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POSITIVE SOLUTIONS FOR A FRACTIONAL BOUNDARY VALUE PROBLEM WITH LIDSTONE LIKE BOUNDARY CONDITIONS

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Abstract. We consider a higher order fractional boundary value problem with Lidstone like boundary conditions, where the nonlinearity is an L^1-Carathèodory function. We first consider the lower order problem. Then, by using a convolution to construct the Green's function for the higher order problem, we are able to apply a recent fixed point theorem to show the existence of positive solutions of the boundary value problem.

1. Introduction

Let $n \in \mathbb{N}$, $n \geq 3$, $n-1 < \alpha \leq n$ and $1 \leq \beta \leq n-1$. We study existence and nonexistence of solutions of the fractional differential equation

$$D^{\alpha}_{0+}u + f(t,u) = 0, \quad t \in (0,1), \tag{1.1}$$

satisfying the boundary conditions

$$u^{(i)}(0) = 0, \quad i = 0,1,\ldots,n-2, \quad D^{\beta}_{0+}u(1) = 0, \tag{1.2}$$

where D^{α}_{0+} and D^{β}_{0+} are the standard Riemann-Liouville derivatives. Here $f : (0,1) \times [0,\infty) \to [0,\infty)$ is an L^1-Caratheòdory function, i.e., f satisfies the following properties:

(a) $f(\cdot, u)$ is a measurable function for all $u \geq 0$;
(b) $f(t, \cdot)$ is continuous for a.e. $t \in (0,1)$ and

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(c) for all $r > 0$ there exists a $\psi_r \in L^1[0,1]$ such that $|f(t,u)| \leq \psi_r(t)$ for a.e. $t \in (0,1)$ and for all $|u| \leq r$.

We then consider a higher order problem with boundary conditions inspired by Lidstone boundary conditions. Let $m \in \mathbb{N}$, $m \geq 3$, $n \in \mathbb{N}$, $2n-1+m < \gamma \leq 2n+m$, $1 \leq \beta \leq n-1$ and consider the boundary value problem

$$D_{0+}^{\gamma} u(t) + (-1)^n g(t,u) = 0, \quad 0 < t < 1, \tag{1.3}$$

satisfying the boundary conditions

$$\begin{aligned} &u^{(i)}(0) = 0, \quad i = 0,1,\dots,m-2, \quad D_{0+}^{\beta} u(1) = 0, \\ &D_{0+}^{\gamma-2l} u(0) = D_{0+}^{\gamma-2l} u(1) = 0, \quad l = 1,\dots,n-1, \end{aligned} \tag{1.4}$$

where $g : (0,1) \times [0,\infty) \to [0,\infty)$ is an L^1-Carathèodory function. To construct the Green's function for this problem, we use a convolution. The Green's function for the higher order problem therefore inherits properties of the Green's function corresponding to (1.1), (1.2) and similar arguments can be made to show the existence of positive solutions of the boundary value problem.

Fixed point theory has been used extensively to study the existence of positive solutions of fractional boundary value problems [2, 7, 8, 10–12, 20, 23, 25] and singular fractional boundary value problems [1, 9, 14, 16, 18, 21, 22, 24, 26] where the nonlinearity may be singular at $t = 0$ or $t = 1$. Of particular interest to this work is the recent paper by Benmezaï, Chentout and Henderson [3], where the authors prove a new fixed point theorem using strongly positive-like operators and then apply their fixed point theorem to a fractional boundary value problem. The use of convolution to construct Green's functions for higher order problems can be found first in [6]. In [15], the authors used convolution to study positive solutions of some different higher order fractional boundary value problems.

2. Preliminaries

We start with the definition of the Riemann-Liouville fractional integral and fractional derivative.

Definition 2.1. Let $\nu > 0$. The Riemann-Liouville fractional integral of a function u of order ν, denoted $I_{0+}^{\nu} u$, is defined as

$$I_{0+}^{\nu} u(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} u(s)\,ds,$$

provided the right-hand side exists. Moreover, let n denote a positive integer and assume $n-1 < \alpha \leq n$. The Riemann-Liouville fractional derivative of order α of the function $u : [0,1] \to \mathbb{R}$, denoted $D_{0+}^{\alpha} u$, is defined as

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s)\,ds = D^n I_{0+}^{n-\alpha} u(t),$$

provided the right-hand side exists. We refer to [4,13,17,19] for a more in depth study of fractional calculus and fractional differential equations.

Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is said to be a cone provided

(i) $\alpha u + \beta v \in \mathcal{P}$ for all $u, v \in \mathcal{P}$ and all $\alpha, \beta \geq 0$ and
(ii) $u \in \mathcal{P}$ and $-u \in \mathcal{P}$ implies $u = 0$.

Cones generate a natural partial ordering on a Banach space. Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. If $u, v \in \mathcal{B}$, $u \preceq v$ if $v - u \in \mathcal{P}$, $u \prec v$ if $v - u \in \mathcal{P}$, $u \neq v$, and $u \not\preceq v$ if $v - u \notin \mathcal{P}$. If both $M, N : \mathcal{B} \to \mathcal{B}$ are continuous mappings, $M \preceq N$ if for all $u \in \mathcal{P}$, $Mu \preceq Nu$. The relations $N \prec M$ and $N \not\preceq M$ are defined similarly. The notation $\succeq$, $\succ$ and $\not\succeq$ define the reverse situations.

Definition 2.2. An operator $L \in L_C(\mathcal{B})$, where $L_C(\mathcal{B})$ is the set of all linear compact self-mappings of B, is said to be positive if $L : \mathcal{P} \to \mathcal{P}$ and strongly positive if $\mathcal{P}^\circ \neq \emptyset$ and $L : \mathcal{P} \setminus \{0\} \to \mathcal{P}^\circ$.

Definition 2.3. Let $L \in L_C(\mathcal{B})$ be positive. L is said to be lower bounded if $\inf\{\|Lu\| : u \in \mathcal{P} \cap \partial B(0,1)\} > 0$.

For all positive operators $L \in L_C(\mathcal{B})$, define the subsets

$$\Lambda_L = \{\lambda \geq 0 : \text{ there exists } u \succ 0_{\mathcal{B}} \text{ such that } Lu \succeq \lambda u\}$$

and

$$\Gamma_L = \{\lambda \geq 0 : \text{ there exists } u \succ 0_{\mathcal{B}} \text{ such that } Lu \preceq \lambda u\}.$$

The proof of the following lemma can be found in [3].

Lemma 2.1. *Let $L \in L_C(\mathcal{B})$ be strongly positive. Then*

$$r(L) = \sup \Lambda_L = \inf \Gamma_L.$$

Definition 2.4. A positive operator $L \in L_C(\mathcal{B})$ is said to be a strong positive-like operator if $r(L) = \sup \Lambda_L = \inf \Gamma_L > 0$.

The following two theorems are the model for which our main result is based. The proofs can be found in the work of Benmezai, Chentout, and Henderson [3]. The first deals with nonexistence of positive fixed points and the second with existence of positive fixed points.

Theorem 2.1. *Let $T : \mathcal{P} \to \mathcal{P}$ be a continuous mapping and let $L \in L_C(B)$ be a strongly positive-like operator. If either*

$$r(L) > 1 \quad \text{and} \quad Tu \succeq Lu, \quad \text{for all } u \in \mathcal{P},$$

or

$$r(L) < 1 \quad \text{and} \quad Tu \preceq Lu, \quad \text{for all } u \in \mathcal{P},$$

then T has no fixed points in $\mathcal{P}$.

Theorem 2.2. *Let $T : \mathcal{P} \to \mathcal{P}$ be a completely continuous mapping and assume that there exist two strongly positive-like operators $L_1, L_2 \in L_c(\mathcal{B})$ and two functions $F_1, F_2 : \mathcal{P} \to \mathcal{P}$ such that L_1 is lower bounded on $\mathcal{P}$, $r(L_2) < 1 < r(L_1)$, and for all $u \in \mathcal{P}$*

$$L_1 u - F_1 u \preceq Tu \preceq L_2 u + F_2 u.$$

If either

$$F_1 u = o(\|u\|) \quad as \quad u \to \infty \quad and \quad F_2 u = o(\|u\|) \quad as \quad u \to 0,$$

or

$$F_1 u = o(\|u\|) \quad as \quad u \to 0 \quad and \quad F_2 u = o(\|u\|) \quad as \quad u \to \infty,$$

then T has a fixed point in $\mathcal{P}$.

