UNIVERSITY OF KRAGUJEVAC
FACULTY OF SCIENCE

Analysis Approximation Applications

JUNE 21–24 VRNJAČKA BANJA

INTERNATIONAL MATHEMATICAL CONFERENCE

BOOK OF ABSTRACTS

2023

Analysis Approximation Applications

UNIVERSITY OF KRAGUJEVAC
FACULTY OF SCIENCE

INTERNATIONAL MATHEMATICAL CONFERENCE
ANALYSIS, APPROXIMATIONS AND APPLICATIONS
(AAA2023)

Dedicated to Academician Gradimir V. Milovanović on the occasion of his 75th anniversary

BOOK OF ABSTRACTS

June 21–24, 2023
Vrnjačka Banja, Serbia

CIP - Каталогизација у публикацији Народна библиотека Србије, Београд

517.5(048)

INTERNATIONAL Mathematical Conference Analysis, Approximations and Applications (2023 ; Vrnjačka Banja)

Book of Abstracts / International Mathematical Conference Analysis, Approximations and Applications (AAA2023), June 21–24, 2023 Vrnjačka Banja, Serbia ; [edited by Aleksandra Milosavljević]. - Kragujevac : Faculty of Science University, 2023 (Niš : Grafika Galeb). - 103 str. ; 25 cm

Tiraž 100. - Bibliografija uz pojedine apstrakte. - Registar.

ISBN 978-86-6009-094-4

a) Теорија функција -- Апстракти

COBISS.SR-ID 118020873

Book of Abstracts
International Mathematical Conference
Analysis, Approximations and Applications (AAA2023)
June 21–24, 2023
Vrnjačka Banja, Serbia
`https://imi.pmf.kg.ac.rs/aaa2023/`
ISBN 978-86-6009-094-4

Published By:

Faculty of Science
University of Kragujevac
Radoja Domanovića 12
34000 Kragujevac
Serbia
Tel.: +381 (0)34 336223
Fax: +381 (0)34 335040
Website: `http://pmf.kg.ac.rs`

For publisher: Prof. Dr. Marija Stanić, dean

Edited By: Aleksandra Milosavljević
Front Cover: Ivan Đorđević
Printed By: Grafika Galeb DOO, Niš, Serbia
Circulation: 100

Year of Publication: 2023

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HOST CITY – VRNJAČKA BANJA

Known from antiquity as the favorite destination of aristocracy, Vrnjačka Banja (eng. Vrnjacka spa) is a health resort in the heart of Serbia, popular nowadays for its springs of water with healthy features.

Vrnjačka Banja was named after the village of Vrnjci where mineral springs were first discovered. It is located in Central Serbia, 200 kilometers south from the city of Belgrade, at forested slopes of mountain Goč and in valleys of the Vrnjačka river and the Lipovačka river.

It has mild continental climate, and it has seven mineral springs, four of which are used for balneological therapy. History says that these mineral springs were discovered in ancient history at the time when this area was populated by the Celtic tribe Scordisci.

Based on archaeological findings, we can say that these springs were used in ancient times and at hot mineral spring in Vrnjačka Banja Romans built the health resort Aquae Orcinae, visited both by legionnaires and aristocracy. The remains of antique pools and terms are preserved to this day.

According to a legend, its healing powers were well known to the Turks as well, as it is known that Turkish sipahi came here looking for a piece of heaven. Local people covered the springs with dirt attempting to hide them.

The modern history of Vrnjačka Banja begins in 1868, when Pavle Mutavdžić, the district chief in Kruševac of the time, formed an appropriate Constitutive association together with some other people of high reputation.

They bought land, ground catchment was performed, a pool was built, and drinking fountains of hot and cold water were set.

In 1883 state ownership was declared, and the inaugural season was opened by the Serbian Prince Mihailo (1867) when the spa became an elite summer resort.

Hot and cold mineral springs, helped by modern medicine, are used for treatment of diseases of the digestive system, diabetes, kidney diseases and urinary tract diseases, cardio vascular diseases and many more. Mineral waters from Vrnjačka Banja are also recommended to healthy people due to their curative effects.

