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A STUDY OF MULTI-TERM TIME-FRACTIONAL DELAY DIFFERENTIAL SYSTEM WITH MONOTONIC CONDITIONS

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Abstract. In this paper, the existence and uniqueness of mild solution for a class of multi-term time-fractional delay differential system have been discussed in ordered Banach space by enforcing monotone iterative technique. The generalized semigroup theory, fractional calculus and measure of noncompactness have been implemented to obtain the required results. A new set of sufficient conditions with the coefficients in the equations satisfying some monotonic properties has been obtained. Finally, an application is given to illustrate the obtained results.

1. Introduction

The fractional differential equations (in brief, FDEs) including Riemann-Liouville and Caputo fractional derivatives have been magnetizing the interest of many researchers, due to demonstrating applications in widespread areas of sciences and engineering such as mathematical modeling, thermal systems, acoustics, modeling of materials or rheology and mechanical systems. The FDEs have been viewed as a beneficial tool, which may describe dynamical behavior of real life phenomena more precisely. In addition, due to the memory and hereditary properties of various materials and processes, in many areas of science like identification systems, signal processing, robotics or control theory, fractional differential operators seem more appropriate in modeling than the classical integer operators. One can also find the various applications of FDEs in models of medicine (modeling of human tissue under

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mechanical loads), electrical engineering (transmission of ultrasound waves), biochemistry (modeling of proteins and polymers) etc. For more knowledge regarding to fractional systems see the papers [2, 8, 9, 11, 12, 28, 32], the monographs [24, 31, 33] and references therein. In addition, fractional delay differential equations have been used frequently in various fields of science and engineering such as panorama of natural phenomena, modeling of equations and porous media etc. For more detail, see the cited papers [2, 3, 19].

It is very difficult to obtain the exact solutions for the nonlinear fractional differential systems in closed forms. To overcome this difficulty, many analytical and numerical techniques have been developed for instance, the Adomian decomposition method [21] and the homotopy analysis method [36], have been applied to investigate various systems of fractional or non-fractional ordered. However, in recent years, considerable work has been reported in the literature by applying monotone iterative technique, which is a flexible and very effective mechanism to study the existence results in a closed set governed by the lower and upper solutions, to investigate the existence of solutions for a class of fractional differential systems. In monotone iterative technique, we construct two monotone sequences by choosing upper and lower solutions as two initial iterations, which converge uniformly to a extremal mild solution of the system between the lower and upper solutions. Due to monotone behavior, the constructed sequences of iterations play an important role in the study of numerical solutions of various initial value and boundary value problems.

From the last few years, multi-term time-fractional differential equations have been generating great interest among the mathematicians and engineers. In [23, 28, 34], a two-term time-fractional differential equation has been studied in the abstract context, which include a concrete example of fractional diffusion-wave problem. In [13] and [29], the multi-term time-fractional diffusion wave equation have been considered with constant and variable coefficients, respectively. Moreover, in [22, 27], the analytical and numerical solutions of multi-term time-fractional diffusion equation have been discussed. In [32], Pardo and Lizama studied the existence of mild solutions of multi-term time-fractional differential equations with nonlocal initial conditions by using Caratheodory type conditions and measure of noncompactness technique. In last few years, many authors repeatedly apply the monotone iterative technique coupled with lower and upper solutions to various functional differential equations of integer order as well as fractional order, see [4–7, 25, 26, 35] and the references therein. However, in the best of authors' knowledge, no work is reported to the multi-term time-fractional differential system in the literature, by enforcing monotone iterative technique.

In this paper, monotone iterative technique coupled with method of lower and upper solutions has been applied to analyze the existence of mild solution for the following multi-term time-fractional delay differential system

$$\begin{cases} {}^cD^{1+\beta}y(t) + \sum_{j=1}^{n} \alpha_j {}^cD^{\gamma_j}y(t) = Ay(t) + F\left(t, y_t, \int_0^t h(t,s,y_s)ds\right), & t \in \mathcal{J}, \\ y(t) = \phi(t) \in \mathfrak{B}, \quad t \in (-\infty, 0], \quad y'(0) = \chi, \end{cases} \tag{1.1}$$

where ${}^cD^\eta$ stands for the Caputo fractional derivative of order $\eta > 0$ and operational interval $\mathcal{J} = [0, T]$, $T < \infty$. $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ is a closed linear operator on a Banach space $(\mathbb{X}, \|\cdot\|)$. All γ_j, $j = 1, 2, \ldots, n$, $n \in \mathbb{N}$, are positive real numbers such that $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$. The nonlinear functions $F : \mathcal{J} \times \mathfrak{B} \times \mathbb{X} \to \mathbb{X}$ and $h : \Delta \times \mathfrak{B} \to \mathbb{X}$ satisfies some suitable conditions, which will be mentioned later. $\Delta := \{(t, s) : 0 \leq s \leq t \leq T\}$. The delay function $y_t : (-\infty, 0] \to \mathbb{X}$ is characterized by $y_t(s) = y(t + s)$ for $s \in (-\infty, 0]$.

The system (1.1) is a general system, which includes recent investigations in this subject [13, 23, 28, 29, 32, 34]. Anticipating a great interest in the problems modeled as the system (1.1), this paper contributes in study of the existence results for mild solutions by applying monotone iterative technique coupled with the method of lower and upper solutions. It should be noticed that, the semigroup theory may not be directly used to solve problem (1.1). However, we construct a mild solution, which is based on the theory of resolvent families [32], which will provide an effective way to deal such problems.

This paper is organized as follows: In Section 2, some basics of fractional calculus and measure of noncompactness have been discussed which will be employed to obtain mains outcomes. In Section 3, the existence and uniqueness results are obtained for the mild solutions of the system (1.1). In Section 4, an example is provided to show the feasibility of the theory discussed in this paper.

2. Preliminaries

Let $\mathbb{R}$ and $\mathbb{N}$ denote the real and natural numbers, respectively. Let us denote $\mathcal{D}(A)$, $\mathcal{R}(A)$ and $\rho(A)$ by the domain, range and resolvent of a linear operator A on $\mathbb{X}$, respectively. Define a partial ordering in $\mathbb{X}$ introduced by a positive cone $\mathbb{P} = \{y \in \mathbb{X} : y \geq \theta\}$ (where θ symbolizes the zero element of $\mathbb{X}$) such that $x \leq y$ if and only if $y - x \in \mathbb{P}$. If $x \leq y$ and $x \neq y$, then $x < y$. A cone $\mathbb{P}$ is called a normal cone if there exists a constant $N > 0$ (called normal constant) such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$. A cone $\mathbb{P} \subset \mathbb{X}$ is said to be regular cone if every increasing, bounded above sequence is convergent, i.e., if $\{w_n\}$ be a sequence such that

$$w_1 \leq w_2 \leq \cdots \leq w_n \leq \cdots \leq z,$$

for some $z \in \mathbb{X}$, then there is a $w \in \mathbb{X}$ such that $\|w_n - w\| \to 0$ as $n \to \infty$. Equivalently, a cone $\mathbb{P} \subset \mathbb{X}$ is said to be regular if every bounded below and decreasing sequence is convergent. It should be notice that a regular cone is a normal cone. For more details regarding to the cone $\mathbb{P}$, see [14]. The Banach space of all continuous $\mathbb{X}$-valued functions is represented by $\mathcal{C}(\mathcal{J}, \mathbb{X})$, on the interval $\mathcal{J}$ equipped with norm $\|u\|_{\mathcal{C}} = \sup_{t \in \mathcal{J}} \|u(t)\|$.

To facilitate the discussion, due to infinite delay an axiomatic definition of the phase space $\mathfrak{B}$ has been introduced by Hale and Kato [16]. Recall, the axioms of the phase space $\mathfrak{B}$, by following the terminology used by Hino et al. in [19] so, we omit the details here.

A linear space $\mathfrak{B}$ consists of all functions defined from $(-\infty, 0]$ into $\mathbb{X}$ equipped with the seminorm $\|\cdot\|_{\mathfrak{B}}$ satisfying the following axioms.

