manifolds, while in 1992, the same notion was independently studied by I. Mihai and R. Rosca [19] and they obtained several results on this manifold. The Lorentzian para-Sasakian manifolds have also been studied by various authors such as [1, 2, 7–9, 14, 26] and many others.

We present our work as follows. In Section 2, we collect the basic results and some basic definitions of Lorentzian para-Sasakian manifolds. The $*$-Ricci solitons and gradient $*$-Ricci solitons on Lorentzian para-Sasakian manifolds are discussed in Section 3 and Section 4, respectively. We present a 4-dimensional non-trivial example of Lorentzian para-Sasakian manifold admitting a $*$-Ricci soliton in Section 5.

2. Preliminaries

Let M be an n-dimensional smooth manifold equipped with a quartet (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ is the unit timelike vector field, η is a 1-form and a Lorentzian metric g on M such that [5, 20]

$$\phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \tag{2.1}$$

which implies

$$\phi\xi = 0, \quad \eta(\phi U) = 0, \quad \text{rank}(\phi) = n-1, \tag{2.2}$$

for all $U \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the collection of all smooth vector fields of M. The manifold M is said to have an almost para-contact metric structure (ϕ, ξ, η, g) when it admits a Lorentzian metric g, such that

$$g(\phi U, \phi V) = g(U,V) + \eta(U)\eta(V), \quad g(U,\xi) = \eta(U), \tag{2.3}$$

for all $U, V \in \mathfrak{X}(M)$.

If moreover,

$$(\nabla_U \phi)V = \eta(V)\phi^2 U + g(\phi U, \phi V)\xi, \tag{2.4}$$

$$\nabla_U \xi = \phi X \Leftrightarrow (\nabla_U \eta)V = g(\phi U, V) = g(U, \phi V), \tag{2.5}$$

where ∇ denotes the Levi-Civita connection of the manifold.

An n-dimensional Lorentzian para-Sasakian manifold satisfies the following relations (see [9]):

$$g(R(U,V)W,\xi) = g(V,W)\eta(U) - g(U,W)\eta(V), \tag{2.6}$$

$$R(U,V)\xi = \eta(V)U - \eta(U)V, \tag{2.7}$$

$$S(U,\xi) = (n-1)\eta(U) \Leftrightarrow Q\xi = (n-1)\xi, \tag{2.8}$$

for all $U, V, W \in \mathfrak{X}(M)$, where R denotes the curvature tensor and S denotes the Ricci tensor of M such that $S(U,V) = g(QU,V)$ for all $U, V \in \mathfrak{X}(M)$.

A Lorentzian para-Sasakian manifold M is said to be a generalized η-Einstein [3] if its non-vanishing Ricci tensor S is of the form

$$S(U,V) = \rho_1 g(U,V) + \rho_2 \eta(U)\eta(V) + \rho_3 g(\phi U, V), \tag{2.9}$$

where ρ_1, ρ_2 and ρ_3 are smooth functions on M. If $\rho_3 = 0$ (resp. $\rho_2 = \rho_3 = 0$), then M is called an η-Einstein (resp. Einstein) manifold.

Lemma 2.1. *An n-dimensional Lorentzian para-Sasakian manifold satisfies the following relations*

$$(\nabla_U Q)\xi = (n-1)\phi U - Q\phi U, \tag{2.10}$$

$$(\nabla_\xi Q)U = -2Q\phi U + 2aU + 2a\eta(U)\xi, \tag{2.11}$$

where Q is the Ricci operator.

