Kragujevac Journal of Mathematics
Volume 48(2) (2024), Pages 225–239.

ON VERTEX-EDGE AND EDGE-VERTEX CONNECTIVITY INDICES OF GRAPHS

SHILADHAR PAWAR[1], AHMED MOHSEN NAJI[1], NANDAPPA D. SONER[2], ALI REZA ASHRAFI[3], AND ALI GHALAVAND[3*]

Abstract. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The vertex-edge degree of the vertex v, $d_G^e(v)$, equals to the number of different edges that are incident to any vertex from the open neighborhood of v. Also, the edge-vertex degree of the edge $e = uv$, $d_G^v(e)$, equals to the number of vertices of the union of the open neighborhood of u and v. In this paper, the vertex-edge connectivity index, ϕ_v, and the edge-vertex connectivity index, ϕ_e, of a graph G were introduced. These are defined as $\phi_v(G) = \sum_{v\in V(G)} d_G^e(v)d_G(v)$ and $\phi_e(G) = \sum_{e=uv\in E(G)} d_G(e)d_G^v(e)$, where $d_G(v)$ is the degree of a vertex $v \in V(G)$ and $d_G(e)$ is the number of edges in $E(G)$ that are adjacent to e. In this paper, we will study the main properties of $\phi_v(G)$, $\phi_e(G)$ and establish some upper and lower bounds for them. The numbers ϕ_v and ϕ_e for titania nanotubes are also computed.

1. Basic Definitions and Notations

In this paper we study some aspects of the vertex-edge degree of a vertex and we are concerned only with simple graphs, i.e., finite graphs having no loops, multiple and directed edges. Let $G = (V(G), E(G))$ be such a graph with vertex set $V(G)$ and edge set $E(G)$. As usual, the number of vertices and edges in G are denoted by $n = |V|$ and $m = |E|$, respectively. The distance $d_G(u, v)$ between two vertices u and v of a graph G is equal to the length of (number of edges in) a shortest path connecting them. For a vertex $v \in V(G)$, the open neighborhood of v is denoted by $N(v, G)$ and is defined as $N(v, G) = \{u \in V(G) \mid uv \in E(G)\}$. The degree of a vertex

Key words and phrases. Vertex-edge degree, edge-vertex degree, vertex-edge connectivity index, edge-vertex connectivity index.
2020 *Mathematics Subject Classification.* Primary: 05C09. Secondary: 05C07, 05C35.
DOI 10.46793/KgJMat2402.225P
Received: October 12, 2020.
Accepted: March 31, 2021.

v in G is denoted by $d_G(v)$ and is defined as the number of neighbours of the vertex v in G, i.e., $\deg_G(v) = |N(v, G)|$. The minimum and maximum degree of vertices in the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For any terminology or notation not mention here, we refer to [17].

A topological index of a graph is a graph invariant calculated from a graph representing a molecule and applicable in chemistry. The Zagreb indices have been introduced, more than fifty years ago, by Gutman and Trinajestić [15], in 1972, and elaborated in [16]. They are defined as $M_1(G) = \sum_{uv \in E(G)}[d_G(u) + d_G(v)] = \sum_{v \in V(G)} d_G(v)^2$ and $M_2(G) = \sum_{uv \in E} d_G(u)d_G(v)$. Furtula and Gutman [12] introduced the forgotten index of G, $F(G)$, as $F(G) = \sum_{uv \in E(G)}[d_G(u)^2 + d_G(v)^2] = \sum_{v \in V(G)} d_G(v)^3$. For properties of the two Zagreb indices see [3, 7, 14, 15, 24, 25, 30] and the references therein.

In recent years, some novel variants of ordinary Zagreb indices introduced and studied, such as Zagreb coincides [1, 16], multiplicative Zagreb indices [13, 29, 30], multiplicative sum Zagreb index [10] and multiplicative Zagreb coincides [31].

In 2017, Naji et al. [22], have introduced a new distance-degree-based topological indices conceived depending on the second degrees of vertices (number of their second neighbours), and are so-called leap Zagreb indices of a graph G. For properties and more detail on leap Zagreb indices, we refer to [2, 22, 23] and [26].

For a vertex v in $V(G)$ the ve-dominates are every edge incident to v as well as every edge adjacent to these incident edges. Also, for an edge $e = uv$ in $E(G)$, the ev-dominates are the vertices of the set $N(v, G) \cup N(u, G)$. There is a natural duality between ve-dominates and ev-dominates for any graph G: a vertex $v \in V$ is an ev-dominates for edge $e \in E$ if and only if the edge e is an ve-dominates for vertex v [6].

Definition 1.1 ([4])**.** Let G be a connected graph and $v \in V(G)$. The vertex-edge degree of the vertex v, $d^e_G(v)$, equals the number of different edges that incident to any vertex from the open neighborhood of v. Also, the edge-vertex degree of the edge $e = uv$, $d^v_G(e)$, equals the number of vertices of the union of the open neighborhoods of u and v.

The concepts of vertex-edge domination and edge-vertex domination were introduced by Peters [21] in his Ph.D. thesis and studied further in [4, 9, 18, 19, 27]. The following fundamental results which will be used in many of our subsequent considerations are found in the earlier papers [28] and [32].