Vrnjačka Banja is one of the most visited spas in Serbia, but what it also offers are numerous possibilities for sports and recreation.

A *must see* event organized in Vrnjačka Banja is The Carnival of Vrnjci, and those who enjoy other kinds of festivities can visit The Film Script Festival, The Great Summer Film Marathon and The Great Poetic Meeting.

The Carnival of Vrnjci is held every July and gives you a chance to join numerous costume balls and see many interesting exhibitions, concerts and theatrical plays.

While staying in Vrnjačka Banja you should visit some of the medieval monuments located near the spa. The monasteries Studenica, Žiča, Gradac, Djurdjevi Stupovi and Sopoćani are situated along the valley of the Ibar river.

The unforgettable experience awaits you at the medieval cities and fortresses of Ras, Maglic and Koznik. The magnificent manor house Belimarković, including the museum of Vrnjačka Banja, enriches the beauties of this town.

How to get to Vrnjačka Banja?

If you are traveling from the direction of Belgrade, you can get to Vrnjačka Banja using the Belgrade–Kraljevo road or the Belgrade–Niš–Skopje highway via Pojate. The road is around 200 km long.

If you are coming by bus all the major cities have bus connections to Vrnjačka Banja. Also you can go to Kraljevo, and then take a local bus to the spa and the distance is 30 km.

SPECIAL SESSION DEDICATED TO GRADIMIR V. MILOVANOVIĆ – ABSTRACT

The life path and scientific career of Academician Professor Gradimir V. Milovanović

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The life path and rich scientific career of the world numerician and mathematician, scientist, professor and Serbian academician Gradimir V. Milovanović will be presented by his former PhD students and permanent collaborators.

SPECIAL GUEST – ABSTRACT

The Ramanujan integral and its derivatives: Computation and analysis

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The principal tool of computation used in this paper is classical Gaussian quadrature on the interval $[0, 1]$, which happens to be particularly effective here. Explicit expressions are found for the derivatives of the Ramanujan integral and it is proved that the latter is completely monotone on $(0, \infty)$. A new series expansion for the incomplete gamma function is found and conjectured to converge alternately from above and below. The paper also pays attention to another famous integral, the Euler integral – better known as the gamma function – revitalizing a largely neglected part of the function, the part corresponding to negative values of the argument, which plays a prominent role in our work.

PLENARY LECTURES – ABSTRACTS

L^p bounds for orthogonal polynomials and applications

Alexander Aptekarev[1]

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The classical Steklov problem deals with bounds of the Tchebyshëv norm $\|p_n\|^\infty$ for the polynomials $p_n(x)$, orthonormal on the interval Δ with respect to the strictly positive wieght function: $w \in L^1(\Delta) \bigcap S_\delta$, $S_\delta := \{w : w(x) \geq \delta > 0, x \in \Delta\}$. Modern applications (in particular, to the information entropy of quantum systems) motivate us to consider also the estimates of L^p norms: $\|p_n\|_w^p(\Delta)$ for the Steklov weight functions $w \in X(\Delta) \bigcap S_\delta$ from the various classes $X := L^\infty$, S — (the Szego class), BMO, A_p — (the Muckenhoupt class).

Our talk is based on the joint papaer with Sergey Denisov and Michel Alexis [1]. Thus, we focus on $\|p_n\|_w^p$, $p > 2$, for $w \in A_2 \bigcap S_\delta$.

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Semiclassical orthogonal polynomials, Painlevé equations and applications to quadrature formulae

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In this lecture we overview the theory of semiclassical orthogonal polynomials following [5]. Some illustrative examples (see [1], [2]) are presented with a special emphasis on the Laguerre-Freud equations associated with the coefficients of the three term recurrence relation they satisfy. The connection with discrete and continuous Painlevé equations has been deeply analyzed in [6],

The truncated Hermite case (see [2]) is related to the so called Rys polynomials, whose zeros and quadrature formulas have been studied in [4], among others. In the framework of random matrices, Gaussian unitary ensembles with two jump discontinuities have been studied in [3], among others.