3. Eigenvalue Criteria

Let $E = C[0,1]$ be the Banach space of continuous functions with the usual supremum norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Define the Banach space X as

$$X = \left\{ u \in C[0,1] : \lim_{t\to 0} \frac{u(t)}{t^{\alpha-1}} \text{ exists} \right\}$$

endowed with the norm

$$\|u\|_X = \sup_{t\in[0,1]} \left| \frac{u(t)}{t^{\alpha-1}} \right|.$$

Fix $\delta \in (0,1)$. Define the cones

$$\begin{aligned} E^+ =& \{u \in E : u(t) \geq 0 \text{ for all } t \in [0,1]\}, \\ \mathcal{P} =& \{u \in E^+ : u(t) \geq \delta^{\alpha-1} \|u\|_0 \text{ for all } t \in [\delta, 1]\} \end{aligned}$$

and

$$X^+ = \{u \in X : u(t) \geq 0 \text{ for all } t \in [0,1]\}.$$

Define the sets

$$\mathbb{L}^1_+ = \{m \in \mathbb{L}^1(0,1) : m(t) \geq 0 \text{ a.e. } t \in [0,1]\}$$

and

$$\mathbb{L}^1_{++} = \{m \in \mathbb{L}^1_+ : m > 0 \text{ on a subset of positive measure}\}.$$

We also introduce the subset $S \subset X$ by

$$S = \left\{ u \in X : u(t) > 0 \text{ for all } t \in (0,1] \text{ and } \lim_{t\to 0} \frac{u(t)}{t^{\alpha-1}} > 0 \right\}.$$

The following theorem is given in [3].

Lemma 3.1. *S is open in X.*

The Green's function for $-D^{\alpha}_{0+}u = 0$ satisfying the boundary conditions (1.2) is given by (see, for example, [5])

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s < t \le 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}, & 0 \le t \le s < 1. \end{cases} \tag{3.1}$$

Therefore, u is a solution of (1.1), (1.2) if and only if

$$u(t) = \int_0^1 G(t,s)f(s,u(s))ds, \quad 0 \le t \le 1.$$

Define $v(t,s)$ by

$$v(t,s) = \begin{cases} \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} - \frac{\left(1-\frac{s}{t}\right)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s < t \le 1, \\ \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}, & 0 \le t \le s < 1. \end{cases}$$

Notice $G(t,s) = t^{\alpha-1}v(t,s)$. The following lemma gives sign properties of G and v.

The proof of (1)–(3) of the following lemma can be found in [15]. The proof of (4) is trivial.

Lemma 3.2. *Let G be defined as in* (3.1)*.*

(1) $G(t,s) \in C([0,1] \times [0,1))$ *with* $G(t,s) > 0$ *for* $(t,s) \in (0,1] \times (0,1)$.
(2) $t^{\alpha-1}G(1,s) \le G(t,s) \le G(1,s)$ *for* $(t,s) \in [0,1] \times [0,1)$.
(3) $G(t,s) \ge \delta^{\alpha-1}G(1,s)$ *for all* $t \in [\delta,1]$ *and all* $s \in [0,1)$.
(4) $v(0,s) > 0$ *for all* $s \in [0,1)$.

Let $m \in \mathbb{L}^1_{++}$. Define $L_m : E \to E$ by

$$L_m u(t) = \int_0^1 G(t,s)m(s)u(s)ds.$$

For $u \in X$, define $L^X_x : X \to E$ by $L^X_m u = L_m u$.

Lemma 3.3. *For $m \in \mathbb{L}^1_{++}$, the operator L_m is compact and positive. Moreover, $L_m : E^+ \to \mathcal{P}$.*

Proof. The proof that L_m is compact is standard. Let $u \in E^+$. Then $u(t) \ge 0$ for $t \in [0,1]$. Since $m > 0$ for a.e. $t \in [0,1]$, then by Lemma 3.2 (1),

$$L_m u(t) = \int_0^1 G(t,s)m(s)u(s)ds \ge 0.$$

So $L_m u \in E^+$ and $L_m : E^+ \to E^+$. Furthermore, Lemma 3.2 (3) gives that

$$\|L_m u\| = |L_m u(1)|_0$$

and

$$L_m u(t) = \int_0^1 G(t,s)m(s)u(s)ds \ge \delta^{\alpha-1}\int_0^1 G(1,s)m(s)u(s)ds = \delta^{\alpha-1}\|L_m u\|.$$

So $L_m u \in \mathcal{P}$ and $L_m : E^+ \to \mathcal{P}$. $\square$

Lemma 3.4. *For $m \in \mathbb{L}^1_{++}$, L_m is a strongly positive-like operator which is lower bounded on the cone $\mathcal{P}$.*

Proof. We start by proving that for $m \in \mathbb{L}^1_+[0,1] \cap C[0,1]$, L^X_m is a strongly positive operator. Using the Arzelà-Ascoli theorem, similar to the argument in [3], we have that L^X_m compact. Next, let $u \in X^+ \setminus \{0\}$. For all $t \in (0,1]$, by Lemma 3.2,

$$L^X_m u(t) = \int_0^1 G(t,s)m(s)u(s)ds > 0.$$

Also,

$$\lim_{t \to 0} \frac{L^X_m u(t)}{t^{\alpha-1}} = \int_0^1 v(0,s)m(s)u(s)ds > 0.$$