Let G be a graph. The total ev-degree, T_e, total ve-degree, T_v, ev-degree Zagreb index, S, first ve-degree Zagreb alpha index, S^α, first ve-degree Zagreb beta index, S^β, second ve-degree Zagreb index, S^μ, of graph G are defined by Chellali et al. [6] as:

$$T_e(G) = \sum_{e \in E(G)} d^v_G(e), \quad T_v(G) = \sum_{v \in V(G)} d^e_G(v),$$

$$S(G) = \sum_{e \in E(G)} d_G^v(e)^2, \quad S^\alpha(G) = \sum_{v \in V(G)} d_G^e(v)^2,$$
$$S^\beta(G) = \sum_{e=uv \in E(G)} [d_G^e(v) + d_G^e(u)], \quad S^\mu(G) = \sum_{e=uv \in E(G)} d_G^e(v) d_G^e(u).$$

Let $\eta(G)$ be the number of triangles in graph G. Authors in [6] have proved that:

$$T_e(G) = T_v(G) = M_1(G) - 3\eta(G), \quad \text{where } G \text{ is an arbitrary graph,} \tag{1.1}$$
$$S(G) = F(G) + 2M_2(G), \quad \text{where } G \text{ is a triangle free connected graph,}$$
$$S^\beta(T) = 2M_2(T), \quad \text{where } T \text{ is an arbitrary tree.}$$

In [8], Ediz defined ve-degree atom-bond connectivity, ve-degree geometric - arithmetic, ve-degree harmonic and ve-degree sum-connectivity indices as parallel to their corresponding classical degree versions. Moreover, the mathematical properties were studied in it.

Titania nanotubes which have been produced fifteen years ago have many applications on the very broad of science from medicine to electronics [20]. Computing certain topological indices of titania nanotubes have been started recently. Since 2015, there are many studies to compute the exact value of some topological indices of titania nanotubes [5, 11].

2. Main Results

Define the ev-degree connectivity index, ϕ_e, and ve-degree connectivity index, ϕ_v, of a graph G as:

$$\phi_e(G) = \sum_{e=uv \in E(G)} d_G(e) d_G^v(e),$$
$$\phi_v(G) = \sum_{v \in V(G)} d_G(v) d_G^e(v),$$

where for $e = uv \in E(G)$, $d_G(e) = d_G(u) + d_G(v) - 2$.

Proposition 2.1. *Let P_n, C_n, S_n, K_n and $K_{a,b}$ be path, cycle, star, complete and bipartite graphs on $n \geq 4$ vertices, respectively. Then $(a + b = n)$*

$$\phi_e(P_n) = 8n - 18, \quad \phi_v(P_n) = 8(n-2), \quad \phi_e(C_n) = \phi_v(C_n) = 8n,$$
$$\phi_e(S_n) = n(n-1)(n-2), \quad \phi_v(S_n) = 2(n-1)^2,$$
$$\phi_e(K_n) = n^2(n-1)(n-2), \quad \phi_v(K_n) = \frac{n^2(n-1)^2}{2},$$
$$\phi_e(K_{a,b}) = ab(n^2 - 2n), \quad \phi_v(K_{a,b}) = 2a^2b^2.$$

Proof. By definitions,

$$\phi_e(P_n) = \sum_{e=uv \in E(P_n)} d_{P_n}(e) d_{P_n}^v(e) = 2(1 \times 3) + (n-3)(2 \times 4) = 8n - 18,$$

$$\phi_v(P_n) = \sum_{v\in V(P_n)} d_{P_n}(v)d^e_{P_n}(v) = 2(1\times 2) + 2(2\times 3) + (n-4)(2\times 4) = 8(n-2).$$

The proof of other cases are similar and we omit them. □

Proposition 2.2. *Let G be a triangle free graph. Then*

$$\phi_e(G) = F(G) + 2M_2(G) - 2M_1(G) \quad \text{and} \quad \phi_v(G) = 2M_2(G).$$

Proof. By definitions,

$$\begin{aligned}\phi_e(G) &= \sum_{e=uv\in E(G)} d_G(e)d^v_G(e) = \sum_{e=uv\in E(G)} d_G(e)[d_G(u)+d_G(v)]\\ &= \sum_{e=uv\in E(G)} [d_G(u)+d_G(v)-2][d_G(u)+d_G(v)]\\ &= F(G) + 2M_2(G) - 2M_1(G),\\ \phi_v(G) &= \sum_{v\in V(G)} d_G(v)d^e_G(v) = \sum_{v\in V(G)} d_G(v) \sum_{uv\in E(G)} d_G(u)\\ &= \sum_{v\in V(G)} d_G(v) \sum_{uv\in E(G)} d_G(u) = 2\sum_{uv\in E(G)} d_G(u)d_G(v)\\ &= 2M_2(G).\end{aligned}$$

Hence, the result is obtained. □

Let G be a graph with n vertices and m edges and let $n_i = |\{v \in V(G) \mid d_G(v) = i\}|$, for all integers i, $1 \le i \le n-1$. By definition,

$$n = n_1 + n_2 + \cdots + n_{n-1}. \tag{2.1}$$

Also, it is well-known,

$$2m = n_1 + 2n_2 + \cdots + (n-1)n_{n-1}. \tag{2.2}$$

Therefore, by (2.1), (2.2) and some simple calculations,

$$n_1 = 2n - 2m + \sum_{i=3}^{n-1}(i-2)n_i. \tag{2.3}$$

Theorem 2.1. *Let G be a triangle free graph. Then $\phi_e(G) - \phi_v(G) \ge 2(m-n)$ and equality holds if and only if $\{d_G(v) \mid v \in V(G)\} \subseteq \{1,2\}$.*

Proof. By Proposition 2.2,

$$\begin{aligned}\phi_e(G) - \phi_v(G) = F(G) - 2M_1(G) &= \sum_{v\in V(G)} d_G(v)^2[d_G(v)-2]\\ &= \sum_{i=1}^{n-1} i^2(i-2)n_i = -n_1 + \sum_{i=3}^{n-1} i^2(i-2)n_i,\end{aligned}$$

and by (2.3),

$$\phi_e(G) - \phi_v(G) = 2m - 2n - \sum_{i=3}^{n-1}(i-2)n_i + \sum_{i=3}^{n-1} i^2(i-2)n_i$$

$$= 2m - 2n + \sum_{i=3}^{n-1}(i-1)(i-2)(i+1)n_i.$$

Therefore, $\phi_e(G) - \phi_v(G) \geq 2(m-n)$ and equality holds if and only if $\{d_G(v) \mid v \in V(G)\} \subseteq \{1,2\}$. ☐