Some open problems will be stated.

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Polynomial inequalities

Vilmos Totik[1]

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The talk will discuss Bernstein and Markov type polynomial inequalities in various settings (general sets, integral norms). The main emphasis will be on the exact constants in the inequalities.

Multiple orthogonal polynomials for special function weights

Walter Van Assche[1]

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Multiple orthogonal polynomials are polynomials in one variable that satisfy orthogonality conditions with respect to r measures. They appear naturally in Hermite-Padé approximation to r functions. The case $r = 1$ corresponds to the usual orthogonal polynomials. Several systems of multiple orthogonal polynomials have been constructed using classical weight functions (multiple Hermite, multiple Laguerre, multiple Jacobi polynomials). In this talk I will use weight functions given by special functions satisfying a differential equation. The r weights then appear by writing the differential equation as a system of first order equations, which then generalizes the Pearson equation for classical orthogonal polynomials. The weights are in terms of modified Bessel functions $K_\nu(2\sqrt{x})$ [3], modified Bessel functions $I_\nu(2\sqrt{x})$ [4], hypergeometric functions [1] and confluent hypergeometric functions [2], and the exponential integral [5]. We give some applications where these multiple orthogonal polynomials appear, such as the eigenvalues of products of random matrices, non-intersecting Brownian motions, and rational approximations to real numbers.

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INVITED LECTURERS FOR THE SESSION ANALYSIS – ABSTRACTS

The Beverton–Holt equation

Martin Bohner[1], Jaqueline Mesquita[2], Sabrina Streipert[3]

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In this talk, we will present the Beverton–Holt equation as used in fisheries and other population models, in many different scenarios (discrete case, continuous case, time scales case, quantum case, periodic case, with and without harvesting etc.).

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On almost sequence spaces defined by infinite matrix

Ekrem Savaş[1]

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Let w denote the set of all real and complex sequences $x = (x_k)$. By l_∞ and c, we denote the Banach spaces of bounded and convergent sequences $x = (x_k)$ normed by $\|x\| = \sup_n |x_n|$, respectively. A linear functional L on l_∞ is said to be a Banach limit [1] if it has the following properties:

1. $L(x) \geq 0$ if $n \geq 0$ (i.e. $x_n \geq 0$ for all n),
2. $L(e) = 1$ where $e = (1, 1, \ldots)$,
3. $L(Dx) = L(x)$, where the shift operator D is defined by $D(x_n) = \{x_{n+1}\}$.

Let B be the set of all Banach limits on l_∞. A sequence $x \in \ell_\infty$ is said to be almost convergent if all Banach limits of x coincide. Let $\hat{c}$ denote the space of almost convergent sequences.

Definition 1. Let $A = [a_{n,k}]$ denote a summability transformation [4] that maps complex sequences x into the sequence Ax where the n-th term of Ax is as follows:

$$[Ax_n] = \sum_{k=1}^{\infty} a_{n,k} x_k$$

The purpose of this paper is to present the new deferred almost sequence space which emerges naturally from the concepts of almost convergence and an infinite matrix. Further some theorems are proved.

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SESSION 1 - ANALYSIS – ABSTRACTS

Dynamical sampling for shift-preserving operators acting on finitely generated shift-invariant subspaces of Sobolev spaces

Suzana Aleksić[1], Stevan Pilipović[2], Aleksandar Aksentijević[3]