(a) If $y : (-\infty, T] \to \mathbb{X}$, $T > 0$ is continuous on $\mathfrak{I}$ and $y_0 \in \mathfrak{B}$, then for every $t \in \mathfrak{I}$ the accompanying conditions hold:
   (i) y_t is a $\mathfrak{B}$-valued continuous function;
   (ii) $\|y(t)\| \le K\|y_t\|_{\mathfrak{B}}$;
   (iii) $\|y_t\|_{\mathfrak{B}} \le K_1(t)\sup_{s\in[0,t]}\|y(s)\| + K_2(t)\|y_0\|_{\mathfrak{B}}$, where $K \ge 0$ is a constant and $K_1(\cdot) : [0,\infty) \to [0,\infty)$ is continuous, $K_2(\cdot) : [0,\infty) \to [0,\infty)$ is locally bounded and K_1, K_2 are independent of $y(\cdot)$.

(b) The space $\mathfrak{B}$ is complete.

Now, recall some definitions and basic results on fractional calculus. Define $g_\eta(t)$ for $\eta > 0$ by

$$g_\eta(t) = \begin{cases} \frac{1}{\Gamma(\eta)}t^{\eta-1}, & t > 0, \\ 0, & t \le 0, \end{cases}$$

where Γ denotes gamma function. The function g_η has the properties $(g_a * g_b)(t) = g_{a+b}(t)$ for $a, b > 0$ and $\widehat{g_\eta}(\lambda) = \frac{1}{\lambda^\eta}$ for $\eta > 0$ and $\operatorname{Re}\lambda > 0$, where $\widehat{(\cdot)}$ and $(\cdot * \cdot)(\cdot)$ denote the Laplace transformation and convolution, respectively.

Definition 2.1. The Riemann-Liouville fractional integral of a function $f \in L^1_{loc}$ $([0,\infty), \mathbb{X})$ of order $\eta > 0$ with lower limit zero is defined as follows

$$I^\eta f(t) = \int_0^t g_\eta(t-s)f(s)ds, \quad t > 0,$$

and $I^0 f(t) = f(t)$.

This fractional integral satisfies the properties $I^\eta \circ I^b = I^{\eta+b}$ for $b > 0$, $I^\eta f(t) = (g_\eta * f)(t)$ and $\widehat{I^\eta f}(\lambda) = \frac{1}{\lambda^\eta}\widehat{f}(\lambda)$ for $\operatorname{Re}\lambda > 0$.

Definition 2.2. Let $\eta > 0$ be given and denote $m = \lceil \eta \rceil$. The Caputo fractional derivative of order $\eta > 0$ of a function $f : [0,\infty) \to \mathbb{X}$ with lower limit zero is given by

$${}^cD^\eta f(t) = I^{m-\eta}D^m f(t) = \int_0^t g_{m-\eta}(t-s)D^m f(s)ds,$$

and ${}^cD^0 f(t) = f(t)$, where $D^m = \frac{d^m}{dt^m}$. In addition, we have ${}^cD^\eta f(t) = (g_{m-\eta} * D^m f)(t)$ and the Laplace transformation of Caputo fractional derivative is given by

$$\widehat{{}^cD^\eta f}(t) = \lambda^\eta \widehat{f}(\lambda) - \sum_{d=0}^{m-1} f^{(d)}(0)\lambda^{\eta-1-d}, \quad \lambda > 0. \tag{2.1}$$

Remark 2.1. Let $m - 1 < \eta \le m,\ m \in \mathbb{N}$, then

$$(I^\eta \circ {}^cD^\eta)f(t) = f(t) - \sum_{d=0}^{m-1} f^{(d)}(0)g_{d+1}(t), \quad t > 0. \tag{2.2}$$

If $f^{(d)}(0) = 0$, for $d = 1, 2, 3, \ldots, m-1$, then $(I^\eta \circ {}^cD^\eta)f(t) = f(t)$ and $\widehat{{}^cD^\eta f}(t) = \lambda^\eta \widehat{f}(\lambda)$.

To give a appropriate representation of mild solution in terms of certain family of bounded and linear operators, we define following family of operators.

Definition 2.3 ([32])**.** Let A be a closed linear operator on a Banach space $\mathbb{X}$ with the domain $\mathcal{D}(A)$ and let $\beta > 0$, γ_j, α_j be the real positive numbers. Then A is called the generator of a $(\beta, \gamma_j)-$ resolvent family if there exists $\omega > 0$ and a strongly continuous function $\mathcal{S}_{\beta,\gamma_j} : [0, \infty) \to \mathcal{L}(\mathbb{X})$ (the space of bounded linear operators on $\mathbb{X}$) such that $\{\lambda^{\beta+1} + \sum_{j=1}^n \alpha_j \lambda^{\gamma_j} : \operatorname{Re}\lambda > \omega\} \subset \rho(A)$ and

$$\lambda^\beta \left(\lambda^{\beta+1} + \sum_{j=1}^n \alpha_j \lambda^{\gamma_j} - A\right)^{-1} y = \int_0^\infty e^{-\lambda t} \mathcal{S}_{\beta,\gamma_j}(t) y dt, \quad \operatorname{Re}\lambda > \omega,\, y \in \mathbb{X}. \tag{2.3}$$

The following result guarantees the existence of (β, γ_j)-resolvent family under some suitable conditions.

Theorem 2.1 ([32])**.** *Let $0 < \beta \le \gamma_n \le \cdots \le \gamma_1 \le 1$ and $\alpha_j \ge 0$ be given and let A be a generator of a strongly continuous and bounded cosine family $\{C(t)\}_{t\in\mathbb{R}}$. Then A generates a bounded (β, γ_j)-resolvent family $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\ge 0}$.*

Let Ω be the set defined by

$$\Omega = \{y \in \mathcal{C}((-\infty, T], \mathbb{X}) : \text{ such that } y_{|_{(-\infty,0]}} \in \mathfrak{B} \text{ and } y_{|_{[0,T]}} \in \mathbb{X}\}.$$

In order to define the mild solution for the system (1.1), we associate system (1.1) with an integral equation, by comparison with the fractional differential system given in [32]. Consider the following definition of mild solution for the system (1.1).

Definition 2.4. Let $0 < \beta \le \gamma_n \le \cdots \le \gamma_1 \le 1$ and $\alpha_j \ge 0$ be given and let A be a generator of a bounded (β, γ_j)-resolvent family $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\ge 0}$. Then a function $y \in \Omega$ is called the mild solution of the system (1.1) if $y'(0) = \chi$ and satisfies the equation

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \mathcal{S}_{\beta,\gamma_j}(t)\phi(0) + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi & \\ + \sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)\phi(0) ds & \\ + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) F\left(s, y_s, \int_0^s h(s,\tau,y_\tau)d\tau\right) ds, & t \in \mathcal{J}, \end{cases} \tag{2.4}$$

where $\mathcal{T}_{\beta,\gamma_j}(t) = (g_\beta * \mathcal{S}_{\beta,\gamma_j})(t)$.

Definition 2.5. The resolvent family $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\ge 0}$ is said to be positive on $\mathbb{X}$, if the order inequality $\mathcal{S}_{\beta,\gamma_j}(t)y \ge \theta$ holds for all $y \ge \theta$, $y \in \mathbb{X}$ and $t \ge 0$.

Lemma 2.1 ([17])**.** (Generalized Gronwall inequality). *Assume $\gamma \ge 0$, $\delta > 0$ and $c(t)$ is a nonnegative and locally integrable function on $0 \le t < T < +\infty$ and let $z(t)$ be nonnegative and locally integrable on $0 \le t < T + \infty$ such that*

$$z(t) \le c(t) + \gamma \int_0^t (t-s)^{\delta-1} z(s) ds,$$

then

$$z(t) \leq c(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(\gamma\Gamma(\delta))^n}{\Gamma(n\delta)} (t-s)^{n\delta-1} c(s) \right] ds, \quad 0 \leq t < T.$$

Let $\mathcal{C}^{1+\beta}((-\infty,T],\mathbb{X}) = \{y \in \mathcal{C}((-\infty,T],\mathbb{X}) : {}^cD^{1+\beta}y(t)$ exists and continuous on $\mathcal{J}$ and $y(t) \in \mathcal{D}(A)$ for all $t \geq 0\}$. An abstract function $y(t) \in \mathcal{C}^{1+\beta}((-\infty,T],\mathbb{X})$ is said to be a solution of (1.1) of if $y(t)$ satisfies the system (1.1).