Proof. Differentiating $Q\xi = (n-1)\xi$ along U and using (2.5), we get (2.10). Next differentiating (2.7) then using (2.5), we find

$$(\nabla_E R)(V,W)\xi = -R(V,W)\phi E + g(\phi E, W)V - g(\phi E, V)W. \tag{2.12}$$

Let $\{e_i\}_{i=1}^n$ be a local orthonormal basis on M. Putting $V = E = e_i$ in (2.12) and summing over i leads to

$$\begin{aligned}\sum_{i=1}^{n} \epsilon_i g((\nabla_{e_i} R)(e_i, W)\xi, U) =& S(W, \phi U) + (n-1)g(\phi W, U)\\ &- 2ag(W,U) - 2a\eta(V)\eta(W),\end{aligned} \tag{2.13}$$

where $\epsilon_i = g(e_i, e_i)$ and $a = \operatorname{tr}\phi$. Here tr stands for trace. From Bianchi's second identity, we can easily obtain that

$$\sum_{i=1}^{n} \epsilon_i g((\nabla_{e_i} R)(U,\xi)W), e_i) = (\nabla_U S)(\xi, W) - (\nabla_\xi S)(U, W). \tag{2.14}$$

By considering (2.13) in (2.14), equation (2.11) follows. □

On a Lorentzian para-Sasakian manifold (M, ϕ, ξ, η, g), we have the following lemmas.

Lemma 2.2. *On a Lorentzian para-Sasakian manifold* (M,ϕ,ξ,η,g)*, we have*

$$\begin{aligned}\bar{R}(U,V,\phi W,\phi E) =& \bar{R}(U,V,W,E) - g(U,W)g(V,E) + g(V,W)g(U,E)\\ &+ 2[g(V,W)\eta(U)\eta(E) - g(U,W)\eta(V)\eta(E)\\ &+ g(U,E)\eta(V)\eta(W) - g(V,E)\eta(U)\eta(W)]\\ &+ g(U,\phi W)g(V,\phi E) - g(V,\phi W)g(U,\phi E),\end{aligned} \tag{2.15}$$

for any U,V,W,E *on* M*, where* $\bar{R}(U,V,W,E) = g(R(U,V)W,E)$*.*

Proof. By virtue of the well-known definition of curvature tensor, we can write

$$\bar{R}(U,V,\phi W,\phi E) = g(\nabla_U\nabla_V\phi W,\phi E) - g(\nabla_V\nabla_U\phi W,\phi E) - g(\nabla_{[U,V]}\phi W,\phi E). \tag{2.16}$$

By making use of (2.2), (2.4) and (2.5), (2.16) takes the form

$$\begin{aligned}\bar{R}(U,V,\phi W,\phi W) =& g(R(U,V)W,E) + \eta(R(U,V)W)\eta(E)\\ &+ g(V,W)g(\phi U,\phi E) - g(U,W)g(\phi V,\phi E)\\ &+ 2g(U,E)\eta(V)\eta(W) - 2g(V,E)\eta(U)\eta(W)\\ &+ g(U,\phi W)g(V,\phi E) - g(V,\phi W)g(U,\phi E),\end{aligned}$$

which in view of (2.3) and (2.6) leads to (2.15). This completes the proof. □

Lemma 2.3. *The* $*$*-Ricci tensor of an* n*-dimensional Lorentzian para-Sasakian manifold* (M,ϕ,ξ,η,g) *is given by*

$$S^*(V,W) = S(V,W) + (n-2)g(V,W) - g(V,\phi W)a + (2n-3)\eta(V)\eta(W), \tag{2.17}$$

for any $V,W \in \mathfrak{X}(M)$*.*

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of the tangent space at each point of the manifold. By the definition of $*$-Ricci tensor, from (2.15), we have

$$\begin{aligned}S^*(V,W) =& \sum_{i=1}^n \epsilon_i \bar{R}(e_i,V,\phi W,\phi e_i)\\ =& \sum_{i=1}^n \epsilon_i \bar{R}(e_i,V,W,e_i) + \sum_{i=1}^n \epsilon_i[g(V,W)g(e_i,e_i) - g(e_i,W)g(V,e_i)]\\ &+ 2\sum_{i=1}^n \epsilon_i[g(V,W)\eta(e_i)\eta(e_i) - g(e_i,W)\eta(V)\eta(e_i)\\ &+ g(e_i,e_i)\eta(V)\eta(W) - g(V,e_i)\eta(e_i)\eta(W)]\\ &+ \sum_{i=1}^n \epsilon_i[g(e_i,\phi W)g(V,\phi e_i) - g(V,\phi W)g(e_i,\phi e_i)],\end{aligned}$$

which leads to (2.17), where $\epsilon_i = g(e_i,e_i)$, i.e., $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_{n-1} = 1$, $\epsilon_n = -1$. □

3. Lorentzian Para-Sasakian Manifolds Admitting $*$-Ricci Solitons

In this section, we characterize the properties of Lorentzian para-Sasakian manifold endowed with $*$-Ricci solitons. Now, we prove the following.