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We analyze shift-invariant spaces V_s, subspaces of Sobolev spaces $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, generated by the set of generators φ_i, $i \in I$, I is countable at most, by the use of range functions and characterize Bessel sequences, frames and Riesz basis of such spaces. We show that an $f \in \mathcal{D}'_{L^2}(\mathbb{R}^n)$ belongs to V_s if and only if its Fourier transform has the form $\widehat{f} = \sum_{i\in I} f_i g_i$, $f_i = \widehat{\varphi}_i \in L^2_s(\mathbb{R}^n)$, $\{\varphi_i(\cdot + k) : k \in \mathbb{Z}^n, i \in I\}$ is a frame and $g_i = \sum_{k\in\mathbb{Z}^n} a^i_k \mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,k\rangle}$, with $(a^i_k)_{k\in\mathbb{Z}^n} \in \ell^2$. Moreover, connecting two different approaches to shift-invariant spaces V_s and $\mathcal{V}^2_s$, $s > 0$, under the assumption that the finite number of generators belongs to $H^s \cap L^2_s$, we give the characterization of elements in V_s through the expansions with coefficients in ℓ^2_s. We also give the representation for shift-preserving operators $L : V_s \to V_s$ in terms of range operators. Using a range operator approach, we derive a result about dual frames and solve the dynamical sampling problem for a class of shift-preserving operators acting on a finitely generated shift-invariant space V_s.

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Fixed point algorithms: convergence, stability and data dependence results

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In this talk, we discuss a newly introduced two step fixed point iterative algorithm. We prove a strong convergence result for weak contractions. We also prove stability and data dependency of a proposed iterative algorithm. Furthermore, we utilize our main result to approximate the solution of a nonlinear functional Volterra integral equation. If time permits, then we will discuss Image recovery problem as well.

Theorem 1. Let $T : C \to C$ be a weak contraction satisfying (1.9), where C is a nonempty, closed and convex subset of a Banach space X. Then proposed iterative algorithm (1.21) is almost T-stable.

Theorem 2. Let S be an approximate operator of a weak contraction mapping T satisfying (1.9), $\{x_n\}$ be a sequence generated by proposed iterative algorithm (1.21) for T and define a sequence $\{u_n\}$ for S as follows:

$$\begin{cases} u_0 = u \in C, \\ u_{n+1} = Sv_n, \\ v_n = S((1-a_n)u_n + a_n S u_n), \; n \in \mathbb{Z}_+, \end{cases} \tag{1}$$

where $\{a_n\}$ is a sequence in $(0,1)$ satisfying $\frac{1}{2} \leq a_n$ for all $n \in \mathbb{Z}_+$ and $\sum_{n=0}^{\infty} a_n = \infty$. If $Tp = p$ and $Sq = q$ such that $u_n \to q$ as $n \to \infty$, then we have

$$\|p - q\| \leq \frac{5\epsilon}{1-\delta},$$

where $\epsilon > 0$ is a fixed number.

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H^p theory for separately (α, β)-harmonic functions

Miloš Arsenović[1], Jelena Gajić[2], Miodrag Mateljević[1]

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We investigate spaces $sh^p_{\alpha,\beta}$ of separately (α, β) harmonic functions on the unit polydisc. The Dirichlet problem with continuous boundary data on the distinguished boundary T^n for the corresponding system of PDEs is solved. Also, representation and extension theorems are proved for the full range of exponents $1 \leq p \leq +\infty$. Results on convergence in norm and in weak-star topolgy at the boundary are obtained in analogy with the classical case of Hardy spaces. In addition, properties of so called slice functions are investigated. These are obtained by fixing k variables, for example $u(z_1, \ldots, z_{n-k}) = v(z_1, \ldots, z_{n-k}, \zeta_1, \ldots, \zeta_k)$

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On a monomial decomposition of a complex polynomial

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Let $p(z)$ be a complex polynomial of degree n. We study conditions for representing this polynomial as the arithmetic mean of monomials $(z - z_i)^n$, $1 \leq i \leq n$. We prove the corresponding uniqueness and existence reslut. We also give estimation of the parameters z_i in terms of the coefficients of the given polynomial $p(z)$.