Definition 2.6. The function $y^{(0)} \in \mathcal{C}^{1+\beta}((-\infty,T],\mathbb{X})$ is said to be a lower solution of the system (1.1), if it satisfies the following inequalities

$$\left\{ \begin{array}{ll} {}^cD^{1+\beta}y^{(0)}(t) + \sum_{j=1}^n \alpha_j {}^cD^{\gamma_j}y^{(0)}(t) \leq Ay^{(0)}(t) & \\ +F\left(t, y_t^{(0)}, \int_0^t h(t,s,y_s^{(0)})ds\right), & t \in \mathcal{J}, \\ y^{(0)}(t) \leq \phi(t) \in \mathfrak{B}, & t \in (-\infty,0],\ y'^{(0)}(0) \leq \chi. \end{array} \right. \tag{2.5}$$

If all the inequalities of (2.5) are reversed, then solution is called upper solution denoted by $z^{(0)}$.

Now, we recall some basic definitions and properties of Kuratowski measure of noncompactness. For more details, we refer to the monograph [14] and paper [10, 18].

Definition 2.7. Let $\mathbb{F}$ be a bounded subset of a Banach space $\mathbb{X}$. The Kuratowski measure of noncompactness denoted by $\mu(\cdot)$ of $\mathbb{F}$ is defined by

$$\mu(\mathbb{F}) := \inf\{\delta > 0 : \mathbb{F} = \cup_{i=1}^n \mathbb{F}_i \text{ with } \operatorname{diam}(\mathbb{F}_i) \leq \delta \text{ for } i = 1,2,3,\ldots,n\}.$$

Lemma 2.2. *Let $\mathbb{X}$ be a Banach space, and let $\mathbb{F} \subset \mathcal{C}([a_1,a_2],\mathbb{X})$ be bounded and equicontinuous. Then $\mu(\mathbb{F}(t))$ is continuous on $[a_1,a_2]$ and*

$$\mu_{\mathcal{C}}(\mathbb{F}) = \sup_{t \in [a_1,a_2]} \mu(\mathbb{F}(t)).$$

Lemma 2.3. *Let $\{y_n\}_{n=1}^{\infty} \subset L^1(\mathcal{J},\mathbb{X})$ be a sequence and there exists $g \in L^1(\mathcal{J},\mathbb{X})$ such that $\|y_n(t)\| \leq g(t)$, a.e. $t \in \mathcal{J}$, then $\mu(\{y_n(t)\}_{n=1}^{\infty})$ is integrable and*

$$\mu\left(\left\{ \int_0^t y_n(s)ds \right\}_{n=1}^{\infty}\right) \leq 2\int_0^t \mu(\{y_n(s)\}_{n=1}^{\infty}ds.$$

Lemma 2.4. *If $\mathbb{F}$ is bounded subset of $\mathbb{X}$, then there exists $\{y_n\}_{n=1}^{\infty} \subset \mathbb{F}$, such that $\mu(\mathbb{F}) \leq 2\mu(\{y_n\}_{n=1}^{\infty})$.*

3. Main Results

Throughout in this section, we denote $S_0 = \sup_{t\in[0,T]} \|\mathcal{S}_{\beta,\gamma_j}(t)\|$. We consider the following assumptions.

(A1) The functions $h : \Delta \times \mathfrak{B} \to \mathbb{X}$ and $F : \mathcal{J} \times \mathfrak{B} \times \mathbb{X} \to \mathbb{X}$, satisfy Carathéodory type conditions, i.e.,

 (i) $h(t,s,\cdot) : \mathfrak{B} \to \mathbb{X}$ is continuous for $(t,s) \in \Delta$ and $h(\cdot,\cdot,v) : \Delta \to \mathbb{X}$ is strongly measurable for all $v \in \mathfrak{B}$;

(ii) $F(t,\cdot,\cdot):\mathfrak{B}\times\mathbb{X}\to\mathbb{X}$ is continuous for each $t\in\mathcal{I}$ and $F(\cdot,u,v):\mathcal{I}\to\mathbb{X}$ is strongly measurable for all $(u,v)\in\mathfrak{B}\times\mathbb{X}$.

(A2) For lower and upper solutions $y^{(0)},z^{(0)}\in\mathcal{C}^{1+\beta}((-\infty,T],\mathbb{X})$ of the system (1.1) such that $y^{(0)}\le z^{(0)}$ the following conditions hold:

(i) $F(t,v_1,w_1)\le F(t,v_2,w_2)$ for all $t\in\mathcal{I}$, and $v_1,v_2\in\mathfrak{B}$ satisfying $y_t^{(0)}\le v_1\le v_2\le z_t^{(0)}$ and $w_1,w_2\in\mathbb{X}$ such that $\int_0^t h(t,s,y_s^{(0)})ds\le w_1\le w_2\le \int_0^t h(t,s,z_s^{(0)})ds$;

(ii) $h(t,s,v_1)\le h(t,s,v_2)$ for all $(t,s)\in\Delta$ and $v_1,v_2\in\mathfrak{B}$ such that $y_t^{(0)}\le v_1\le v_2\le z_t^{(0)}$.

(A3) The functions F,h satisfy the followings conditions.

(i) For $G\subset\mathfrak{B}$ and $H\subset\mathbb{X}$, where $G(r)=\{\varphi(r):r\in(-\infty,0],\varphi\in G\}$ there exists a constant $L>0$ such that

$$\mu(F(t,G,H))\le L\left[\sup_{-\infty<r\le 0}\mu(G(r))+\mu(H)\right],\quad \text{a.e. } t\in\mathcal{I}.$$

(ii) For each bounded set $G\subset\mathfrak{B}$, there exists an integrable function $\xi:\Delta\to[0,\infty)$ such that

$$\mu(h(t,s,G))\le\xi(t,s)\sup_{-\infty<r\le 0}\mu(G(r)),$$

for a.e. $(t,s)\in\Delta$. For convenience, we denote $\xi^*=\max\int_0^t\xi(t,s)ds$.

In order to give operator theoretical approach, we define a operator $Q:\Omega\to\Omega$ by

$$(3.1)\quad (Qy)(t)=\begin{cases}\phi(t), & t\in(-\infty,0],\\ \mathcal{S}_{\beta,\gamma_j}(t)\phi(0)+(g_1*\mathcal{S}_{\beta,\gamma_j})(t)\chi & \\ +\sum_{j=1}^n\alpha_j\int_0^t\frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)\phi(0)ds & \\ +\int_0^t\mathcal{T}_{\beta,\gamma_j}(t-s)F\left(s,y_s,\int_0^s h(s,\tau,y_\tau)d\tau\right)ds, & t\in\mathcal{I}.\end{cases}$$

It is clear to see that Q is well defined.