Proposition 2.3. *Let G be a triangle free connected graph with n vertices and m edges. Then $\phi_e(G) \leq mn(n-2)$ and equality holds if and only if $G \cong K_{k,n-k}$.*

Proof. By definition of triangle free graph G, $d_G(u)+d_G(v) \leq n$ for all $e = uv \in E(G)$. Thus,

$$\begin{aligned}\phi_e(G) &= \sum_{uv\in E(G)} \Big(d_G(u) + d_G(v)\Big)\Big(d_G(u) + d_G(v) - 2\Big)\\ &\leq \sum_{uv\in E(G)} n(n-2) = mn(n-2).\end{aligned}$$

Equality holds if and only if $G \cong K_{k,n-k}$. ☐

A graph G is said to be ve-regular graph if and only if $|\{d^e_G(v) \mid v \in V(G)\}| = 1$ and is said to be ev-regular graph if and only if $|\{d^v_G(e) \mid e \in E(G)\}| = 1$.

Theorem 2.2. *For any graph G with n vertices and m edges*

$$S^\alpha(G) \geq \frac{(M_1(G) - 3\eta(G))^2}{n}. \tag{2.4}$$

Equality holds if and only if G is a ve-regular graph. Moreover,

$$\phi_v(G) \leq \sqrt{S^\alpha(G)M_1(G)}. \tag{2.5}$$

Equality holds if and only if there exists a real number c such that $d_G(v) = cd^e_G(v)$ for all $v \in V(G)$ and

$$\phi_e(G) \leq \sqrt{S(G)\Big(F(G) + 2M_2(G) - 4M_1(G) + 4m\Big)}. \tag{2.6}$$

Equality holds if and only if there exists a real number l such that $d_G(e) = ld^v_G(e)$ for all $e \in E(G)$.

Proof. Let G be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. Nest we will use Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right). \tag{2.7}$$

To prove (2.4), we put in (2.7), $a_i = d^e_G(v_i)$ and $b_i = 1$. Then by (1.1)

$$(M_1(G) - 3\eta(G))^2 = T_v(G)^2 = \left(\sum_{i=1}^{n} d^e_G(v_i)\right)^2 \leq \left(\sum_{i=1}^{n} d^e_G(v_i)^2\right)\left(\sum_{i=1}^{n} 1\right) = S^\alpha(G)n.$$

Therefore, $S^\alpha(G) \geq \frac{(M_1(G)-3\eta(G))^2}{n}$ and equality holds in Cauchy-Schwartz inequality if and only if $(a_1, a_2, \ldots, a_n) = c(b_1, b_2, \ldots, b_n)$, where c is a real number. Hence equality holds in (2.4) if and only if G is a ve-regular graph.

To prove (2.5), we put in (2.7), $a_i = d_G^e(v_i)$ and $b_i = d_G(v_i)$. Then we obtain

$$\phi_v(G)^2 = \left(\sum_{i=1}^{n} d_G^e(v_i) d_G(v_i)\right)^2 \leq \left(\sum_{i=1}^{n} d_G^e(v_i)^2\right)\left(\sum_{i=1}^{n} d_G(v_i)^2\right) = S^\alpha(G) M_1(G).$$

Therefore, $\phi_v(G) \leq \sqrt{S^\alpha(G)M_1(G)}$ and equality holds in Cauchy-Schwartz inequality if and only if $(a_1, a_2, \ldots, a_n) = c(b_1, b_2, \ldots, b_n)$, where c is a real number. Hence equality holds in Equation (2.5) if and only if there exists real number c such that $d_G(v) = c d_G^e(v)$ for all $v \in V(G)$.

To prove (2.6), again by Cauchy-Schwartz inequality,

$$\begin{aligned}
\phi_e^2(G) &= \left(\sum_{e=uv\in E(G)} d_G^v(e) d_G(e)\right)^2 \leq \left(\sum_{e=uv\in E(G)} d_G^v(e)^2\right)\left(\sum_{e=uv\in E(G)} d_G(e)^2\right) \\
&= S(G) \sum_{e=uv\in E(G)} \Big(d_G(u) + d_G(v) - 2\Big)^2 \\
&= S(G)\Big(F(G) + 2M_2(G) - 4M_1(G) + 4m\Big).
\end{aligned}$$

Thus $\phi_e(G) \leq \sqrt{S(G)\Big(F(G) + 2M_2(G) - 4M_1(G) + 4m\Big)}$ and equality holds in (2.6) if and only if there exists real number l such that $d_G(e) = l d_G^v(e)$ for all $e \in E(G)$. □

If G is a triangle free r-regular graph, then for all $v \in V(G)$, $d_G^e(v) = \sum_{uv\in E(G)} r = r^2$ and for all $e \in E(G)$, $d_G^v(e = uv) = d_G(u) + d_G(v) = 2d_G(v)$. If G is a complete graph then $d_G^e(v) = n(n-1)/2$, $v \in V(G)$ and $d_G^v(e = uv) = n$ for all $e \in E(G)$. Therefore, the Equalities (2.4), (2.5) and (2.6) hold for triangle free regular graphs and also complete graphs.