Relations for Bernoulli–Barnes numbers and Barnes zeta functions

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The *Barnes ζ-function* is

$$\zeta(z, x; \mathbf{a}) := \sum_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^n} \frac{1}{(x + m_1 a_1 + \cdots + m_n a_n)^z}$$

defined for $\Re(x) > 0$ and $\Re(z) > n$ and continued meromorphically to $\mathbb{C}$. Specialized at negative integers $-k$, the Barnes ζ-function gives

$$\zeta(-k, x; \mathbf{a}) = \frac{(-1)^n k!}{(k+n)!} B_{k+n}(x; \mathbf{a})$$

where $B_k(x; \mathbf{a})$ is a *Bernoulli–Barnes polynomial*, which can be also defined through a generating function that has a slightly more general form than that for Bernoulli polynomials. Specializing $B_k(0; \mathbf{a})$ gives the *Bernoulli–Barnes numbers*. We exhibit relations among Barnes ζ-functions, Bernoulli–Barnes numbers and polynomials, which generalize various identities of Agoh, Apostol, Dilcher, and Euler.

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Existence of approximating solution of second order nonlinear integro-differential equations via iteration method

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In this paper the existence theorem for the second order functional integro-differential equations in Banach algebras is proved under the mixed generalized Lipschitz and Caratheodory conditions. The existence of extremal solutions is also proved under certain monotonicity conditions usings Dhage iteration Method. As a generalization of ordinary integro-differential equations, there is a series of papers dealing with the abstract measure integro-differential equations in which ordinary derivative is replaced by the derivative of set functions, namely, the Radon-Nokodym derivative of a measure with respect to another measure. See Bellale [10], Dhage [4], Dhage and Bellale [8] and the references therein. The above mentioned papers also include some already known abstract measure differential equations those considered in P. C. Das and Sharma [2] , Shendge and Joshi studied as special cases. The origin of the quadratic integral equations appears in the works of Chandrasekhar's H-equation in radioactive heat transfer, but the study of nonlinear integral equations via operator theoretic techniques seems to have been started by Dhage in the year 1988. Similarly, the study of nonlinear quadratic differential equations is relatively new and initiated by Dhage and O'Regan in the year 2000. In this paper we discussed the following problem for external solution For a given closed and bounded interval J = [0; a] in $\mathbb{R}$, the set of real numbers, consider the following integro-differential equation.

$$(2)\qquad \begin{cases} \left(\dfrac{x(t)}{f(t,x(t))}\right)' = g\left(t, x(t), \int\limits_0^t k(s,x(s))\mathrm{d}s\right) \quad \text{a.e. } t \in J \\ x(0) = x_0 \in \mathbb{R} \end{cases}$$

where $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is continuous, $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $k : J \times \mathbb{R} \to \mathbb{R}$.

The existence of the solutions to (2) is proved in Dhage and Ballale by using a new nonlinear alternative of Leray-Schauder type developed in same paper. In this paper we apply a nonlinear alternative of Leray-Schauder type due to Dhage and Bellale [7] involving the product of two operators in a Banach algebra under some weaker conditions than that given in Dhage and Regan [9] to a quadratic abstract measure differential equation related to (2) for proving the existence results. The existence of extremal solutions is also proved using a fixed point theorem of Banas [1] in ordered Banach algebras.

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Completion of operator matrices and some interesting applications

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We will discuss certain problems on completions of different types of operator matrices and using these results we will give certain necessary and sufficient conditions for the existence of solutions of some operator equations that belong to prescribed classes of operators. Appropriate representations of solutions for each class will be given. Also, we will discuss the existence of a positive solution of the operator equation $AXB = C$ which is still an open problem considered before only under additional conditions including that of regularity, as well as under certain range conditions such as $R(B) \subseteq \overline{R(A^*)}$. In this talk we will consider the existence of a positive solution of the operator equation $AXB = C$ without any additional range or regularity assumptions using two well-known results of Douglas and Zoltán. Also we will give a general form of a positive solution and consider some possible applications.

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Different approaches in the study of φ-contraction

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The study of φ-contractions dates to 1968. and results of eminent researchers like F. E. Browder, D. W. Boyd and J. S. Wong. In the following half of century various modifications and improvements of this concept were in the spotlight of research in the area of Metric Fixed Point Theory. We intend to discuss on the relation of Boyd-Wong contraction, Matkowski contraction, Meir-Keeler contraction, Proinov contraction, F-contraction and θ-contraction. It is also of great importance to study possible applications and benefits of applying nonlinear contractions over Banach contraction and its modifications.