Let us define a function $u(\cdot):(-\infty,T]\to\mathbb{X}$ by

$$u(t)=\begin{cases}\phi(t), & t\in(-\infty,0],\\ 0, & t\in\mathcal{I}.\end{cases}$$

For a function $v:(-\infty,T]\to\mathbb{X}$ such that $v(0)=0$, we define the function $\overline{v}$ by

$$\overline{v}(t)=\begin{cases}0, & t\in(-\infty,0],\\ v(t), & t\in\mathcal{I}.\end{cases}$$

If $y(\cdot)$ is a solution of (2.4), then it can be decompose $y(\cdot)$ as $y(t)=u(t)+\overline{v}(t)$, $t\in(-\infty,T]$ and $v(\cdot)$ satisfies

$$\begin{aligned}v(t)=&\mathcal{S}_{\beta,\gamma_j}(t)\phi(0)+(g_1*\mathcal{S}_{\beta,\gamma_j})(t)\chi+\sum_{j=1}^n\alpha_j\int_0^t\frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)\phi(0)ds\\ &+\int_0^t\mathcal{T}_{\beta,\gamma_j}(t-s)F\left(s,u_s+\overline{v}_s,\int_0^s h(s,\tau,u_\tau+\overline{v}_\tau)d\tau\right)ds.\end{aligned}$$

Define $\mathbb{X}_0 = \{v \in \Omega : v_0 = 0\}$. For any $v \in \mathbb{X}_0$,

$$\|v\|_{\mathbb{X}_0} = \sup_{t\in\mathcal{J}} \|v(t)\| + \|v_0\|_{\mathfrak{B}} = \sup_{t\in\mathcal{J}} \|v(t)\|.$$

Clearly, $\mathbb{X}_0$ is a Banach space equipped with the norm $\|\cdot\|_{\mathbb{X}_0}$. We assume that $(\mathbb{X}_0, \|\cdot\|_{\mathbb{X}_0})$ stands for a ordered Banach space with partial order $\leq$ induced by a positive normal cone $\mathbb{P}_0 = \{v \in \mathbb{X}_0 : v(t) \geq \theta\}$ with the normal constant N_0. Evidently $\mathcal{C}((-\infty, T], \mathbb{X}_0)$ is also an ordered Banach space with the partial order $\leq$ reduced by a positive normal cone $\mathbb{P}_0 = \{v \in \mathbb{X}_0 : v(t) \geq \theta, t \in (-\infty, T]\}$ with normal constant N_0. For $v, w \in \mathcal{C}((-\infty, T], \mathbb{X}_0)$ such that $v \leq w$, $[v, w]$ denotes a ordered interval $\{x \in \mathcal{C}((-\infty, T], \mathbb{X}_0) : v \leq x \leq w\}$ in $\mathcal{C}((-\infty, T], \mathbb{X}_0)$ and $[v(t), w(t)]$ denotes the ordered interval $\{x \in \mathcal{C}((-\infty, T], \mathbb{X}_0) : v(t) \leq x(t) \leq w(t)\}$ in $\mathbb{X}_0$.

Theorem 3.1. *Let $\mathbb{X}_0$ be an ordered Banach space with a positive normal cone $\mathbb{P}_0$. Suppose that the system* (1.1) *admits lower and upper solutions denoted by $v^{(0)}, w^{(0)} \in \mathcal{C}^{1+\beta}(\mathcal{J}, \mathbb{X})$ such that $v^{(0)} \leq w^{(0)}$, $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$ is a positive operator and the assumptions* (A1)-(A3) *are satisfied. Then the system* (1.1) *admits maximal and minimal mild solutions between $w^{(0)}$ and $v^{(0)}$.*

Proof. Let $D = [v^{(0)}, w^{(0)}] = \{u \in \mathcal{C}(\mathcal{J}, \mathbb{X}_0) : v^{(0)} \leq u \leq w^{(0)}\}$. Define a map $\tilde{Q} : D \to \mathbb{X}_0$ by

(3.2)

$$(\tilde{Q}v)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \mathcal{S}_{\beta,\gamma_j}(t)\phi(0) + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi & \\ + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)\phi(0)ds & \\ + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) F\left(s, u_s + \overline{v}_s, \int_0^s h(s,\tau,u_\tau + \overline{v}_\tau)d\tau\right) ds, & t \in \mathcal{J}. \end{cases}$$

From (A1)-(A2) for any $v \in D$ we have

$$\begin{aligned} F\left(t, u_t + v_t^{(0)}, \int_0^t h(t,\tau,u_\tau + v_\tau^{(0)})d\tau\right) \leq & F\left(t, u_t + \overline{v}_t, \int_0^t h(t,\tau,u_\tau + \overline{v}_\tau)d\tau\right) \\ \leq & F\left(t, u_t + w_t^{(0)}, \int_0^t h(t,\tau,u_\tau + w_\tau^{(0)})d\tau\right). \end{aligned}$$

Now, using normality of the cone $\mathbb{P}_0$, there exists a constant $\mathfrak{K} > 0$ such that

$$\left\| f\left(t, u_t + \overline{v}_t, \int_0^t g(t,\tau,u_\tau + \overline{v}_\tau)d\tau\right) \right\| \leq \mathfrak{K}, \quad v \in D.$$

For convenience, we divide the proof in the following steps.

Step 1. The map $\tilde{Q}$ is continuous map on D.

Let $\{v^{(n)}\}$ be a sequence in D such that $\{v^{(n)}\} \to v \in D$ as $n \to \infty$. For $t \in (-\infty, 0]$ we get

$$\|\tilde{Q}v^{(n)}(t) - \tilde{Q}v(t)\| = 0.$$

Also, from (A1) for $t \in \mathcal{J}$ and as $n \to \infty$, we have

(i) $\int_0^s h(s,\tau,u_\tau+\overline{v}_\tau^{(n)})d\tau \to \int_0^s h(s,\tau,u_\tau+\overline{v}_\tau)d\tau$;

(ii) $F\left(s,u_s+\overline{v}_s^{(n)},\int_0^s h(s,\tau,u_\tau+\overline{v}_\tau^{(n)})d\tau\right) \to F\left(s,u_s+\overline{v}_s,\int_0^s h(s,\tau,u_\tau+\overline{v}_\tau)d\tau\right)$.

Now, by applying Lebesgue Dominated Convergence Theorem for $t\in\mathcal{J}$, we have

$$\begin{aligned}\|\tilde{Q}v^{(n)}(t)-\tilde{Q}v(t)\| \le &\int_0^t \left\|\mathcal{T}_{\beta,\gamma_j}(t-s)\right\|\bigg\|F\left(s,u_s+\overline{v}_s^{(n)},\int_0^s h(s,\tau,u_\tau+\overline{v}_\tau^{(n)})d\tau\right)\\ &-F\left(s,u_s+\overline{v}_s,\int_0^s h(s,\tau,u_\tau+\overline{v}_\tau)d\tau\right)\bigg\|ds\\ \to&0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

Thus map $\tilde{Q}$ is continuous on D.

Step 2. $\tilde{Q}$ is a increasing monotonic operator.

Consider x, $y\in D$ with $x\le y$ then $x(t)\le y(t)$ for $t\in\mathcal{J}$. Therefore, x_t, y_t belong to the ordered Banach space $\mathbb{X}_0$ such that $x_t\le y_t$ for $t\in\mathcal{J}$. Using (A2) and positivity of $\mathcal{S}_{\beta,\gamma_j}(t)$, we obtain

$$\tilde{Q}x\le\tilde{Q}y. \tag{3.3}$$

Now, we show that $v^{(0)}\le\tilde{Q}v^{(0)}$ and $\tilde{Q}w^{(0)}\le w^{(0)}$. For this, let

$$g(t)={}^cD^{1+\beta}v^{(0)}(t)+\sum_{j=1}^n\alpha_j{}^cD^{\gamma_j}v^{(0)}(t)-Av^{(0)}(t)$$

subject to the conditions $v^{(0)}(0)=y_0$, $v'^{(0)}(0)=y_1$.

Then by definition of lower solution, we obtain $g(t)\le F\left(t,y_t,\int_0^t h(t,s,y_s)ds\right)$ for $t\in\mathcal{J}$. Since $v^{(0)}(t)$ is a lower solution of (1.1), we get

$$\begin{aligned}v^{(0)}(t)=&\mathcal{S}_{\beta,\gamma_j}(t)y_0+(g_1*\mathcal{S}_{\beta,\gamma_j})(t)y_1+\sum_{j=1}^n\alpha_j\int_0^t\frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)y_0ds\\ &+\int_0^t\mathcal{T}_{\beta,\gamma_j}(t-s)g(s)ds\\ \le&\mathcal{S}_{\beta,\gamma_j}(t)\phi(0)+(g_1*\mathcal{S}_{\beta,\gamma_j})(t)\chi+\sum_{j=1}^n\alpha_j\int_0^t\frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)\phi(0)ds\\ &+\int_0^t\mathcal{T}_{\beta,\gamma_j}(t-s)F\left(s,u_s+v_s^{(0)},\int_0^s h(s,\tau,u_\tau+v_\tau^{(0)})d\tau\right)ds\\ \le&\tilde{Q}v^{(0)}(t),\quad t\in\mathcal{J},\end{aligned}$$

and also $v^{(0)}(t)\le\phi(t)$, $v'^{(0)}(0)\le\chi$. Therefore, $v^{(0)}(t)\le\tilde{Q}v^{(0)}(t)$ for all $t\in(-\infty,T]$. Similarly, we can show that $w^{(0)}(t)\ge\tilde{Q}w^{(0)}(t)$ for all $t\in(-\infty,T]$. Thus, $\tilde{Q}$ is a increasing monotonic operator.