Theorem 2.3. *Let G be an r-regular graph. Then*

$$\phi_v(G) = r\Big[M_1(G) - 3\eta(G)\Big] \quad \textit{and} \quad \phi_e(G) = 2(r-1)\Big[M_1(G) - 3\eta(G)\Big].$$

Proof. Let G be an r-regular graph. Then (1.1) gives

$$\begin{aligned}
\phi_v(G) &= \sum_{v\in V(G)} d_G^e(v) d_G(v) = \sum_{v\in V(G)} d_G^e(v) r \\
&= r \sum_{v\in V(G)} d_G^e(v) = r\Big[M_1(G) - 3\eta(G)\Big], \\
\phi_e(G) &= \sum_{e\in E(G)} d_G^v(e) d_G(e) = \sum_{e\in E(G)} d_G^v(e) 2(r-1) \\
&= 2(r-1) \sum_{e\in E(G)} d_G^v(e) = 2(r-1)\Big[M_1(G) - 3\eta(G)\Big],
\end{aligned}$$

as desired. ☐

Theorem 2.4. *Let G be graph.*

(a) *If G is a ve-regular graph with $d^e_G(v) = c$ for all $v \in V(G)$, then $\phi_v(G) = 2cm$.*
(b) *If G is an ev-regular graph with $d^v_G(e) = k$ for all $e \in E(G)$, then $\phi_e(G) = k[M_1(G) - 2m]$.*

Proof. Let G be ve-regular graph with $d^e_G(v) = c$ for all $v \in V(G)$. Then

$$\phi_v(G) = \sum_{v \in V(G)} d^e_G(v) d_G(v) = c \sum_{v \in V(G)} d_G(v) = 2cm.$$

Now, let G be ev-regular graph, with $d^v_G(e) = k$, for all $e \in E(G)$. Then

$$\phi_e(G) = \sum_{e \in E(G)} d^v_G(e) d_G(e) = k \sum_{e=uv \in E(G)} [d_G(u) + d_G(v) - 2] = k[M_1(G) - 2m].$$

This completes our argument. ☐

Lemma 2.1. *Let G be a connected graph with given vertices u and v such that $uv \notin E(G)$. If $G' = G + uv$, then $T_v(G) = T_e(G) \leq T_v(G') = T_e(G') - 2$.*

Proof. Let $x = M_1(G') - 3\eta(G')$ and $y = M_1(G) - 3\eta(G)$. By definition,

$$\begin{aligned} x - y =&(d_G(u) + 1)^2 + (d_G(v) + 1)^2 - 3\Big(\eta(G) + |N(u, G) \cap N(v, G)|\Big) \\ &- \Big[d_G(u)^2 + d_G(v)^2 - 3\eta(G)\Big] \\ =&2d_G(u) + 2d_G(v) + 2 - 3|N(u, G) \cap N(v, G)| \\ \geq&4|N(u, G) \cap N(v, G)| + 2 - 3|N(u, G) \cap N(v, G)| \geq 2. \end{aligned}$$

The proof follows from (1.1). ☐

Let G be a graph. The path $P_k := v_0v_2 \dots v_k$ is called a pendant path in G if $\{v_0, v_1, \dots, v_k\} \subseteq V(G)$, $d_G(v_0) \geq 3$, $d_G(v_k) = 1$, $\{v_iv_{i+1} \mid 0 \leq i \leq k-1\} \subseteq E(G)$, and $d_G(v_1) = \cdots = d_G(v_{k-1}) = 2$, when $k \geq 2$.

Lemma 2.2. *Let G be a graph with two pendant paths $P_k := v_0v_2 \dots v_k$ and $Q_l := u_0u_2 \dots u_l$. If $G' = G - v_0v_1 + u_lv_1$, then $T_v(G') = T_e(G') < T_v(G) = T_e(G) - 2$.*

Proof. Let $x = M_1(G') - 3\eta(G')$ and $y = M_1(G) - 3\eta(G)$. By definition,

$$y - x = d_G(v_0)^2 + 1 - \Big[(d_G(v_0) - 1)^2 + 4\Big] = 2d_G(v_0) - 4 \geq 2,$$

and (1.1) gives the result. ☐

Lemmas 2.1 and 2.2 give the following result.

Corollary 2.1. *Let G be a connected graph with n vertices. Then*

$$4n - 6 \leq T_v(G) = T_e(G) \leq \frac{1}{2}n^2(n-1).$$

Equality in left holds if and only if $G \cong P_n$ and equality in right holds if and only if $G \cong K_n$.

Corollary 2.2. *Let G be a connected graph with n vertices. Then*

$$\phi_v(G) \leq \frac{n^2(n-1)^2}{2} \quad and \quad \phi_e(G) \leq n^2(n-1)(n-2).$$

Equalities hold if and only if $G \cong K_n$.

Proof. By definitions,

$$\phi_v(G) = \sum_{v \in V(G)} d_G^e(v) d_G(v) \leq (n-1) \sum_{v \in V(G)} d_G^e(v) = (n-1)T_v(G),$$

$$\phi_e(G) = \sum_{e \in E(G)} d_G^v(e) d_G(e) \leq (2n-4) \sum_{e \in E(G)} d_G^v(e) = (2n-4)T_e(G).$$

Now, Corollary 2.1 gives the results. □

For positive integer $n \geq 4$, let $C_3 := v_1v_2v_3v_1$ and $P_{n-3} := u_1u_2 \ldots u_{n-3}$ be cycle and path graph on 3 and $n-3$ vertices, respectively. Then the graph C_3^{n-3} is obtained from C_3 and P_{n-3} by attaching vertices v_1 and u_1. By (1.1),

$$T_v(C_3^{n-3}) = T_e(C_3^{n-3}) = 4n - 1. \tag{2.8}$$

Lemma 2.3. *Let G be a graph with $n \geq 4$ vertices and minimum degree at least 2. Then $T_v(G) = T_e(G) \geq 4n$, with equality if and only if $G \cong C_n$.*

Proof. If $G \cong C_n$, then $T_v(G) = T_e(G) = 4n$ and lemma holds. Otherwise, by using Lemmas 2.1, 2.2 and (2.8), $T_v(G) = T_e(G) \geq 4n + 1$ which gives the lemma. □

Corollary 2.3. *Let G be a graph with $n \geq 4$ vertices and minimum degree at least 2. Then*

$$\phi_v(G) \geq 8n \quad and \quad \phi_e(G) \geq 8n.$$

Equalities hold if and only if $G \cong C_n$.