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The essentially left and right generalized Drazin invertible operators and generalized Saphar decomposition

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In this paper we define and study the classes of the essentially left and right generalized Drazin invertible operators and of the left and right Weyl-g-Drazin invertible operators by means of the analytical core and the quasinilpotent part of an operator. We show that the essentially left (right) generalized Drazin invertible operator can be represented as a sum of a left (right) Fredholm and a quasinilpotent operator. Analogously, the left (right) Weyl-g-Drazin invertible operator can be represented as a sum of a left (right) Weyl and a quasinilpotent operator. We also characterize these operators in terms of their generalized Saphar decompositions, accumulation and interior points of various spectra of operator pencils. Furthermore, we expand the results from [7], on generalized left and right Drazin invertible operators. Special attention is devoted to the investigation of the corresponding spectra of operator pencils.

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Double stochastic operators on L^1

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We prove some results on double stochastic operators on L^1 spaces. Thus, we extend some results from a discrete case to a more general settings.

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Isolated points of the extended spectrum of a linear relation

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In the case of linear operators in a Banach space, the study of isolated points in the spectrum is strongly connected with the theory of generalzied inverses. For example, if zero is an isolated point in the spectrum of a linear bounded operator, its has a generalized Drazin inverse. For a linear relation (multivalued linear operator) in a Banach spaces the direct sum decoposition of it represents a powerful tool for determining the existence of a generalized inverses.

In this talk, we introduce the notions of $R-$invariance and $R-$restriction that is the best way to study the spectral decomposition associated with an isolated points in the spectrum of a linear relation. Our main attention is focused on the study the case when 0 or ∞ are an isolated points in the spectrum of a linear relation or, even more, when there are a pole of the resolvent function.

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The classical Karamata's theory of regular variability and the index function operator

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The classical Karamata's theory of regular variability has a functional and sequence forms. In this paper, we will consider asymptotic equivalence relations (strong and weak asymptotic equivalence and asymptotic similarity) in terms of Karamata's theory (see [1]). We will also use the index function operator, and prove new statements of the same type (in function and sequence variants) (see [2]).

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A series expansion of separately (α, β)-harmonic function in the unit polydisc

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We define separately $(\alpha, \beta)-$ harmonic functions in the unit polydisc which satisfy a system of elliptic PDEs. Also, we derive series expansion of such functions. Our result extend earlier results for (α, β)-harmonic functions in the disc and for n-harmonic functions in $\mathbb{D}^n$. This is joint work with Miloš Arsenović and Miodrag Mateljević and this is work in progress.

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Hypercyclic operators on non-unital C*-algebras

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The dynamics of wedge operators on the (non-commutative) C^*-algebra of compact operators on a Hilbert space has been considered in for instance [2], whereas the dynamics of the weighted composition operators on the (commutative) C^*-algebra of continuous functions has been considered in for instance [1]. In this talk we present an algebraic generalization of these results to the case of arbitrary non-unital C^*-algebras. More precisely, we let $\mathcal{A}$ be a non-unital C^*-algebra such that $\mathcal{A}$ is a closed two-sided ideal in a unital C^*-algebra $\mathcal{A}_1$ and we let Φ be an isometric $*$-isomorphism of $\mathcal{A}_1$ such that $\Phi(\mathcal{A}) = \mathcal{A}$. For an invertible element $b \in \mathcal{A}_1$ we let $T_{\Phi,b}$ be the operator on $\mathcal{A}_1$ defined by $T_{\Phi,b}(a) = b \cdot \Phi(a)$ for all $a \in \mathcal{A}_1$. Then $T_{\Phi,b}$ is a bounded linear operator on $\mathcal{A}_1$ and since $\mathcal{A}$ is an ideal in $\mathcal{A}_1$, it follows that $T_{\Phi,b}(\mathcal{A}) \subseteq \mathcal{A}$ because $\Phi(\mathcal{A}) = \mathcal{A}$.