Step 3. $\tilde{Q}$ is an equicontinuous operator.

For any $v\in D$ and t_1, $t_2\in(-\infty,0]$ such that $t_1<t_2$, we have

$$\|\tilde{Q}v(t_2)-\tilde{Q}v(t_1)\|=0.$$

Further for $v \in D$ and $t_1, t_2 \in \mathcal{J}$ such that $t_1 < t_2$, we have

$$\begin{aligned}
\|\tilde{Q}v(t_2) - \tilde{Q}v(t_1)\| \leq & \|\mathcal{S}_{\beta,\gamma_j}(t_2)\phi(0) - \mathcal{S}_{\beta,\gamma_j}(t_1)\phi(0)\| \\
& + \|(g_1 * \mathcal{S}_{\beta,\gamma_j})(t_2) - (g_1 * \mathcal{S}_{\beta,\gamma_j})(t_1)\|\|\chi\| \\
& + \sum_{j=1}^{n} \alpha_j S_0 \bigg\| \int_0^{t_2} \frac{(t_2 - s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} ds \\
& - \int_0^{t_1} \frac{(t_1 - s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} ds \bigg\| \|\phi(0)\| \\
& + \int_0^{t_1} \|\mathcal{T}_{\beta,\gamma_j}(t_2 - s) - \mathcal{T}_{\beta,\gamma_j}(t_1 - s)\| \\
& \times \left\| F\left(s, u_s + \overline{v}_s, \int_0^s h(s,\tau,u_\tau + \overline{v}_\tau) d\tau\right) \right\| ds \\
& + \int_{t_1}^{t_2} \|\mathcal{T}_{\beta,\gamma_j}(t_2 - s)\| \left\| F\left(s, u_s + \overline{v}_s, \int_0^s h(s,\tau,u_\tau + \overline{v}_\tau) d\tau\right) \right\| ds. \\
= & \sum_{i=1}^{5} J_i.
\end{aligned}$$

We have

$$\begin{aligned}
J_2 &= \|(g_1 * \mathcal{S}_{\beta,\gamma_j}(t_2) - (g_1 * \mathcal{S}_{\beta,\gamma_j})(t_1)\|\|\chi\| \\
&= \left\| \int_0^{t_2} g_1(t_2 - \tau)\mathcal{S}_{\beta,\gamma_j}(\tau) d\tau - \int_0^{t_1} g_1(t_1 - \tau)\mathcal{S}_{\beta,\gamma_j}(\tau) d\tau \right\| \|\chi\| \\
&\leq \int_{t_1}^{t_2} \|\mathcal{S}_{\beta,\gamma_j}(\tau)\| d\tau \|\chi\| \\
&\leq S_0 \|\chi\|(t_2 - t_1) \\
&\to 0 \quad \text{as} \quad t_1 \to t_2
\end{aligned}$$

and

$$\begin{aligned}
J_3 \leq & \sum_{j=1}^{n} \alpha_j S_0 \left\| \int_0^{t_2} \frac{(t_2 - s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} ds - \int_0^{t_1} \frac{(t_1 - s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} ds \right\| \|\phi(0)\| \\
\leq & \sum_{j=1}^{n} \alpha_j S_0 \left| \frac{t_2^{1+\beta-\gamma_j} - t_1^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} \right| \|\phi(0)\| \\
\to & 0 \quad \text{as} \quad t_1 \to t_2.
\end{aligned}$$

From the expressions J_2 and J_3, we can easily deduce that $J_4 \to 0$ and $J_5 \to 0$ as $t_1 \to t_2$ independently of $u \in D$. Therefore, $\|\tilde{Q}v(t_2) - \tilde{Q}v(t_1)\| \to 0$ as $t_1 \to t_2$ independently of $u \in D$. Hence, $Q(D)$ is equicontinuous on $\mathcal{J}$.

Step 4. Now, we will show $\mu(\{\tilde{Q}v^{(n)}\}_{n=1}^{\infty}) = 0$.

Define the sequences

$$v^{(n)} = \tilde{Q}v^{(n-1)}, \quad w^{(n)} = Qw^{(n-1)}, \quad n = 1, 2, \ldots \tag{3.4}$$

It follows from monotonicity of $\tilde{Q}$ that

$$v^{(0)} \leq v^{(1)} \leq \cdots \leq v^{(n)} \leq \cdots \leq w^{(n)} \leq \cdots \leq w^{(1)} \leq w^{(0)}. \tag{3.5}$$

Next, we will show that $\{v^{(n)}\}$ and $\{w^{(n)}\}$ convergent uniformly in $\mathcal{J}$.

We set $\mathbb{B} = \{v^{(n)} : n \in \mathbb{N}\}$ and $\mathbb{B}_0 = \{v^{(n-1)} : n \in \mathbb{N}\}$. Using normality of cone $\mathbb{P}_0$, we obtain that $\mathbb{B}$ and $\mathbb{B}_0$ are bounded. Since $\mathbb{B}_0 = \mathbb{B} \cup \{v^{(0)}\}$, it follows that $\mu(\mathbb{B}_0(t)) = \mu(\mathbb{B}(t))$ for $t \in (-\infty, T]$. Let

$$\varphi(t) := \mu(\mathbb{B}_0(t)) = \mu(\mathbb{B}(t)), \quad t \in (-\infty, T].$$

Since $\mathbb{B} = \tilde{Q}(\mathbb{B}_0)$, we have

$$\mu(\mathbb{B}(t)) = \mu(\tilde{Q}(\mathbb{B}_0)(t)).$$

For $t \in (-\infty, 0]$, $\varphi(t) := \mu(\tilde{Q}(\mathbb{B}_0)(t)) = 0$. For $t \in \mathcal{J}$, we have

$$\begin{aligned}
\varphi(t) =& \mu(\tilde{Q}(\mathbb{B}_0)(t) \\
\leq & 2\mu(\tilde{Q}\{v^{(n-1)}(t)\}) \\
\leq & 2\mu\Bigg[\mathcal{S}_{\beta,\gamma_j}(t)\phi(0) + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)\phi(0)ds \\
& + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) F\left(s, u_s + \overline{v}_s^{n-1}, \int_0^s h(s,\tau,u_\tau + \overline{v}_\tau^{n-1})d\tau\right)ds\Bigg] \\
\leq & 2\mu\left[\int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) F\left(s, u_s + \overline{v}_s^{n-1}, \int_0^s h(s,\tau,u_\tau + \overline{v}_\tau^{n-1})d\tau\right)ds\right] \\
\leq & \frac{4S_0}{\Gamma(1+\beta)}\left[\int_0^t (t-s)^\beta \mu\left\{F\left(s, u_s + \overline{v}_s^{n-1}, \int_0^s h(s,\tau,u_\tau + \overline{v}_\tau^{n-1})d\tau\right)\right\}ds\right] \\
\leq & \frac{4S_0 L}{\Gamma(1+\beta)}\Bigg[\int_0^t (t-s)^\beta \Bigg\{\sup_{-\infty<r\leq 0} \mu(\overline{v}^{n-1}(s+r)) \\
& + \mu\left(\int_0^s h(s,\tau,u_\tau + \overline{v}_\tau^{n-1})d\tau\right)\Bigg\}ds\Bigg] \\
\leq & \frac{4S_0 L}{\Gamma(1+\beta)}\Bigg[\int_0^t (t-s)^\beta \Bigg\{\sup_{0\leq z\leq s} \mu(\overline{v}^{n-1}(z)) \\
& + 2\int_0^s \xi(s,\tau) \sup_{-\infty<r\leq 0} \mu(\overline{v}^{n-1}(\tau+r))d\tau\Bigg)\Bigg\}ds\Bigg] \\
\leq & \frac{4S_0 L}{\Gamma(1+\beta)}(1+2\xi^*) \int_0^t (t-s)^\beta \sup_{0\leq z\leq s} \mu(\overline{v}^{n-1}(z))ds \\
\leq & \frac{4S_0 L}{\Gamma(1+\beta)}(1+2\xi^*) \int_0^t (t-s)^\beta \varphi(s)ds.
\end{aligned}$$