Proof. By definitions,

$$\phi_v(G) = \sum_{v \in V(G)} d_G^e(v) d_G(v) \geq 2 \sum_{v \in V(G)} d_G^e(v) = 2T_v(G),$$

$$\phi_e(G) = \sum_{e \in E(G)} d_G^v(e) d_G(e) \geq 2 \sum_{e \in E(G)} d_G^v(e) = 2T_e(G).$$

Now, Lemma 2.3 gives the results. □

Lemma 2.4 (Diaz-Metcalf inequality)**.** *Let the real numbers $a_i \neq 0$, b_i, $1 \leq i \leq n$, satisfy*

$$l \leq \frac{b_i}{a_i} \leq L.$$

Then

$$\sum_{i=1}^n b_i^2 + lL \sum_{i=1}^n a_i^2 \leq (L+l) \sum_{i=1}^n a_i b_i.$$

Equality holds if and only if $b_i = la_i$ or $b_i = La_i$.

Theorem 2.5. *Let G be a graph with n vertices, m edges, minimum degree $\delta \geq 1$ and maximum degree Δ. Then*

(i) $\phi_v(G) \geq \frac{1}{2\Delta+\delta+1}\left[2S^{\alpha}(G) + (\delta+1)\Delta M_1(G)\right]$ *and equality holds if and only if* $d_G^e(v) = \frac{1}{2}(\delta+1)d_G(v)$ *or* $d_G^e(v) = \Delta\, d_G(v)$ *for all* $v \in V(G)$*;*

(ii) $\phi_e(G) \geq \frac{1}{3}\left[S(G) + 2F(G) + 4M_2(G) - 6M_1(G) + 18\eta(G)\right]$ *and equality holds if and only if* $d_G^v(e) = d_G(e) + 2$ *or* $2d_G^v(e) = d_G(e) + 2$ *for all* $e \in E(G)$*.*

Proof. Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. To prove (i), by setting $a_i = d_G(v_i)$ and $b_i = d_G^e(v_i)$ for all $i = 1, 2, \ldots, n$, $L = \Delta$ and $l = \frac{1}{2}(\delta+1)$ in Diaz-Metcalf inequality we get

$$\sum_{i=1}^{n} d_G^e(v_i)^2 + \frac{1}{2}(\delta+1)\Delta \sum_{i=1}^{n} d_G(v_i)^2 \leq \left(\frac{1}{2}(\delta+1) + \Delta\right) \sum_{i=1}^{n} d_G(v_i) d_G^e(v_i),$$

which implies that

$$S^{\alpha}(G) + \frac{1}{2}(\delta+1)\Delta M_1(G) \leq \left(\frac{1}{2}(\delta+1) + \Delta\right) \phi_v(G).$$

Therefore,

$$\phi_v(G) \geq \frac{1}{2\Delta+\delta+1}\Big[2S^{\alpha}(G) + (\delta+1)\Delta M_1(G)\Big],$$

and equality holds if and only if $d_G^e(v) = \frac{1}{2}(\delta+1)d_G(v)$ or $d_G^e(v) = \Delta\, d_G(v)$ for all $v \in V(G)$.

To prove (ii), setting $a_i = d_G^v(e_i)$ and $b_i = d_G(e_i) + 2$ for all $i = 1, 2, \ldots, m$, $L = 2$ and $l = 1$ in Diaz-Metcalf inequality we get

$$\sum_{i=1}^{m} d_G^v(e_i)^2 + 2\sum_{i=1}^{m} (d_G(e_i)+2)^2 \leq 3\sum_{i=1}^{m} (d_G(e_i)+2) d_G^v(e_i),$$

which implies that

$$S(G) + 2\Big(F(G) + 2M_2(G)\Big) \leq 3\phi_e(G) + 6T_e(G).$$

Therefore, by (1.1),

$$\phi_e(G) \geq \frac{1}{3}\Big[S(G) + 2F(G) + 4M_2(G) - 6M_1(G) + 18\eta(G)\Big],$$

and equality holds if and only if $d_G^v(e) = d_G(e) + 2$ or $2d_G^v(e) = d_G(e) + 2$ for all $e \in E(G)$. This completes the proof. □

If G is a triangle free r-regular graph, then for all $v \in V(G)$, $d_G^e(v) = r^2$ and for all $e = uv \in E(G)$, $d_G^v(e) = d_G(e) + 2$. Therefore, by Theorem 2.5,

$$\phi_v(G) = \frac{1}{3r+1}\Big[2S^{\alpha}(G) + (r+1)rM_1(G)\Big],$$
$$\phi_e(G) = \frac{1}{3}\Big[S(G) + 2F(G) + 4M_2(G) - 6M_1(G)\Big].$$

Theorem 2.6. *Let G be a graph with n vertices and m edges. Then*
(i) $\phi_v(G) \geq \frac{2m}{n}[M_1(G) - 3\eta(G)]$;
(ii) $\phi_e(G) \geq \frac{1}{m}[M_1(G) - 2m][M_1(G) - 3\eta(G)]$.
The bounds attain on the cycle C_n, $n \geq 3$, and the star $K_{1,n-1}$, $n \geq 2$.

Proof. Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. Chebyshev's inequality states that, for any non-increasing sequences $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$, we have

$$n\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i.$$

Suppose $a_i = d_G(v_i)$ and $b_i = d_G^e(v_i)$, for $i = 1, 2, \ldots, n$. By (1.1), we obtain

$$n\sum_{i=1}^{n} d_G(v_i) d_G^e(v_i) \geq \sum_{i=1}^{n} d_G(v_i) \sum_{i=1}^{n} d_G^e(v_i),$$

and hence, $\phi_v(G) \geq \frac{2m}{n}[M_1(G) - 3\eta(G)]$. This proves (i).

To prove (ii), we define $a_i = d_G(e_i)$ and $b_i = d_G^v(e_i)$, for $i = 1, 2, \ldots, m$. By (1.1), we obtain

$$m\sum_{i=1}^{m} d_G(e_i) d_G^v(e_i) \geq \sum_{i=1}^{m} d_G(e_i) \sum_{i=1}^{m} d_G^v(e_i),$$

and hence, $\phi_e(G) \geq \frac{1}{m}[M_1(G) - 2m][M_1(G) - 3\eta(G)]$. □

It is well-known that $M_1(G) \geq 4n - 6$, with equality if and only if $G \cong P_n$. Therefore, Theorem 2.6, Corollary 2.1 and $M_1(G) \geq 4n - 6$ give the following results.