Theorem 3.1. *If an n-dimensional Lorentzian para-Sasakian manifold admits a $*$-Ricci soliton (g, F, λ), then the $*$-Ricci soliton is steady.*

Proof. By using (2.17) in (1.1), we have

$$(\pounds_F g)(U,V) = -2S(U,V) - 2[\lambda + (n-2)]g(U,V) - 2(2n-3)\eta(U)\eta(V) + 2g(U,\phi V)a. \tag{3.1}$$

Taking covariant differentiation of (3.1) with respect to W, we get

$$\begin{aligned}(\nabla_W \pounds_F g)(U,V) = &-2(\nabla_W S)(U,V) - 2(2n-3)[g(\phi W,U)\eta(V) \\ &+ g(\phi W,V)\eta(U)] + 2[g(V,W)\eta(U) \\ &+ g(U,W)\eta(V) + 2\eta(U)\eta(V)\eta(W)]a.\end{aligned} \tag{3.2}$$

Following Yano [27], the following formula

$$(\pounds_F \nabla_U g - \nabla_U \pounds_F g - \nabla_{[F,U]} g)(V,W) = -g((\pounds_F \nabla)(U,V),W) - g((\pounds_F \nabla)(U,W),V)$$

is well-known for any U, V, W on M. As g is parallel with respect to ∇, the above relation becomes

$$(\nabla_U \pounds_F g)(V,W) - g((\pounds_F \nabla)(U,V),W) - g((\pounds_F \nabla)(U,W),V) = 0, \tag{3.3}$$

for any U, V, W. Since $\pounds_F \nabla$ is a symmetric tensor of type $(1,2)$, then from (3.3) it follows that

$$g((\pounds_F \nabla)(U,V),W) = \frac{1}{2}(\nabla_V \pounds_F g)(U,W) + \frac{1}{2}(\nabla_U \pounds_F g)(V,W) - \frac{1}{2}(\nabla_W \pounds_F g)(U,V). \tag{3.4}$$

Using (3.2) in (3.4), we have

$$\begin{aligned}g((\pounds_F \nabla)(U,V),W) = &(\nabla_W S)(U,V) - (\nabla_V S)(W,U) - (\nabla_U S)(V,W) \\ &- 2(2n-3)g(\phi U,V)\eta(W) + 2g(\phi U,\phi V)\eta(W)a,\end{aligned}$$

which by putting $V = \xi$ reduces to

$$g((\pounds_F \nabla)(U,\xi),W) = (\nabla_W S)(U,\xi) - (\nabla_U S)(\xi,W) - (\nabla_\xi S)(W,U). \tag{3.5}$$

By considering (2.10) and (2.11) in (3.5), we obtain

$$(\pounds_F \nabla)(U,\xi) = 2Q\phi U - 2aU - 2a\eta(U)\xi. \tag{3.6}$$

Taking the covariant derivative of (3.6) with respect to V, we have

$$\begin{aligned}(\nabla_V \pounds_F \nabla)(U,\xi) = &2(\nabla_V Q)\phi U - (\pounds_F \nabla)(U,\phi V) + 2Q(\nabla_V \phi)U \\ &- 2ag(U,\phi V)\xi - 2a\eta(U)\phi V.\end{aligned}$$

Again from [27], we have

$$(\pounds_F R)(U,V)W + (\nabla_V \pounds_F \nabla)(U,W) - (\nabla_U \pounds_F \nabla)(V,W) = 0.$$