So $L^X_m : X \setminus \{0\} \to S \subset X^{+\circ}$. So L^X_m is strongly positive, and by Lemma 2.1,

$$r(L^X_m) = \sup \Lambda_{L^X_m} = \inf \Gamma_{L^X_m}.$$

Since L^X_m is an embedding of the operator L_m into X, $\Lambda_{L^X_m} \subset \Lambda_{L_m}$ and $\Gamma_{L^X_m} \subset \Gamma_{L_m}$. Next, let $\lambda \geq 0$ and $u \in E^+ \setminus \{0\}$ be such that $L_m u \succeq \lambda u$. Then, from an argument similar to that above, $U = L_m u \in X^+ \setminus \{0\}$. Now

$$L^X_m U = L^X_m (L_m u) = L_m (L_m u) \succeq \lambda L_m U.$$

So, $\lambda \in \Lambda_{L^X_m}$, and $\Lambda_{L^X_m} = \Lambda_{L_m}$. Similarly, $\Gamma_{L^X_m} = \Gamma_{L_m}$. So,

$$r(L_m) = \sup \Lambda_{L_m} = \inf \Gamma_{L_m}.$$

So, L_m is a strongly positive-like operator.

Finally, for $u \in \mathcal{P}$,

$$\|L_m u\| = L_m u(1) = \int_0^1 G(1,s)m(s)u(s)ds \geq \delta^{\alpha-1} \int_0^1 G(1,s)m(s)\delta^{\alpha-1} ds \|u\|.$$

So L_m is lower bounded on the cone $\mathcal{P}$. □

4. Existence and Nonexistence Results

Define the operator $T : E^+ \to E$ by

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds.$$

Notice that u is a solution of the boundary value problem (1.1), (1.2) if and only if u is a fixed point of T.

We have the following lemma.

Lemma 4.1. *$T : E^+ \to E$ is compact and $T : E^+ \to \mathcal{P}$.*

Proof. The fact that T is compact is a standard application of the Arzela-Ascoli theorem. Next, let $u \in E^+$. Then by Lemmma 3.2 (1) and (3),

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds \geq 0,$$

and, since $\|Tu\| = Tu(1)$,

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds \geq \delta^{\alpha-1}\int_0^1 G(1,s)f(s,u(s))ds = \delta^{\alpha-1}\|u\|.$$

So, $T : E^+ \to \mathcal{P}$. □

Let $m \in \mathbb{L}^1_{++}$. Consider the linear boundary value problem

$$D^\alpha_{0^+}u(t) + \mu m(t)u(t) = 0, \quad \text{a.e. } t \in (0,1), \tag{4.1}$$

satisfying the boundary conditions (1.2), where μ is a real parameter.

Lemma 4.2. *For all $m \in \mathbb{L}^1_{++}$, (4.1), (1.2) admit a unique positive eigenvalue $\mu_\alpha(m)$.*

Proof. Now (μ, u) is a solution of (4.1), (1.2) if and only if $L_m u = \mu^{-1}u$. Lemma 3.4 gives that $\mu^{-1} = r(L_m)$ is the unique positive eigenvalue of L_m. Thus, $\mu_\alpha(m) = 1/r(L_m)$ is the unique positive eigenvalue of (4.1), (1.2). □

Theorem 4.1. *Assume that there exists $m \in \mathbb{L}^1_+$ such that one of the following hypotheses is satisfied:*

$$\mu_\alpha(m) < 1 \quad \textit{and} \quad f(t,u) \geq m(t)u, \quad \textit{for all } u \geq 0 \textit{ and a.e. } t \in (0,1), \tag{4.2}$$

$$\mu_\alpha(m) > 1 \quad \textit{and} \quad f(t,u) \leq m(t)u, \quad \textit{for all } u \geq 0 \textit{ and a.e. } t \in (0,1), \tag{4.3}$$

Then (1.1), (1.2) has no positive solutions.