We study the dynamics of the operator $T_{\Phi,b}$ and we provide the necessary and sufficient conditions for a finite family of such operators to be disjoint hypercyclic on $\mathcal{A}$. Moreover, we illustrate our result in the various cases of some concrete non-unital C^*-algebras.

In addition, we study the dynamics of the generalized weighted bilateral shift operators on the standard Hilbert C^*-module, and we provide concrete examples. This talk is partly based on [3, 4].

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On generalized metric properties of the hyperspace $\mathcal{F}(X)$

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We present some results concerning several generalized metric properties of the space $\mathcal{F}(X)$ of finite subsets of a space X endowed with the Vietoris topology. In particular, we consider such properties (P) for which $\mathcal{F}(X)$ has (P) if and only if X has (P). Also, we identify a few properties (Q) such that X has (Q) whenever $\mathcal{F}(X)$ has (Q), but the converse is not true.

The subsequences of a power sequence in a Banach algebra

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We consider the covergence of the sequences a^{qn} and a^{q^n}, where a is an arbitrary element of a complex Banach algebra with the unit, and q is an integer such that $q \geq 2$.

We give necessary and sufficient conditions such that $\lim_{n\to\infty} a^{qn} = d$ exists, and give the explicit form of d. The final result of this limit has the same form as the one obtained by Chen and Hartwig [1], but we use a different method to prove it. We, also, consider the existence and the form of the limit $\lim_{n\to\infty} a^{q^n}$ for the same q.

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Asymptotical behavior of solutions of Emden-Fowler equation

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We will show the existence of infinitely many solutions of differential equation of Emden-Fowler type $y'' + x^a y^\sigma = 0$, for $a \in \mathbb{R}$ and $\sigma < 0$, which are tending to 0 as $x \to 0+$. Also, we will describe the conditions on parameters a and σ which assure that equation of Emden-Fowler type $y'' - x^a y^\sigma = 0$, for $a \in \mathbb{R}$ and $\sigma < 0$ has infinitely many solutions defined in some neighborhood of 0 and the conditions which guarantee existence of infinitely solutions with certain asymptotic behavior.

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Linear preservers of the extended majorization relations on $\ell^p(I)$

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Linear preservers of the extended majorization relations on the discrete Lebegue spaces $\ell^p(I)$, where I is an arbitrary non-empty set and $p \in [1, \infty)$, are disscused. Necessary and sufficient conditions under which a bounded linear operator on $\ell^p(I)$ is a linear preserver of the selected majorization relation, are given.

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Lipschitz continuity for harmonic functions and solutions of the $\bar{\alpha}$-Poisson equation

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We study Lipschitz continuity for harmonic functions, gradient harmonic functions and solutions of the $\bar{\alpha}$-Poisson equation in planar case and several variables. We also review some recently obtained results. It turns out that the gradients of hyperbolic harmonic functions behave differently from those of euclidean harmonic functions. A similar conclusion is obtained for the family of T_α-harmonic functions. Namely, unlike the space of harmonic functions, the solution of the Dirichlet problem in the space of T_α-harmonic functions is shown to be Lipschitz-continuous when so is the boundary function, for $\alpha > 0$.

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Generalization of the Levin-Stečkin inequality

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V. I. Levin and S. B. Stečkin proved the following theorem

Theorem 1. Let f be defined on $[0,1]$ satisying the conditions:

$$f(x) \text{ is nondecreasing for } 0 \leq x \leq \frac{1}{2}, \tag{3}$$

and

$$f(x) = f(1-x), x \in [0,1]. \tag{4}$$

Then for any convex function ϕ we have

$$\int_0^1 f(x)\phi(x)dx \leq \int_0^1 f(x)dx \int_0^1 \phi(x)dx. \tag{5}$$

Using the Green function we obtain some interesting results concerning the difference of the integral arithmetic means

$$\frac{\int_a^b f(x)d\lambda_2(x)}{\int_a^b d\lambda_2(x)} - \frac{\int_a^b f(x)d\lambda_1(x)}{\int_a^b d\lambda_1(x)}, \tag{6}$$

where $\lambda_1, \lambda_2 \colon [a,b] \to \mathbb{R}$ are continuous functions or the functions of bounded variation, such that $\lambda_i(a) \neq \lambda_i(b)$, $i = 1, 2$.
First we derive results for integral means with general measures, using them we obtain the results for integral weighted means with functions as weights and with unique measure, and finally we show that we have really obtained generalization of the Levin-Stečkin inequality.