Thus the last two equations give

$$\begin{aligned}(\pounds_F R)(U,V)\xi =& 2(\nabla_U Q)\phi V - 2(\nabla_V Q)\phi U \\ &+ 2Q(\eta(V)U - \eta(U)V) + 2a(\eta(U)\phi V - \eta(V)\phi U) \\ &+ (\pounds_F \nabla)(U,\phi V) - (\pounds_F \nabla)(V,\phi U).\end{aligned} \tag{3.7}$$

Setting $V = \xi$ in (3.7) and making use of (2.11), it follows that

$$(\pounds_F R)(U,\xi)\xi = 2QU + 2Q\eta(U)\xi - 2a\phi U - (\pounds_F \nabla)(\xi,\phi U). \tag{3.8}$$

Taking the Lie derivative of $R(U,\xi)\xi = -U - \eta(U)\xi$ along F, we have

$$(\pounds_F R)(U,\xi)\xi - g(U,\pounds_F\xi)\xi + 2\eta(\pounds_F\xi)U = -(\pounds_F\eta)(U)\xi. \tag{3.9}$$

By using (3.9), (3.8) takes the form

$$\begin{aligned}(\pounds_F\eta)(U)\xi =& -2QU - 2Q\eta(U)\xi + 2a\phi U + (\pounds_F \nabla)(\xi,\phi U) + g(U,\pounds_F\xi)\xi \\ &- 2\eta(\pounds_F\xi)U.\end{aligned} \tag{3.10}$$

Now taking the Lie derivative of $g(U,\xi) = \eta(U)$, we find

$$(\pounds_F\eta)U = g(U,\pounds_F\xi) + (\pounds_F g)(U,\xi). \tag{3.11}$$

By putting $V = \xi$ in (3.1) and using (2.1)–(2.3), we find

$$(\pounds_F g)(U,\xi) = -2\lambda\eta(U). \tag{3.12}$$

Again putting $U = \xi$ in (3.12), we arrive

$$\eta(\pounds_F\xi) = -\lambda. \tag{3.13}$$

By making use of (3.11)-(3.13), we get from (3.10) that

$$(\lambda I - Q)\phi^2 U = -a\phi U - \frac{1}{2}(\pounds_F \nabla)(\xi,\phi U),$$

which by virtue of (3.6) leads to $\lambda = 0$, where $\phi^2 U \neq 0$. This shows that $*$-Ricci soliton on M is steady. This completes the proof. □

Theorem 3.2. *An n-dimensional Lorentzian para-Sasakian manifold endowed with an almost $*$-Ricci soliton (g,ξ,λ) is a generalized η-Einstein. Also, the soliton is steady.*

Proof. Let the Lorentzian metric of an n-dimensional Lorentzian para-Sasakian manifold be an almost $*$-Ricci soliton (g,ξ,λ), then (1.1)) turns into

$$g(\nabla_U\xi,V) + g(U,\nabla_V\xi) + 2S^*(U,V) + 2\lambda g(U,V) = 0, \tag{3.14}$$

for all vector fields U and V on M. By making use of equations (2.5) and (2.17), equation (3.14) transforms to

$$S = \rho_1 g + \rho_2 \eta\otimes\eta + \rho_3 g(\cdot,\phi\,\cdot),$$

where $\rho_1 = -(\lambda + n - 2)$, $\rho_2 = -(2n-3)$ and $\rho_3 = a - 1$. Also, in view of (2.1)–(2.3), (2.8) and the above equation, we can easily find that $\lambda = 0$. This gives the statement of Theorem 3.2. □

Particularly, if we suppose that $a = \operatorname{tr}\phi = 1$, then from Theorem 3.2, we infer that

$$S = \rho_1 g + \rho_2 \eta \otimes \eta. \tag{3.15}$$

Let us consider an orthonormal frame field on a Lorentzian para-Sasakian manifold and contracting (3.15), we lead

$$r = n\rho_1 - \rho_2 = -n^2 + 4n - 3.$$

Now, we state the following.