Proof. Let $u \in \mathcal{P}$, and suppose (4.2) holds. Then $f(t,u) \geq m(t)u$, which implies $Tu \succeq L_m u$. But L_m is a strongly positive-like operator with $r(L_m) = 1/\mu_\alpha(m) > 1$. Theorem 2.1 is therefore satisfied and T has no positive fixed points. A similar argument can be made if (4.3) holds. □

Theorem 4.2. *Assume that there exist $m_1, m_2 \in \mathbb{L}^1_{++}$, $q_1, q_2 \in \mathbb{L}^1_+$, and two functions $\phi_1, \phi_2 : [0,\infty) \to [0,\infty)$ such that $\mu_\alpha(m_1) < 1 < \mu_\alpha(m_2)$ and for all $u \geq 0$ and a.e. $t \in (0,1)$,*

$$m_1(t)u - q_1(t)\phi_1(u) \leq f(t,u) \leq m_2(t)u + q_2(t)\phi_2(u). \tag{4.4}$$

If either

- (H1) *$\phi_1(u) = o(\|u\|)$ as $u \to \infty$, $\phi_2(u) = o(\|u\|)$ as $u \to 0$, ϕ_1 is nondecreasing, and ϕ_2 is nondecreasing near 0 or*
- (H2) *$\phi_1(u) = o(\|u\|)$ as $u \to 0$, $\phi_2(u) = o(\|u\|)$ as $u \to \infty$, ϕ_1 is nondecreasing near 0, and ϕ_2 is nondecreasing,*

then (1.1), (1.2) has at least one positive solution.

Proof. For $i = 1,2$, let $F_i : \mathcal{P} \to \mathcal{P}$ be defined by

$$F_i u(t) = \int_0^1 G(t,s)\phi_i(u(s))ds.$$

From (4.4), we have that for all $u \in \mathcal{P}$,

$$L_{m_1}u - F_1 u \preceq Tu \preceq L_{m_2}u + F_2 u,$$

with

$$r(L_{m_2}) = \frac{1}{\mu_\alpha(m_2)} < 1 < r(L_{m_1}) = \frac{1}{\mu_\alpha(m_1)}.$$

Suppose (H1) holds. Then, we have,

$$\frac{\|F_i u\|_\infty}{\|u\|_\infty} = \sup_{t\in[0,1]} \frac{F_i u(t)}{\|u\|_\infty} \leq \int_0^1 G(1,s)q_i(s)\frac{\phi_i(u(s))}{\|u\|_\infty}ds \leq \int_0^1 G(1,s)q_i(s)ds,$$

which progresses to our conclusion,

$$F_1 u = o(\|u\|) \quad \text{as} \quad u \to \infty \quad \text{and} \quad F_2 u = o(\|u\|) \quad \text{as} \quad u \to 0.$$

We therefore have from Theorem 2.2 that T has a fixed point, which finally is a positive solution to (1.1), (1.2). The case for (H2) is similar. □

5. An Extension to a Higher Order Problem

In this section, we consider the fractional boundary value problem (1.3), (1.4), motivated by the two-point Lidstone boundary value problem for ordinary differential equations. Define $G_0(t,s) = G(t,s)$ from (3.1) to be the Green's function for $-D_{0+}^{\alpha} u = 0$, $u^{(i)}(0) = 0$, $i = 0,1,\dots,m-2$, $D_{0+}^{\beta} u(1) = 0$. Denote by $G_n(t,s)$ the Green's function for the BVP $-D_{0+}^{\gamma} u = 0$, (1.4).

The construction for $G_n(t,s)$ is similar to the construction in [6] and is given here for completeness. Define $G_k(t,s)$ by

$$G_k(t,s) = -\int_0^1 G_{k-1}(t,r)G_{conj}(r,s)dr, \tag{5.1}$$

$k = 2,\dots,n-1$, where

$$G_{conj}(t,s) = \begin{cases} t(1-s), & 0 \leq t < s \leq 1, \\ s(1-t), & 0 \leq s < t \leq 1, \end{cases} \tag{5.2}$$

is the Green's function for $-u'' = 0$, $u(0) = u(1) = 0$. Thus the Green's function $G_n(t,s)$ for (1.3), (1.4) is of the form

$$G_n(t,s) = -\int_0^1 G_{n-1}(t,r)G_{conj}(r,s)dr,$$

where $G_{n-1}(t,s)$ is the Green's function for

$$\begin{aligned} &D_{0+}^{\gamma-2} u(t) + h(t) = 0, \quad 0 < t < 1, \\ &u^{(i)}(0) = 0, \quad i = 0,1,\dots,m-2, \quad D_{0+}^{\beta} u(1) = 0, \\ &D_{0+}^{\gamma-2l} u(0) = D_{0+}^{\gamma-2l} u(1) = 0, \quad l = 1,\dots,n-2. \end{aligned}$$