Corollary 2.4. *Let G be a graph with n vertices and m edges. Then*

$$\phi_v(G) \geq \frac{2m}{n}(4n - 6) \text{ and } \phi_e(G) \geq \frac{1}{m}[4n - 2m - 6][4n - 6].$$

Lemma 2.5 (Ozeki-Izumino-Mori-Seo type inequality). *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be n-tuples of real numbers satisfying $0 \leq r_1 \leq a_i \leq R_1$ and $0 \leq r_2 \leq b_i \leq R_2$, $i = 1, \ldots, n$. Then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \frac{n^2}{3}\left(R_1 R_2 - r_1 r_2\right)^2.$$

Theorem 2.7. *Let G be a connected graph with n vertices and m edges. Then*

(i) $\phi_v(G) \geq \sqrt{M_1(G)S^\alpha(G) - \frac{n^2}{3}\left(\Delta^3 - \delta(\delta + 1)\right)^2}$;

(ii) $\phi_e(G) \geq \sqrt{\left(F(G) + 2M_2(G) - 4M_1(G) + 4m\right)S(G) - \frac{16}{3}m^2\left(\Delta(\Delta - 1) - \delta(\delta - 1)\right)^2}$.

Proof. Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. To prove (i), we put $a = \left(d_G(v_1), d_G(v_2), \ldots, d_G(v_n)\right)$, $b = \left(d_G^e(v_1), d_G^e(v_2), \ldots, d_G^e(v_n)\right)$, $r_1 = \delta$, $R_1 = \Delta$, $r_2 = \delta + 1$ and $R_2 = \Delta^2$. By Ozeki-Izumino-Mori-Seo type inequality we get

$$M_1(G)S^\alpha(G) - \phi_v(G)^2 \leq \frac{n^2}{3}\left(\Delta^3 - \delta(\delta + 1)\right)^2,$$

which implies that

$$\phi_v(G) \geq \sqrt{M_1(G)S^\alpha(G) - \frac{n^2}{3}\Big(\Delta^3 - \delta(\delta+1)\Big)^2}.$$

To prove (ii), we set $a = \Big(d_G(e_1), d_G(e_2), \dots, d_G(e_m)\Big)$, $b = \Big(d_G^v(e_1), d_G^v(e_2), \dots, d_G^v(e_m)\Big)$, $r_1 = 2(\delta-1)$, $R_1 = 2(\Delta-1)$, $r_2 = 2\delta$ and $R_2 = 2\Delta$. Again by Ozeki-Izumino-Mori-Seo type inequality we get

$$\Big(F(G) + 2M_2(G) - 4M_1(G) + 4m\Big)S(G) - \phi_e(G)^2 \leq \frac{m^2}{3}\Big(4\Delta(\Delta-1) - 4\delta(\delta-1)\Big)^2,$$

which implies that

$$\phi_e(G) \geq \sqrt{\Big(F(G) + 2M_2(G) - 4M_1(G) + 4m\Big)S(G) - \frac{16}{3}m^2\Big(\Delta(\Delta-1) - \delta(\delta-1)\Big)^2}.$$

This completes our argument. □

Corollary 2.5. *Let G be a connected graph with n vertices and m edges. Then*

$$\phi_v(G) \geq \frac{1}{3}\sqrt{\frac{9(4n-6)^3}{n} - 3n^2\Big(n^3 - 3n^2 + 3n - 3\Big)^2}.$$

Proof. By (1.1), (2.4) and Corollary 2.1, $S^\alpha \geq \frac{(4n-6)^2}{n}$. Therefore, by $M_1(G) \geq 4n-6$ and Theorem 2.7,

$$\phi_v(G) \geq \frac{1}{3}\sqrt{\frac{9(4n-6)^3}{n} - 3n^2\Big(n^3 - 3n^2 + 3n - 3\Big)^2},$$

as desired. □

3. Examples

Let G be a simple graph. The notation $m_{i,j}$, $1 \leq i \leq j \leq n-1$, denote the number of edges of G connecting a vertex of degree i with a vertex of degree j.

It is preferred to show titania nanotubes as $TiO_2[m,n]$, where m and n denote the number of octagons in a row and in a column, respectively. See Figure 1 for details. The $TNT_3[m,n]$ is the two-parametric chemical graph of three-layered titania nanotubes, where m and n represent the number of titanium atoms in each row and column, respectively, Figure 2. Finally, $TNT_6[m,n]$ is the two-parametric chemical graph of a six-layered single-walled titania nanotube, where m and n represent the number of titanium atoms in each column and row, respectively, Figure 3.

The following proposition is a result of Table 1 and Proposition 2.2 in which the *ve*-degree and *ev*-degree connectivity indices of $TiO_2[m,n]$, $TNT_3[m,n]$ and $TNT_6[m,n]$ are given.