Corollary 3.1. *If an n-dimensional Lorentzian para-Sasakian manifold admits an almost $*$-Ricci soliton (g, ξ, λ), with $\operatorname{tr}\phi = 1$, then it has constant scalar curvature.*

A non-flat semi-Riemannian manifold is called pseudo Ricci symmetric and denoted by $(PRS)_n$ if the non-zero Ricci tensor S of type $(0, 2)$ of the manifold satisfies the condition [28]

$$(\nabla_U S)(V, W) = 2A(U)S(V, W) + A(V)S(U, W) + A(W)S(U, V), \tag{3.16}$$

where A is a non-zero 1-form such that $g(U, \sigma) = A(U)$, for all vector fields $U; \sigma$ being the vector field corresponding to the associated 1-form A. In partcular, if $A = 0$, then the manifold is called Ricci symmetric.

Taking the covariant derivative of (3.15) leads to

$$(\nabla_U S)(V, W) = \rho_2[g(\phi U, V)\eta(W) + g(\phi U, W)\eta(V)]. \tag{3.17}$$

Now using (3.15) and (3.17), (3.16) becomes

$$\begin{aligned} \rho_2[g(\phi U, V)\eta(W) + g(\phi U, W)\eta(V)] =& 2A(U)[\rho_1 g(V, W) + \rho_2\eta(V)\eta(W)] \\ &+ A(V)[\rho_1 g(U, W) + \rho_2\eta(U)\eta(W)] \\ &+ A(W)[\rho_1 g(U, V) + \rho_2\eta(U)\eta(V)]. \end{aligned} \tag{3.18}$$

Taking $U = W = \xi$ in (3.18), we get $A(V) = 3A(\xi)\eta(V)$, which by putting $V = \xi$ gives $A(\xi) = 0$. This implies that $A(V) = 0$. Thus we have the following.

Theorem 3.3. *A pseudo Ricci symmetric Lorentzian para-Sasakian manifold admitting an almost $*$-Ricci soliton (g, ξ, λ), with $\operatorname{tr}\phi = 1$ is Ricci symmetric.*

4. Gradient $*$-Ricci Solitons on η-Einstein Lorentzian Para-Sasakian Manifolds

This section is concerned with the study of gradient $*$-Ricci solitons within the context of η-Einstein Lorentzian para-Sasakian manifolds.

Let an n-dimensional Lorentzian para-Sasakian manifold be η-Einstein, then it is noticed that the equation (2.9) takes the form

$$S = \rho_1 g(U, V) + \rho_2 \eta(U) \otimes \eta(V). \tag{4.1}$$

Setting $V = U = e_i$ in (4.1), where $\{e_i\}_{i=1}^n$ represents a set of orthonormal frame field of M, and taking the summation over i, $1 \leq i \leq n$, we have

$$r = \rho_1 n - \rho_2. \tag{4.2}$$

On the other hand, putting $U = V = \xi$ in (4.1) and making use of (2.1) and (2.3), we also have

$$-(n-1) = -\rho_1 + \rho_2. \tag{4.3}$$

Hence, it follows from (4.2) and (4.3) that

$$\rho_1 = \frac{r}{n-1} - 1, \quad \rho_2 = \frac{r}{n-1} - n.$$

Thus, the Ricci tensor S of an η-Einstein Lorentzian para-Sasakian manifold is given by

$$S(U,V) = \left(\frac{r}{n-1} - 1\right) g(U,V) + \left(\frac{r}{n-1} - n\right) \eta(U)\eta(V). \tag{4.4}$$

Definition 4.1. A semi-Riemannian metric g of a semi-Riemannian manifold M is called a gradient $*$-Ricci soliton if it satisfies

$$\mathrm{Hess} f + S^* + \lambda g = 0, \tag{4.5}$$

for some smooth function f, where $\mathrm{Hess}\, f$ (Hessian f) is defined by $\mathrm{Hess}\, f = \nabla\nabla f$. It is noticed that if we choose $F = Df$ in equation (1.1), where D denotes the gradient operator of g, then we get (4.5).