To see this, for the base case, first consider the linear differential equation

$$D_{0+}^{\alpha+2} u(t) + h(t) = 0,$$

satisfying the boundary conditions

$$u^{(i)}(0) = 0, \quad i = 0,1,\dots,m-2, \quad D_{0+}^{\beta} u(1) = 0,$$

$$D_{0^+}^{\gamma-2(n-1)}u(0) = 0, \quad D_{0^+}^{\gamma-2(n-1)}u(1) = 0.$$

Make the change of variable $v(t) = D_{0^+}^{\alpha+2-2}u(t)$. Then $D^2v(t) = D^2D_{0^+}^{\alpha+2}u(t) = D_{0+}^{\alpha}u(t) = -h(t)$. Since $\alpha = \gamma - 2n + 2$, $v(0) = D_{0+}^{\alpha}u(0) = 0$ and $v(1) = D_{0+}^{\alpha}u(1) = 0$. Thus v satisfies the Dirichlet boundary value problem

$$\begin{aligned}
&v'' + h(t) = 0, \quad 0 < t < 1,\\
&v(0) = 0, \quad v(1) = 0.
\end{aligned}$$

Also, u now satisfies a lower order boundary value problem,

$$\begin{aligned}
&D_{0+}^{\alpha}u(t) = v(t), \quad 0 < t < 1,\\
&u^{(i)}(0) = 0, \quad i = 0, 1, \ldots, m-2, \quad D_{0+}^{\beta}u(1) = 0,
\end{aligned}$$

and so,

$$\begin{aligned}
u(t) &= \int_0^1 G_0(t,s)(-v(s))ds\\
&= \int_0^1 \left(-\int_0^1 G_0(t,s)G_{conj}(s,r)ds\right) h(r)dr\\
&= \int_0^1 G_1(t,s)h(s)ds,
\end{aligned}$$

where $G_1(t,s) = -\int_0^1 G_0(t,r)G_{conj}(r,s)dr$.

For the inductive step, consider

$$D_{0^+}^{\gamma}u(t) + k(t) = 0,$$

satisfying (1.4). The argument here is similar to above. Make the change of variable $v(t) = D_{0^+}^{\gamma-2}u(t)$. Thus $D^2v(t) = D^2D_{0^+}^{\gamma-2}u(t) = D_{0+}^{\gamma}u(t) = -k(t)$. Since $v(0) = D_{0+}^{\gamma-2}u(0) = 0$ and $v(1) = D_{0+}^{\gamma-2}u(1) = 0$, then v satisfies the Dirichlet boundary value problem

$$\begin{aligned}
&v'' + k(t) = 0, \quad 0 < t < 1,\\
&v(0) = 0, \quad v(1) = 0.
\end{aligned}$$

Here u now satisfies a lower order boundary value problem,

$$\begin{aligned}
&D_{0+}^{\gamma-2}u(t) = v(t), \quad 0 < t < 1,\\
&u^{(i)}(0) = 0, \quad i = 0, 1, \ldots, m-2, \quad D_{0+}^{\beta}u(1) = 0,\\
&D_{0^+}^{\gamma-2l}u(0) = 0, \quad D_{0^+}^{\gamma-2l}u(1) = 0, \quad l = 2, \ldots, k,
\end{aligned}$$

and by the induction hypothesis,

$$\begin{aligned}
u(t) &= \int_0^1 G_{n-1}(t,s)(-v(s))ds\\
&= \int_0^1 \left(-\int_0^1 G_{n-1}(t,s)G_{conj}(s,r)ds\right) k(r)dr
\end{aligned}$$

$$= \int_0^1 G_n(t,s)k(s)ds,$$

where $G_n(t,s) = -\int_0^1 G_{n-1}(t,r)G_{conj}(r,s)dr$.

Define $v_n(t,s)$ so that $t^{\alpha-1}v_n(t,s) = G_n(t,s)$. The following lemma follows from Lemma 3.2.

Lemma 5.1. *Let G_n be defined inductively as above.*

(1) $G_n(t,s) \in C([0,1]\times[0,1))$ *with* $(-1)^n G_n(t,s) > 0$ *for* $(t,s) \in (0,1]\times(0,1)$.
(2) $t^{\alpha-1}(-1)^n G(1,s) \le (-1)^n G(t,s) \le (-1)^n G(1,s)$ *for* $(t,s) \in [0,1]\times[0,1)$.
(3) $(-1)^n G(t,s) \ge (-1)^n \delta^{\alpha-1} G(1,s)$ *for all* $t \in [\delta,1]$ *and all* $s \in [0,1)$.
(4) $(-1)^n v(0,s) \ge 0$ *for all* $s \in [0,1)$.