Now, by the Gronwall's inequality, $\varphi(t) \equiv 0$ on $\mathcal{J}$. So $\mu\{v^{(n)} : n \in \mathbb{N}\} = 0$. This implies that the set $\{v^{(n)} : n \in \mathbb{N}\}$ is relatively compact in D. So, we conclude that the sequence $\{v^{(n)}\}$ admits a convergent subsequence in D. Further by (3.5), we observe that $\{v^{(n)}\}$ itself is convergent sequence in $\mathbb{X}$. So, there exists $v^* \in \mathbb{X}$ satisfying $v^{(n)} \to v^*$ as $n \to \infty$. By (3.2) and (3.4), we have

(3.6)
$$v^{(n)}(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \mathcal{S}_{\beta,\gamma_j}(t)\phi(0) + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi & \\ + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)\phi(0)ds + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) & \\ \times F\left(s, u_s + \overline{v}_s^{(n-1)}, \int_0^s h(s,\tau,u_\tau + \overline{v}_\tau^{(n-1)})d\tau\right) ds, & t \in \mathcal{J}. \end{cases}$$

As $n \to \infty$, then applying Lebesgue Dominated Convergence Theorem, we have

(3.7)
$$v^*(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \mathcal{S}_{\beta,\gamma_j}(t)\phi(0) + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi & \\ + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)\phi(0)ds + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) & \\ \times F\Bigg(s, u_s + \overline{v}_s^{*}, \int_0^s h(s,\tau,u_\tau + \overline{v}_\tau^{*})d\tau\Bigg) ds, & t \in \mathcal{J}. \end{cases}$$

Then $v^* \in \mathcal{C}(\mathcal{J}, \mathbb{X})$ and $v^* = \tilde{Q}v^*$. Thus v^* is a fixed point of $\tilde{Q}$ and hence v^* will be the solution of (3.2). Similarly, there exists $w^* \in \mathcal{C}(\mathcal{J}, \mathbb{X})$ in such a way $w^{(n)} \to w^*$ as $n \to \infty$ and $w^* = \tilde{Q}w^*$. If $v \in D$ be a fixed point of $\tilde{Q}$ then by (3.3), we get $v^{(1)} \le Qv^{(0)} \le Qv = v \le Qw^{(0)} \le Qw^{(1)}$. Now, by induction principle $v^{(n)} \le v \le w^{(n)}$. In view of (3.5) and as $n \to \infty$, we obtain $v^{(0)} \le v^* \le v \le w^* \le w^{(0)}$. Hence, w^* and v^* are the maximal and minimal mild solutions of the system (1.1) in D, respectively. □

Corollary 3.1. *Let $\mathbb{X}_0$ be an ordered Banach space with a positive regular cone $\mathbb{P}_0$. Suppose that the system* (1.1) *admits lower and upper solutions denoted by $v^{(0)}, w^{(0)} \in \mathcal{C}^{1+\beta}(\mathcal{J}, \mathbb{X})$ such that $v^{(0)} \le w^{(0)}$, $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\ge 0}$ is a positive operator and the assumptions* (A1)-(A2) *are satisfied. Then the system* (1.1) *admits maximal and minimal mild solutions between $w^{(0)}$ and $v^{(0)}$.*

Proof. By regularity of the cone $\mathbb{P}_0$, we have that any ordered-bounded and ordered-monotonic sequence in $\mathbb{X}_0$ is convergent. Let $\{y^n\}$ be an increasing or decreasing sequence in D. Then using assumption $(A2)$, $F(t, y_t^n, \int_0^t h(t,s,y_s^n)ds)$ is ordered-bounded and ordered-monotonic sequence in $\mathbb{X}_0$ and hence $\{F(t, y_t^n, \int_0^t h(t,s,y_s^n)ds)\}$ is convergent. Therefore, $\mu(\{F(t, y_t^n, \int_0^t h(t,s,y_s^n)ds)\}) = 0$. Hence, assumption (A3) holds. Now, by Theorem 3.1, we conclude the assertion. □

Corollary 3.2. *Let $\mathbb{X}_0$ be a weakly sequentially complete ordered Banach space with a positive normal cone $\mathbb{P}_0$. Suppose that the system* (1.1) *admits lower and upper solutions denoted by $v^{(0)}, w^{(0)} \in \mathcal{C}^{1+\beta}(\mathcal{J}, \mathbb{X})$ such that $v^{(0)} \le w^{(0)}$, $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\ge 0}$ is a positive operator and the assumptions* (A1)-(A2) *are satisfied. Then the system* (1.1) *admits maximal and minimal mild solutions between $w^{(0)}$ and $v^{(0)}$.*

Proof. Since, in a weakly sequentially complete and ordered Banach space, the normal cone $\mathbb{P}_0$ is regular. Therefore using Corollary 3.1, we can conclude the assertion. □

Corollary 3.3. *We assume that $\mathbb{X}_0$ is a reflexive and ordered Banach space space with positive normal cone $\mathbb{P}_0$. Also, consider that the system* (1.1) *admits lower and upper solutions $v^{(0)}, w^{(0)} \in \mathcal{C}^{1+\beta}(\mathcal{I}, \mathbb{X})$ such that $v^{(0)} \leq w^{(0)}$, $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$ is positive and the assumptions* (A1)-(A2) *are satisfied. Then the system* (1.1) *admits maximal and minimal mild solutions between $w^{(0)}$ and $v^{(0)}$.*

Proof. Since, in a reflexive and ordered Banach space, the normal cone $\mathbb{P}_0$ is regular. Now, by Corollary 3.1, we conclude the assertion. □

Next, we will show the uniqueness of the mild solution for the system (1.1). For this we consider the following assumption.

(A4) The functions $h : \Delta \times \mathfrak{B} \to \mathbb{X}$ and $F : \mathcal{I} \times \mathfrak{B} \times \mathbb{X} \to \mathbb{X}$ are such that

(i) h is continuous and there exists an integrable function $\psi : \Delta \to [0, T]$ such that

$$h(t, s, u_2) - h(t, s, u_1) \leq \psi(t, s)[u_2(r) - u_1(r)],$$

for any $(t, s) \in \Delta$ and $v_t^{(0)} \leq u_1 \leq u_2 \leq w_t^{(0)}$, $r \in (-\infty, 0]$;

(ii) F is continuous and there exists $\kappa \geq 0$ such that

$$F(t, u_2, v_2) - F(t, u_1, v_1) \leq \kappa[(u_2(r) - u_1(r)) + (v_2 - v_1)], \quad r \in (-\infty, 0],$$

for any $t \in \mathcal{I}$, $u_1, u_2 \in \mathfrak{B}$ with $v_t^{(0)} \leq u_1 \leq u_2 \leq w_t^{(0)}$ and $v_1, v_2 \in \mathbb{X}$ with $\int_0^t g(t, s, v_s^{(0)})ds \leq v_1 \leq v_2 \leq \int_0^t g(t, s, w_s^{(0)})ds$.