Let the η-Einstein Lorentzian para-Sasakian manifold M admit a gradient $*$-Ricci soliton. Then from (4.5) it follows that

$$\nabla_U Df + Q^*U + \lambda U = 0, \tag{4.6}$$

for all U on M. First we prove the following lemmas for later use.

Lemma 4.1. *An n-dimensional η-Einstein Lorentzian para-Sasakian manifold satisfies*

$$(\nabla_U Q^*)\xi - (\nabla_\xi Q^*)U = -\left(\frac{r}{n-1} + n - 3\right)\phi U + \left(a - \frac{\xi(r)}{n-1}\right)(U + \eta(U)\xi), \tag{4.7}$$

for all X on M.

Proof. By using (4.4) in (2.17), we find

$$S^*(V,W) = \left(\frac{r}{n-1} + n - 3\right)(g(V,W) + \eta(V)\eta(W)) - g(V,\phi W)a.$$

It yields

$$Q^*V = \left(\frac{r}{n-1} + n - 3\right)(V + \eta(V)\xi) - \phi V a. \tag{4.8}$$

Differentiating (4.8) along U, we get

$$(\nabla_U Q^*)V = \left(\frac{r}{n-1} + n - 3\right)[(\nabla_U \eta)(V)\xi + \eta(V)\nabla_U \xi] \\ - (g(U,V)\xi + \eta(V)U + 2\eta(U)\eta(V)\xi)a + \frac{U(r)}{n-1}(V + \eta(V)\xi), \tag{4.9}$$

which by replacing V by ξ and using (2.1), (2.3) and (2.5) reduces to

$$(\nabla_U Q^*)\xi = -\left(\frac{r}{n-1} + n - 3\right)\phi U + (U + \eta(U)\xi)a. \tag{4.10}$$

Again replacing U by ξ in (4.9) and using same equations, we find

$$(\nabla_\xi Q^*)U = \frac{\xi r}{n-1}(U - \eta(U)\xi). \tag{4.11}$$

By subtracting (4.11) from (4.10), (4.7) follows. □

Lemma 4.2. *If an η-Einstein Lorentzian para-Sasakian manifold admits a gradient $*$-Ricci soliton, then we have*

$$R(U,V)Df = (\nabla_V Q^*)U - (\nabla_U Q^*)V. \tag{4.12}$$

Proof. Differentiating (4.6) covariantly along Y, we have

$$\nabla_V \nabla_U Df + \nabla_V Q^* U + \lambda \nabla_V U = 0, \tag{4.13}$$

which by interchanging U and V becomes

$$\nabla_U \nabla_V Df + \nabla_U Q^* V + \lambda \nabla_U V = 0. \tag{4.14}$$

Also from (4.6), we find

$$\nabla_{[U,V]} Df = -Q^*[U,V] - \lambda[U,V]. \tag{4.15}$$

By making use of (4.13)–(4.15), Lemma 4.2 follows. □

Theorem 4.1. *Let the metric of an η-Einstein Lorentzian para-Sasakian manifold M admit a gradient $*$-Ricci soliton. Then the gradient of the potential function is pointwise collinear with the potential vector field of M.*

Proof. Putting $U = \xi$ in (4.12), we have

$$R(\xi,V)Df = (\nabla_V Q^*)\xi - (\nabla_\xi Q^*)V,$$

which by virtue of the Lemma 4.1 leads to

$$g(R(\xi,V)Df,\xi) = 0. \tag{4.16}$$

By using (2.8), we have

$$g(R(\xi,V)Df,\xi) = -(Vf) - \eta(V)(\xi f). \tag{4.17}$$

From (4.16) and (4.17), we find $(Vf) = -\eta(V)(\xi f)$. This implies that

$$Df = -(\xi f)\xi.$$

This completes the proof. □

Taking the covariant derivative of $Df = -(\xi f)\xi$ along U, we have

$$\nabla_U Df = -(U(\xi f))\xi - (\xi f)\phi U, \tag{4.18}$$

which gives

$$g(\nabla_U Df, \xi) = U(\xi f),$$

where (2.1) and (2.2) are used. Using the last equation in (4.18), we obtain

$$\nabla_U Df = -g(\nabla_U Df, \xi)\xi - (\xi f)\phi U. \tag{4.19}$$

From equations (2.17) and (4.6), we conclude that

$$\nabla_U Df = -QU - (\lambda + n - 2)U - (2n-3)\eta(U)\xi + \phi U a, \tag{4.20}$$