Proposition 3.1. *The following hold:*

$$\phi_v\Big(TiO_2[m,n]\Big) = 4m(65n+31), \quad \phi_e\Big(TiO_2[m,n]\Big) = 4m(107n+47),$$

Table 1. End point degree edges distributions of $TiO_2[m,n]$, $TNT_3[m,n]$ and $TNT_6[m,n]$

symbol	$m_{2,2}$	$m_{2,3}$	$m_{2,4}$	$m_{2,5}$	$m_{2,6}$	$m_{3,4}$	$m_{3,5}$	$m_{3,6}$
$TiO_2[m,n]$	0	0	$6m$	$4mn+2m$	0	$2m$	$6mn-2m$	0
$TNT_3[m,n]$	0	0	$4m$	0	$4m$	$4m$	0	$2m(6n-5)$
$TNT_6[m,n]$	$2m$	$2m$	$6m$	$8mn$	0	$2m$	$2m(6n-5)$	0

$$\phi_v\Big(TNT_3[m,n]\Big) = 8m(54n-13), \quad \phi_e\Big(TNT_3[m,n]\Big) = 2m(378n-101),$$
$$\phi_v\Big(TNT_6[m,n]\Big) = 4m(130n-29), \quad \phi_e\Big(TNT_6[m,n]\Big) = 4m(214n-55).$$

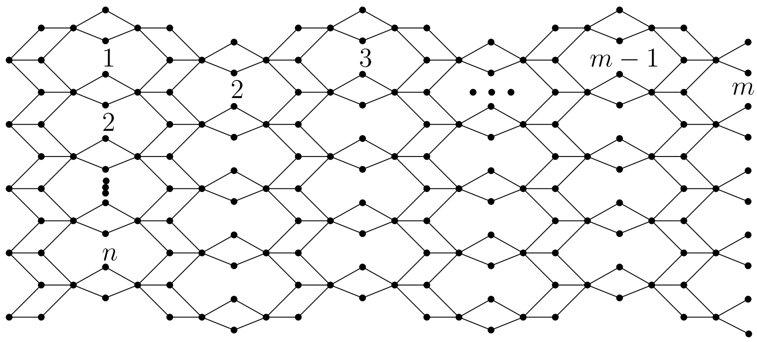

Figure 1. The molecular graph of titania nanotubes.

4. Concluding Remarks

In this paper, two graph invariants of the vertex-edge connectivity index and the edge-vertex connectivity index of a graph G were introduced. The main properties

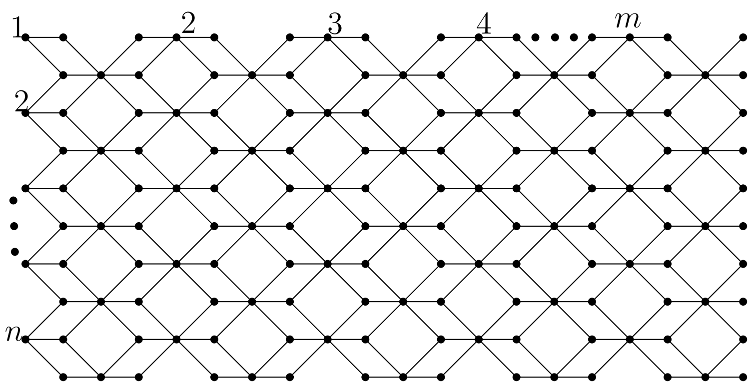

Figure 2. The graph of 3-layered titania nanotube.

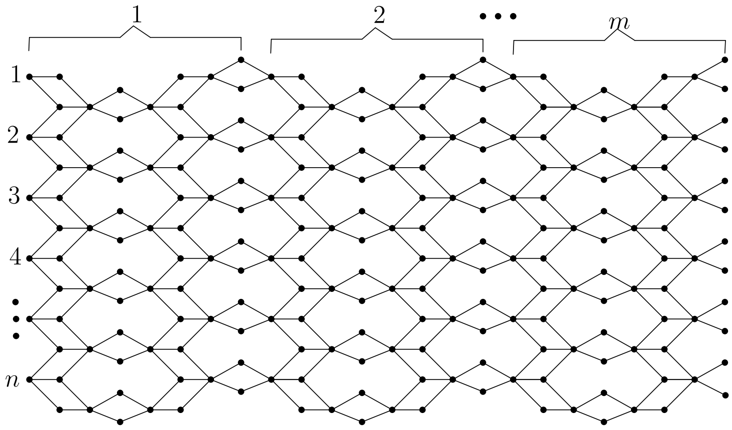

FIGURE 3. The graph of six-layered single walled titania nanotubes.

of these invariants were studied and we established some upper and lower bounds for them. These numbers for titania nanotubes are also computed.

Acknowledgements. The authors are indebted referees for his/her suggestions and careful remarks that leaded us to correct and improve this paper. The research of the first, second and third authors is supported by $UGC - SAP - DRS - II$, No. $F.510/12/DRS - II/2018(SAP - I)$, dated: April 9^{th}, 2018. The research of the fourth author is supported partially by the University of Kashan under grant No. 364988/617.

REFERENCES

[1] A. R. Ashrafi, T. Došlić and A. Hamzeh, *The Zagreb coindices of graph operations*, Discrete Appl. Math. **158** (2010), 1571–1578. `https://doi.org/10.1016/j.dam.2010.05.017`

[2] B. Basavanagoud and E. Chitra, *On the leap Zagreb indices of generalized xyz-point-line transformation graphs $T^{xyz}(G)$ when $z = 1$*, Int. J. Math. Combin. **2** (2018), 44–66.

[3] B. Borovićanin, K. C. Das, B. Furtula and I. Gutman, *Bounds for Zagreb indices*, MATCH Commun. Math. Comput. Chem. **78**(1) (2017), 17–100.

[4] R. Boutrig, M. Chellali, T. W. Haynes and S. T. Hedetniemi, *Vertex-edge domination in graphs,* Aequationes Math. **90**(2) (2016), 355–366. `https://doi.org/10.1007/s00010-015-0354-2`

[5] M. Cancan and M. S. Aldemir, *On ve-degree and ev-degree Zagreb index of titania nanotubes*, American Journal of Chemical Engineering **5**(6) (2017), 163–168. `https://doi.org/10.11648/j.ajche.20170506.18`

[6] M. Chellali, T. W. Haynes, S. T. Hedetniemi and T. M. Lewis, *On ve-degrees and ev-degrees in graphs*, Discrete Math. **340** (2017), 31–38. `https://doi.org/10.1016/j.disc.2016.07.008`

[7] K. C. Das and I. Gutman, *Some properties of the second Zagreb index*, MATCH Commun. Math. Comput. Chem. **52** (2004), 103–112.