Theorem 3.2. *Let $\mathbb{X}_0$ be an ordered Banach space with normal positive cone $\mathbb{P}_0$ with normal constant N_0. Assume that $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$ is positive, the system* (1.1) *has upper and lower solutions $v^{(0)}, w^{(0)} \in \mathcal{C}^{1+\beta}(\mathcal{I}, \mathbb{X})$ such that $v^{(0)} \leq w^{(0)}$ and assumptions* (A2) *and* (A4) *hold. Then the system* (1.1) *has a unique mild solution in $[v^{(0)}, w^{(0)}]$.*

Theorem 3.3. *Let $\mathbb{X}_0$ be an ordered Banach space with a positive normal cone $\mathbb{P}_0$ with normal constant N_0. Suppose that the system* (1.1) *admits lower and upper solutions denoted by $v^{(0)}, w^{(0)} \in \mathcal{C}(\mathcal{I}, \mathbb{X})$ such that $v^{(0)} \leq w^{(0)}$, $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$ is a positive operator and the assumptions* (A2)-(A4) *are satisfied. Then the system* (1.1) *admits a unique mild solution in $[v^{(0)}, w^{(0)}]$.*

Proof. Let $\{x_n\} \in [v_t^{(0)}, w_t^{(0)}]$ and $\{y_n\} \in [v^{(0)}, w^{(0)}]$ be two monotonic increasing sequences. For $m, n = 1, 2, \ldots$, with $m > n$, for some $r_1, r_2 \in (-\infty, 0]$ using (A4), we have

$$\theta \leq h(t, s, x_m) - g(t, s, x_n) \leq \xi(t, s)[x_m(r_1) - x_n(r_1)]$$

and

$$\theta \leq F(t, x_m, y_m) - F(t, x_n, y_n) \leq \kappa[(x_m(r_2) - x_n(r_2)) + (y_m - y_n)].$$

Using the normality of positive cone $\mathbb{P}_0$, we get

$$\|h(t, s, x_m) - h(t, s, x_n)\| \leq N_0\xi(t, s)\|x_m(r_1) - x_n(r_1)\|$$

and

$$\|F(t,x_m,y_m)-F(t,x_n,y_n)\| \leq N_0\kappa\|(x_m(r_2)-x_n(r_2))+(y_m-y_n)\|.$$

Using the property of measure of noncompactness, we have

$$\mu(\{h(t,s,x_m)\}) \leq N_0\xi(t,s) \sup_{-\infty\leqslant r\leqslant 0} \mu(\{x_m(r)\})$$

and

$$\mu(\{F(t,x_m,y_m)\}) \leq N_0\kappa \left[\sup_{-\infty\leqslant r\leqslant 0} \mu(\{x_m(r)\}) + \mu(\{y_m\})\right].$$

Now, we observed that (A4) implies (A1) and (A3). Therefore, by Theorem 3.1, minimal and maximal mild solutions v^* and w^* exist for the system (1.1) on D, respectively.

By (3.2), for any $t \in (-\infty, 0]$, we have

$$\theta \leq w^*(t) - v^*(t) = \tilde{Q}w^*(t) - \tilde{Q}v^*(t) = 0.$$

Using the normality of positive cone $\mathbb{P}_0$, we get $\|v^*(t)-w^*(t)\| \leq 0$, i.e., $v^*(t) = w^*(t)$ for all $t \in (-\infty, 0]$.

To abbreviate the writing, we set $K_0 := \sup_{0\leq t\leq T} K_1(t)$. Now using (A4) and the positivity of operator $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$, for any $t \in \mathcal{J}$, we have

$$\begin{aligned}
\|v^*(t)-w^*(t)\| =& \|\tilde{Q}v^*(t) - \tilde{Q}w^*(t)\| \\
\leq& N_0 \bigg\| \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\bigg[F\bigg(s, u_s+\overline{v}^*_s, \int_0^s h(s,\tau,u_\tau+\overline{v}^*_\tau)d\tau\bigg) \\
&- F\bigg(s, u_s+\overline{w}^*_s, \int_0^s h(s,\tau,u_\tau+\overline{w}^*_\tau)d\tau\bigg)\bigg]ds\bigg\| \\
\leq& N_0\kappa\bigg[\int_0^t \|\mathcal{T}_{\beta,\gamma_j}(t-s)\|\bigg(\|\overline{v}^*_s-\overline{w}^*_s\|_{\mathfrak{B}} \\
&+ \bigg\|\int_0^s h(s,\tau,u_\tau+\overline{v}^*_\tau)d\tau - \int_0^s h(s,\tau,u_\tau+\overline{w}^*_\tau)d\tau\bigg\|\bigg)ds\bigg] \\
\leq& N_0\kappa\bigg[\int_0^t \|\mathcal{T}_{\beta,\gamma_j}(t-s)\|\bigg(\|\overline{v}^*_s-\overline{w}^*_s\|_{\mathfrak{B}} + \int_0^s \xi(s,\tau)\|\overline{v}^*_\tau-\overline{w}^*_\tau\|_{\mathfrak{B}}d\tau\bigg)ds\bigg] \\
\leq& \frac{N_0S_0K_0\kappa}{\Gamma(1+\beta)}\bigg[\int_0^t (t-s)^\beta\bigg\{\sup_{-\infty\leq r\leq 0}\|\overline{v}^*_s(r)-\overline{w}^*_s(r)\| \\
&+ \int_0^s \xi(s,\tau)\sup_{-\infty\leq r\leq 0}\|\overline{v}^*_\tau(r)-\overline{w}^*_\tau(r)\|d\tau\bigg\}ds\bigg] \\
\leq& \frac{N_0S_0K_0\kappa}{\Gamma(1+\beta)}\bigg[\int_0^t (t-s)^\beta\bigg\{\sup_{0\leq z\leq s}\|\overline{v}^*(z)-\overline{w}^*(z)\| \\
&+ \xi^*\sup_{0\leq z\leq s}\|\overline{v}^*(z)-\overline{w}^*(z)\|\bigg\}ds\bigg]
\end{aligned}$$

$$\leq \frac{N_0 S_0 K_0 \kappa}{\Gamma(1+\beta)}(1+\xi^*)\int_0^t (t-s)^{\beta}\|v^*(s)-w^*(s)\|ds.$$

Now, by Lemma 2.1, we get $v^*(t) = w^*(t)$ for all $t \in [0,T]$. So, $v^*(t) = w^*(t)$ for all $t \in (-\infty, T]$. Hence, $v^*(t) = w^*(t) = z^*(t)$(say) for all $t \in (-\infty, T]$ is the unique solution of (3.2). So, we get $y(t) = u(t) + z^*(t)$ is the unique mild solution of the system (1.1). □

4. Example

The fractional order diffusion wave equations have great applications in various fields of science and engineering. These equations represent propagation of mechanical waves through viscoelastic media, charge transport in amorphous semiconductors [15,20,30], and may be used in thermodynamics and shear in fluids, the flow of fluid through fissured rocks [1]. In particular, the fractional delay diffusion wave equations describe the driver reaction time, time taken for a signal traveling to the controlled object, time consume by body to produce red blood cells and cell division time in the dynamics of viral persistence or exhaustion.

Let $\beta, \gamma_j > 0$, $j = 1,2,3,\dots,n$ be given, satisfying $0 < \beta \leq \gamma_n \leq \dots \leq \gamma_1 \leq 1$. Consider the following system

(4.1)
$$\begin{cases} {}^cD^{1+\beta}u(t,\nu) + \sum_{j=1}^{n} \alpha_j {}^cD^{\gamma_j}u(t,\nu) = \Delta u(t,\nu) + L\Big(\frac{|u_t(\theta,\nu)|}{|1+u_t(\theta,\nu)|} \\ \qquad\qquad + \int_0^t (t-s)^{-1/2}s^{-1/2}\int_{-\infty}^0 \xi(\theta)u_t(\theta,\nu)d\theta ds\Big), \\ u(\theta,\nu) = u_0(\theta,\nu), \quad \theta \in (-\infty, 0], \quad \frac{\partial u(t,\nu)}{\partial t}|_{t=0} = z_0, \end{cases}$$

where $\mathbb{X} = L^2([0,1],\mathbb{R})$, $t \in \mathcal{J} = [0,1]$, $T > 0$, $\nu \in [0,1]$, $L \geq 0$, $x_t(\theta,\nu) = x(t+\theta,\nu)$, $t \in \mathcal{J}$, $\xi : (-\infty,0] \to \mathbb{R}^+$, $u_0 : (-\infty,0]\times[0,1] \to \mathbb{R}$ and Δ is the Laplace operator with maximal domain $\{v \in \mathbb{X} : v \in H^2([0,1],\mathbb{R})\}$. Let $\mathbb{P} = \{v \in \mathbb{X} : v(\nu) \geqslant 0 \text{ a.e. } \nu \in [0,1]\}$. Then the cone $\mathbb{P}$ is normal in Banach space $\mathbb{X}$ with normal constant $N = 1$.