which implies that

$$g(\nabla_U Df, \xi) = -\lambda\eta(U). \tag{4.21}$$

Thus from the equations (2.1), (2.2), (2.8), and (4.19)–(4.21), we obtain

$$QU = -(\lambda + n - 2)U - (\lambda + 2n - 3)\eta(U)\xi + (a + (\xi f))\phi U,$$

which informs that the manifold M under the consideration is generalized η-Einstein. Hence, we can state the following.

Corollary 4.1. *Every η-Einstein Lorentzian para-Sasakian manifold of dimension n endowed with a gradient $*$-Ricci metric is generalized η-Einstein.*

5. Example

In this section, we construct a non-trivial example of a Lorentzian para-Sasakian manifold.

We consider the 4-dimensional manifold $M = \{(u, v, w, t) \in \mathbb{R}^4\}$, where (u, v, w, t) are the standard coordinates in $\mathbb{R}^4$. Let ζ_1, ζ_2, ζ_3 and ζ_4 be the vector fields on M given by

$$\zeta_1 = e^t \frac{\partial}{\partial u}, \quad \zeta_2 = e^t \frac{\partial}{\partial v}, \quad \zeta_3 = e^t \left(\frac{\partial}{\partial v} + \frac{\partial}{\partial w}\right), \quad \zeta_4 = -\frac{\partial}{\partial t}.$$

Let g be the semi-Riemannian metric defined by

$$g(e_i, e_j) = \begin{cases} 1, & 1 \le i = j \le 3, \\ -1, & i = j = 4, \\ 0, & 1 \le i \ne j \le 4. \end{cases}$$

Let η be the 1-form on M defined by $\eta(U) = g(U, \zeta_4) = g(U, \xi)$ for all $U \in \mathfrak{X}(M)$. Let ϕ be the $(1,1)$ tensor field on M defined by

$$\phi\zeta_1 = \zeta_1, \quad \phi\zeta_2 = \zeta_2, \quad \phi\zeta_3 = \zeta_3, \quad \phi\zeta_4 = 0.$$

By applying the linearity of ϕ and g, we have

$$\begin{aligned}\eta(\xi) =& -1, \quad \phi^2 U = U + \eta(U)\xi, \quad \eta(\phi U) = 0,\\ g(U,\xi) =&\eta(U), \quad g(\phi U, \phi V) = g(U,V) + \eta(U)\eta(V),\end{aligned}$$

for all $U, V \in \mathfrak{X}(M)$. Then we have

$$\begin{aligned}[\zeta_1,\zeta_2] =&[\zeta_1,\zeta_3] = [\zeta_2,\zeta_3] = 0,\\ [\zeta_1,\zeta_4] =&\zeta_1, \quad [\zeta_2,\zeta_4] = \zeta_2, \quad [\zeta_3,\zeta_4] = \zeta_3.\end{aligned}$$