Proof. We start by showing (1) holds. For the base case, consider that $G_0(t,s) = G(t,s)$ from Lemma 3.2 which does belong to $C([0,1]\times[0,1])$ and is positive. Since

$$G_1(t,s) = -\int_0^1 G_0(t,r)G_{conj}(r,s)ds,$$

and $G_{conj}(r,s) \in C([0,1]\times[0,1])$ and $G_{conj}(t,s) > 0$, it follows that $G_1(t,s) \in C([0,1]\times[0,1])$ and $-G_1(t,s) > 0$. For the inductive step, assume $G_{n-1}(t,s) \in C([0,1]\times[0,1))$ and $(-1)^{n-1}G_{n-1}(t,s) > 0$ for $(t,s) \in (0,1]\times(0,1)$. Then by definition

$$(-1)^n G_n(t,s) = -\int_0^1 (-1)^{n-1} G_{n-1}(t,r)G_{conj}(r,s)dr,$$

we see that since $G_{conj}(t,s) \in C([0,1]\times[0,1])$ and $G_{conj}(t,s) > 0$ for $(t,s) \in (0,1)\times(0,1)$, then $(-1)^n G_n(t,s) > 0$ for $(t,s) \in (0,1]\times(0,1)$ and $G_n(t,s) \in C([0,1]\times[0,1))$.

For (2), similar to the first item, the base case follows from Lemma 3.2. Since for $G_0(t,s) = G(t,s)$, we have

$$t^{\alpha-1}G_0(1,s) \le G_0(t,s) \le G_0(1,s),$$

and by the definition of $G_1(t,s)$ we have

$$\begin{aligned} -t^{\alpha-1}G_1(1,s) &= \int_0^1 t^{\alpha-1}G_0(1,r)G_{conj}(r,s)dr \\ &\le \int_0^1 G_0(t,r)G_{conj}(r,s)dr \\ &= -G_1(t,s) \\ &\le \int_0^1 G_0(1,r)G_{conj}(r,s)dr \\ &= -G_1(1,s). \end{aligned}$$

For the inductive step, in a similar fashion, assume

$$t^{\alpha-1}(-1)^{n-1}G_{n-1}(1,s) \le (-1)^{n-1}G_{n-1}(t,s) \le (-1)^{n-1}G_{n-1}(1,s).$$

Then by the definition of $G_n(t,s)$, we have

$$\begin{aligned}
t^{\alpha-1}(-1)^n G_n(1,s) &= (-1)^{n+1} t^{\alpha-1}\int_0^1 G_{n-1}(1,r)G_{conj}(r,s)dr \\
&\le (-1)^{n+1}\int_0^1 G_{n-1}(t,r)G_{conj}(r,s)dr \\
&= (-1)^n G_n(t,s) \\
&\le (-1)^{n+1}\int_0^1 G_{n-1}(1,r)G_{conj}(r,s)dr \\
&= (-1)^n G_n(1,s).
\end{aligned}$$

Notice that (3) is a direct result of (2), and a proof of (4) can similarly be obtained using induction. □

Define the sets

$$\mathbb{L}^1_{n+} = \{m \in \mathbb{L}^1(0,1) : (-1)^n m(t) \ge 0 \text{ a.e. } t \in [0,1]\}$$

and

$$\mathbb{L}^1_{n++} = \{m \in \mathbb{L}^1_{n+} : (-1)^n m(t) > 0 \text{ on a subset of positive measure}\}.$$

Let $m \in \mathbb{L}^1_{n++}$. Define $L_{nm} : E \to E$ by

$$L_{nm}u(t) = \int_0^1 G_n(t,s)m(s)u(s)ds.$$

Define $L^X_{nm} : X \to E$ by, for $u \in X$, $L^X_{nm}u = L_{nm}u$.