Using the theory of cosine families, we can see that Laplacian Δ generates a bounded cosine function $\{C(t)\}_{t\geq 0}$ on the space $L^2([0,1],\mathbb{R})$. Moreover, by Theorem 2.1 the operator Δ in system (4.1) generates a bounded $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$-resolvent family. Let us assume $S_0 = \sup_{t\in[0,1]} \|\mathcal{S}_{\beta,\gamma_j}(t)\|$.

For $t \in [0,1]$, $\nu \in [0,1]$ and $\theta \in (-\infty,0]$, we set $z_0 = \chi$ and

$$\begin{aligned} y(t) =& u(t,\nu), \\ \phi(\theta) =& u_0(\theta,\nu), \\ h(t,s,y_s) =& (t-s)^{-1/2}s^{-1/2}\int_{-\infty}^0 \xi(\theta)u_t(\theta,\nu)d\theta, \\ F(t,y_t,\int_0^t h(t,s,y_s)ds) =& L\left[\frac{|u_t(\theta,\nu)|}{1+|u_t(\theta,\nu)|} + \int_0^t h(t,s,y_s)ds\right]. \end{aligned}$$

Now, we observe that the system (4.1) has a abstract form of system (1.1). Let $v(t) = 0$ for $t \in [0,1]$. Then $F\left(t, v_t, \int_0^t h(t,s,v_s)ds\right) = 0$ for $t \in [0,1]$ and $\phi(t) \geq v(t)$ for $t \in (-\infty, 0]$. Let us suppose that there exists a function $w(t) \geq 0$ such that (4.2)

$$\begin{cases} {}^cD^{1+\beta}w(t) + \sum_{j=1}^n \alpha_j {}^cD^{\gamma_j}w(t) \geq Aw(t) + F\left(t, w_t, \int_0^t h(t,s,w_s)ds\right), & t \in (0,T], \\ w(t) \geq \phi(t) \in \mathfrak{B}, \quad t \in (-\infty, 0], \quad w'(0) \geq \chi. \end{cases}$$

Thus the system (1.1) admits lower and upper solutions v, w such that $v \leq w$.

Let $\vartheta > 0$ be a constant and

$$\mathfrak{B} = \left\{ y \in \mathcal{C}((-\infty, 0], \mathbb{R}) : \lim_{\theta \to -\infty} e^{\vartheta\theta} y(\theta) \text{ exists in } \mathbb{R} \right\}.$$

The norm of $\mathfrak{B}$ is given by $\|y\|_{\mathfrak{B}} = \sup_{-\infty < \theta \leq 0} e^{\vartheta\theta}|y(\theta)|$. Let $y : (-\infty, 0] \to \mathbb{R}$ such that $y_0 \in \mathfrak{B}$. Then

$$\lim_{\theta \to -\infty} e^{\vartheta\theta} y_t(\theta) = \lim_{\theta \to -\infty} e^{\vartheta\theta} y(t+\theta) = \lim_{\theta \to -\infty} e^{\vartheta(\theta - t)} y(\theta) = e^{-\vartheta t} \lim_{\theta \to -\infty} e^{\vartheta\theta} y_0(\theta) < \infty.$$

Hence, $y_t \in \mathfrak{B}$. Finally, we will show that

$$\|y_t\|_{\mathfrak{B}} \leq K_1(t) \sup_{s \in [0,t]} |y(s)| + K_2(t)\|y_0\|_{\mathfrak{B}},$$

where $K_1 = K_2 = 1$ and $K = 1$. We have $|y_t(\theta)| = |y(t+\theta)|$. If $t + \theta \leq 0$, we obtain

$$|y_t(\theta)| \leq \sup_{s \in (-\infty, 0]} |y(s)|.$$

If $t + \theta \geq 0$, then we get

$$|y_t(\theta)| \leq \sup_{s \in [0,t]} |y(s)|.$$

Thus, for all $(t + \theta) \in [0,1]$ we have

$$|y_t(\theta)| \leq \sup_{s \in (-\infty, 0]} |y(s)| + \sup_{s \in [0,t]} |y(s)|.$$

Then

$$\|y_t\|_{\mathfrak{B}} \leq \|y_0\|_{\mathfrak{B}} + \sup_{s \in [0,t]} |y(s)|.$$

One can easily check that $\mathfrak{B}$ is a Banach space equipped with the norm $\|\cdot\|_{\mathfrak{B}}$ and hence conclude that $\mathfrak{B}$ is a phase space. Clearly, the functions f and h satisfies the assumptions (A1) and (A2). For $t \in [0,1]$, $\varphi_1, \varphi_2 \in \mathfrak{B}$ with $0 \leq \varphi_1 \leq \varphi_2$ and $v_1, v_2 \in X$, we have

$$0 \leq h(t,s,\varphi_2) - h(t,s,\varphi_1) \leq (t-s)^{-1/2} s^{-1/2} \int_{-\infty}^0 \xi(\theta)(\varphi_2(\theta) - \varphi_2(\theta)d\theta$$

and

$$0 \leq F(t,\varphi_2,v_2) - F(t,\varphi_1,v_1) \leq L\left[\frac{|\varphi_2(\theta)|}{1+|\varphi_2(\theta)|} - \frac{|\varphi_1(\theta)|}{1+|\varphi_1(\theta)|} + v_2 - v_1\right].$$

Using normality of cone $\mathbb{P}$, we have

$$\begin{aligned}\|h(t,s,\varphi_2)-h(t,s,\varphi_1)\| \leq &(t-s)^{-1/2}s^{-1/2}\int_{-\infty}^{0}|\xi(\theta)|\|\varphi_2(\theta)-\varphi_1(\theta)\|d\theta,\\ \|F(t,\varphi_2,v_2)-F(t,\varphi_1,v_1)\| \leq &L[\|\varphi_2(\theta)-\varphi_1(\theta)\|+\|v_2-v_1\|].\end{aligned}$$

Now, by the property of measure of noncompactness for $U \subset \mathcal{C}((-\infty,0],\mathbb{X})$ and $V \subset \mathbb{X}$, we have

$$\begin{aligned}\mu(h(t,s,U)) \leq &\xi(t,s)\sup_{-\infty\leq\theta\leq 0}\mu(U(\theta)),\\ \mu(f(t,U,V)) \leq &L[\sup_{-\infty<\theta\leq 0}\mu(U(\theta))+\mu(V)],\end{aligned}$$

where $\xi(t,s) = (t-s)^{-1/2}s^{-1/2}\int_{-\infty}^{0}|\xi(\theta)|d\theta$. Let $\xi^* = \sup_{t,s\in(-\infty,1]}\xi(t,s)$. Thus, assumptions (A3) and (A4) are fulfilled. Now by the Theorem 3.1, the system (4.1) admits extrimal mild solutions lying between the lower solution 0 and the upper solution w. Further, by Theorem 3.3 the system (4.1) admits unique mild solution.

5. Conclusion

The monotone iterative technique has been employed to establish the existence and uniqueness of mild solution for a class of multi-term time-fractional delay differential system in an ordered Banach space. Assuming the existence of the lower and upper solutions of the system (1.1), a new set of sufficient conditions has been obtained in which the nonlinear functions satisfy some monotonic properties. One can extend this idea to establish the existence results for multi-term time-fractional differential system with impulsive conditions.

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