Using Koszul's formula, we can easily calculate

$$\begin{aligned}\nabla_{\zeta_1}\zeta_1 =&\zeta_4, \quad \nabla_{\zeta_1}\zeta_2 = 0, \quad \nabla_{\zeta_1}\zeta_3 = 0, \quad \nabla_{\zeta_1}\zeta_4 = \zeta_1,\\ \nabla_{\zeta_2}\zeta_1 =&0, \quad \nabla_{\zeta_2}\zeta_2 = \zeta_4, \quad \nabla_{\zeta_2}\zeta_3 = 0, \quad \nabla_{\zeta_2}\zeta_4 = \zeta_2,\\ \nabla_{\zeta_3}\zeta_1 =&0, \quad \nabla_{\zeta_3}\zeta_2 = 0, \quad \nabla_{\zeta_3}\zeta_3 = \zeta_4, \quad \nabla_{\zeta_3}\zeta_4 = \zeta_3,\\ \nabla_{\zeta_4}\zeta_1 =&0, \quad \nabla_{\zeta_4}\zeta_2 = 0, \quad \nabla_{\zeta_4}\zeta_3 = 0, \quad \nabla_{\zeta_4}\zeta_4 = 0.\end{aligned}$$

From the above values it can be easily verified that for $\zeta_4 = \xi$, M is a Lorentzian para-Sasakian manifold. We found that the non-vanishing components of curvature tensor are given by

$$\begin{aligned}R(\zeta_1,\zeta_2)\zeta_1 =& -\zeta_2, \quad R(\zeta_1,\zeta_3)\zeta_1 = -\zeta_3, \quad R(\zeta_1,\zeta_4)\zeta_1 = -\zeta_4,\\ R(\zeta_1,\zeta_2)\zeta_2 =&\zeta_1, \quad R(\zeta_2,\zeta_3)\zeta_2 = -\zeta_3, \quad R(\zeta_2,\zeta_4)\zeta_2 = -\zeta_4,\\ R(\zeta_1,\zeta_3)\zeta_1 =&\zeta_1, \quad R(\zeta_2,\zeta_3)\zeta_3 = \zeta_2, \quad R(\zeta_3,\zeta_4)\zeta_3 = -\zeta_4,\\ R(\zeta_1,\zeta_4)\zeta_4 =& -\zeta_1, \quad R(\zeta_2,\zeta_4)\zeta_4 = -\zeta_2, \quad R(\zeta_3,\zeta_4)\zeta_3 = -\zeta_3.\end{aligned}$$

From the above expressions of curvature tensors, we obtain

$$S(\zeta_{1,}\zeta_1) = S(\zeta_2,\zeta_2) = S(\zeta_3,\zeta_3) = 3, \quad S(\zeta_4,\zeta_4) = -3.$$

In view of 2.17, L.H.S. of (1.1) can be expressed as

$$\begin{aligned}(\pounds_F g)(V,W) + 2S^*(V,W) + 2\lambda g(V,W) =&g(\nabla_V F, W) + g(V, \nabla_W F)\\ &+ 2S(V,W) + 4g(V,W)\\ &- 6g(V,\phi W)a + 10\eta(V)\eta(W).\end{aligned}$$

Let $V = \sum_{i=1}^4 V^i e_i$, $W = \sum_{i=1}^4 W^i e_i$ and $F = \sum_{i=1}^4 F^i e_i$, where V^i, W^i and F^i are scalars for $i = 1, 2, 3, 4$ such that

$$F^4 = \frac{F^1(V^1W^4 + W^1V^4) + F^2(V^2W^4 + W^2V^4) + F^3(V^3W^4 + W^3V^4)}{2(V^1W^1 + V^2W^2 + V^3W^3)} - 2,$$

provided $V^1W^1 + V^2W^2 + V^3W^3 \neq 0$. Then by the straight forward calculations, we can notice that

$$\begin{aligned}&2(V^1W^1F^4 + V^2W^2F^4 + V^3W^3F^4) - (V^1F^1W^4 + V^2F^2W^4 + V^3F^3W^4\\ &+ W^1F^1V^4 + W^2F^2V^4 + W^3F^3V^4) + 4(V^1W^1 + V^2W^2 + V^3W^3) = 0,\end{aligned}$$

for $a = 3$ and hence we have $\pounds_F g + 2S^* + 2\lambda g = 0$, provided $\lambda = 0$. Thus, we can say that the Lorentzian para-Sasakian manifold of dimension 4 admits a steady type $*$-Ricci soliton, which proves Theorem 3.1.

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