[8] S. Ediz, *On ve-degree molecular topological properties of silicate and oxygen networks*, Int. J. Comput. Sci. Math. **9**(1) (2018), 1–12. `https://doi.org/10.1504/IJCSM.2018.090730`

[9] S. Ediz, *Predicting some physicochemical properties of octane isomers: A topological approach using evdegree and ve-degree Zagreb indices*, International Journal of Systems Science and Applied Mathematics. **2**(5) (2017), 87–92. https://doi.org/10.11648/j.ijssam.20170205.12

[10] M. Eliasi, A. Iranmanesh and I. Gutman, *Multiplicative versions of first Zagreb index*, MATCH Commun. Math. Comput. Chem. **68** (2012), 217–230.

[11] M. R. Farahani, M. K. Jamil and M. Imran, *Vertex PI_v topological index of Titania nanotubes,* Appl. Math. Nonlinear Sci. **1** (2016), 170–175. https://doi.org/10.21042/AMNS.2016.1.00013

[12] B. Furtula and I. Gutman, *A forgotten topological index*, J. Math. Chem. **53** (2015), 1184–1190. https://doi.org/10.1007/s10910-015-0480-z

[13] I. Gutman, *Multiplicative Zagreb indices of trees*, Bull. Int. Math. Virtual Inst. **18** (2011), 17–23.

[14] I. Gutman, B. Ruščic, N. Trinajstić and C. F. Wilcox, *Graph theory and molecular orbitals, XII. Acyclic polyenes*, J. Chem. Phys. **62** (1975), 3399–3405. https://doi.org/10.1063/1.430994

[15] I. Gutman and N. Trinajstić, *Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons*, Chem. Phys. Lett. **17** (1972), 535–538. https://doi.org/10.1016/0009-2614(72)85099-1

[16] I. Gutman, B. Furtula, Z. K. Vukićević and G. Popivoda, *On Zagreb indices and coindices*, MATCH Commun. Math. Comput. Chem. **74** (2015), 5–16.

[17] F. Harary, *Graph Theory*, Addison-Wesley Publishing Co., Reading, Mass. Menlo Park, London, 1969.

[18] J. Lewis, S. T. Hedetniemi, T. W. Haynes and G. H. Fricke, *Vertex-edge domination,* Util. Math. **81** (2010), 193–213.

[19] J. Lewis, *Vertex-edge and edge-vertex parameters in graphs*, Ph.D. Thesis, Clemson University, 2007.

[20] Y. Z. Li, N. H. Lee, E. G. Lee, J. S. Song and S. J. Kim, *The characterization and photocatalytic properties of mesoporous rutile TiO*2 *powder synthesized through cell assembly of nanocrystals,* Chem. Phys. Lett. **389** (2004), 124–128. https://doi.org/10.1016/j.cplett.2004.03.081

[21] K. W. Peters, *Theoretical and algorithmic results on domination and connectivity* (*Nordhaus-Gaddum, Gallai type results, max-min relationships, linear time, series-parallel*), Ph.D. Thesis, Clemson University, 1986.

[22] A. M. Naji, N. D. Soner and I. Gutman, *On leap Zagreb indices of graphs*, Commun. Comb. Optim. **2**(2) (2017), 99–117. https://doi.org/10.22049/CCO.2017.25949.1059

[23] A. M. Naji and N. D. Soner, *The first leap Zagreb index of some graph opertations*, International Journal of Applied Graph Theory **2**(1) (2018), 7–18.

[24] S. Nikolić, G. Kovačević, A. Milićević and N. Trinajstić, *The Zagreb indices* 30 *years after*, Croatica Chemica Acta **76** (2003), 113–124.

[25] P. Shiladhar, A. M. Naji and N. D. Soner, *Leap Zagreb indices of some wheel related graphs*, J. Comput. Math. Sci. **9**(3) (2018), 221–231.

[26] P. Shiladhar, A. M. Naji and N. D. Soner, *Computation of leap Zagreb indices of some windmill graphs*, International Journal of Mathematics and its Applications **6**(2-B) (2018), 183–191.

[27] B. Sahin and S. Ediz, *On ev-degree and ve-degree topological indices*, Iranian Journal of Mathematical Chemistry **9**(4) (2018), 263–277. https://doi.org/10.22052/IJMC.2017.72666.1265

[28] N. D. Soner and A. M. Naji, *The k-distance neighborhood polynomial of a graph*, Int. J. Math. Comput. Sci. **3**(9) (2016), 2359–2364.

[29] R. Todeschini and V. Consonni, *New local vertex invariants and molecular descriptors based on functions of the vertex degrees*, MATCH Commun. Math. Comput. Chem. **64** (2010), 359–372.

[30] K. Xu and H. Hua, *A unified approach to extremal multiplicative Zagreb indices for trees, unicyclic and bicyclic graphs*, MATCH Commun. Math. Comput. Chem. **68** (2012), 241–256.

[31] K. Xu, K. C. Das and K. Tang, *On the multiplicative Zagreb coindex of graphs*, Opuscula Math. **33**(1) (2013), 197–210. http://dx.doi.org/10.7494/OpMath.2013.33.1.191

[32] S. Yamaguchi, *Estimating the Zagreb indices and the spectral radius of triangle and quadrangle-free connected graphs,* Chemical Physics Letters **458**(4) (2008), 396–398. https://doi.org/10.1016/j.cplett.2008.05.009

[1]Department of Studies in Mathematics,
University of Mysore,
Manasagangotri, Mysuru-570 006, India
Email address: shiladharpawar@gmail.com
Email address: ama.mohsen78@gmail.com

[2]Department of Mathematics, Faculty of Education,
Thamar University,
Thamar, Yemen
Email address: ndsoner@yahoo.co.in

[3]Department of Pure Mathematics, Faculty of Mathematical Sciences,
University of Kashan,
Kashan, I. R. Iran
Email address: ashrafi@kashanu.ac.ir
Email address: alighalavand@grad.kashanu.ac.ir

*Corresponding author