Lemma 5.2. *For $m \in \mathbb{L}^1_{n+}$, the operator L_{nm} is compact and positive. Moreover, $L_{nm} : E^+ \to \mathcal{P}$.*

Proof. Let $u \in E^+$. Then

$$\begin{aligned}
L_{nm}u(t) &= \int_0^1 G_n(t,s)m(s)u(s)ds \\
&= \int_0^1 (-1)^n G_n(t,s)|m(s)|u(s)ds > 0,
\end{aligned}$$

and, since $\|L_{nm}u\| = |L_{nm}u(1)|_0$,

$$\begin{aligned}
L_{nm}u(t) &= \int_0^1 G_n(t,s)m(s)u(s)ds \\
&= \int_0^1 (-1)^n G_n(t,s)|m(s)|u(s)ds \\
&\ge \delta^{\alpha-1}\int_0^1 (-1)^n G_n(1,s)|m(s)|u(s)ds \\
&= \delta^{\alpha-1}\|L_{nm}u\|,
\end{aligned}$$

concluding the proof. □

Lemma 5.3. *For $m \in \mathbb{L}^1_{n++}$, L_{nm} is a strongly positive-like operator which is lower bounded on the cone $\mathcal{P}$.*

Proof. As in the proof of Lemma 3.4, if we can show $L_{nm}^X : X^+ \setminus \{0\} \to S \subset X^{+\circ}$, the result follows. Let $u \in X^+$. First, notice for $t \in (0, 1]$,

$$L_{nm}^X u(t) = \int_0^1 (-1)^n G_n(t,s)|m(s)|u(s)ds > 0.$$

Again, since m and $v_n(0, s)$ have the same sign,

$$\lim_{t\to 0} \frac{L_{mn}^X u(t)}{t^{\alpha-1}} = \int_0^1 (-1)^n v_n(0,s)|m(s)|u(s)ds > 0.$$

So $L_m^X : X \setminus \{0\} \to S \subset X^{+\circ}$, and the result follows. □

Define the operator $T_n : E^+ \to E$ by

$$T_n u(t) = \int_0^1 G_n(t,s)(-1)^n g(s, u(s))ds.$$

Notice that u is a solution of the boundary value problem (1.3), (1.4) if and only if u is a fixed point of T_n.

The following lemma is a direct result of the Arzelà-Ascoli theorem and Lemma 5.1.

Lemma 5.4. *$T_n : E^+ \to E$ is compact and $T_n : E^+ \to \mathcal{P}$.*

The proofs of the main results are similar to the proofs from Section 4 and are therefore omitted.

Let $m \in \mathbb{L}^1_{n++}$. Consider the linear boundary value problem

$$D^\gamma_{0^+} u(t) + \mu m(t)u(t) = 0, \quad \text{a.e. } t \in (0, 1), \tag{5.3}$$

satisfying the boundary conditions (1.4), where μ is a real parameter.

Lemma 5.5. *For all $m \in \mathbb{L}^1_{n++}$,* (5.3), (1.4) *admits a unique positive eigenvalue $\mu_\alpha(m)$.*

Theorem 5.1. *Assume that there exists $m \in \mathbb{L}^1_{n+}$ such that one of the following hypotheses is satisfied.*

$$\mu_\alpha(m) < 1 \quad \text{and} \quad (-1)^n g(t,u) \geq m(t)u, \quad \text{for all } u \geq 0 \text{ and a.e. } t \in (0,1), \tag{5.4}$$

$$\mu_\alpha(m) > 1 \quad \text{and} \quad (-1)^n g(t,u) \leq m(t)u, \quad \text{for all } u \geq 0 \text{ and a.e. } t \in (0,1), \tag{5.5}$$

then (1.3), (1.4) *has no positive solutions.*

Theorem 5.2. *Assume that there exist $m_1, m_2 \in \mathbb{L}^1_{n++}$, $q_1, q_2 \in \mathbb{L}^1_{n+}$, and two functions $\phi_1, \phi_2 : [0,\infty) \to [0,\infty)$ such that $\mu_\alpha(m_1) < 1 < \mu_\alpha(m_2)$ and, for all $u \geq 0$ and a.e. $t \in (0,1)$*

$$m_1(t)u - q_1(t)\phi_1(u) \leq (-1)^n g(t,u) \leq m_2(t)u + q_2(t)\phi_2(u). \tag{5.6}$$

If either

(H1) $\phi_1(u) = o(\|u\|)$ *as* $u \to \infty$, $\phi_2(u) = o(\|u\|)$ *as* $u \to 0$, ϕ_1 *is nondecreasing, and* ϕ_2 *is nondecreasing near* 0 *or*

(H2) $\phi_1(u) = o(\|u\|)$ *as* $u \to 0$, $\phi_2(u) = o(\|u\|)$ *as* $u \to \infty$, ϕ_1 *is nondecreasing near* 0, *and* ϕ_2 *is nondecreasing,*

then (1.3), (1.4) *has at least one positive solution.*

We conclude the paper by remarking that the hypotheses of Theorems 4.2 and 5.2 are similar to the hypotheses of the main theorem in [3]. Therefore, the examples of nonlinearities provided in that work could be easily modified for the problems given in this paper.

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