W-distance

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In 1996, the Japanese mathematicians Kada, Suzuki and Takahashi introduced the new concept of w-distance on a metric space. They proved a new non convex minimization theorem, and by using this theorem, the authors obtained some generalizations of Caristi's fixed point theorem and Ekeland's ϵ-variationalprinciple. As applications, some fixed point theorems due to Ćirić, Kannan and Subrahmanyam were also deduced. Fundamental results in these directions are the papers of Suzuki and Takahashi. There exists a large literature devoted to this subject.

Further results on (b, c)-inverses

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In this talk we present some new results on (b, c)-inverses in rings with a unity. We show a connection between the unity of a ring and the (b, c)-inverses of elements in that ring. Also, a number of properties that hold for some other generalized inverses are extended to (b, c)-inverses.

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Application of multivalued fixed points and fixed figures

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We define some contractions to establish a multivalued fixed point, fixed ellipse, and fixed hyperbola. Further, we discuss some results which contribute to the study of the existence of multivalued fixed points, their geometry, and an application to a boundary value problem with differential inclusion significantly. This work is motivated by the geometry of set-valued fixed points performing a remarkable role in real-world problems and is fascinating and innovative.

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On reverse and forward order laws for the (b, c)-inverse in rings

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The (b, c)-inverse unifies some well known generalized inverses, such as the Moore-Penrose inverse, the Drazin inverse, the core inverse, the inverse along an element, etc. In this talk, the reverse order law for the (b, c)-inverse, in a ring with a unity, is discussed. Also, we introduce the forward order law for the (b, c)-inverse in a ring with a unity. We derive equivalent conditions for these rules to hold and we analyze these rules for different choices of b and c.

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SESSION 2 - APPROXIMATIONS AND APPLICATIONS – ABSTRACTS

Presentation of characteristic models for defining weight coefficients of criteria in multi-criteria analysis

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The paper presents some new procedures for determining weighting coefficients of criteria and presents a comparison with some previously developed techniques. The paper shows the advantages and disadvantages of the observed techniques for determining the weighting coefficients of the criteria and presents recommendations for the successful and efficient application of characteristic models.

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On reverse triangle inequality and some its applications

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In Euclidean space the triangle inequality asserts that the sum of any two sides of a triangle is strictly bigger than the remaining third side. The reverse triangle inequality is its equivalent which say that any side of a triangle is greater than to the difference between the other two sides.

In first part of this talk we shall proved equivalence of triangle inequality and reverse triangle inequality on class of pseudo - metric spaces. In particular we shall presented the following pseudo - metric spaces characterization of Lindenbaum's type [1].

Proposition 1. Let X be non empty set and $d : X \times X \to [0, \infty)$, such that for any $x, y \in X$ $x = y$ implies $d(x, y) = 0$. Then the following statements are equivalent:

1) (X, d) is pseudo - metric space;

2) for any $x, y, z \in X$ holds $d(x, y) \leq d(x, z) + d(y, z)$;

3) for any $x, y, z \in X$ holds $|d(x, z) - d(y, z)| \leq d(x, y)$.

Further, as examles of applications of reverse triangle inequality, we shell proved one inequality of Serbian mathematician D. D. Adamović (see [2] page 281.) and some its generalizations. The main result of this section is:

Proposition 2. If sequences $(a_n), (b_n) \in L^p$ and $\|(b_n)\|_{L^p} = B$, then

$$\left| \left(\sum_{i=1}^{\infty} |a_n|^p \right)^{\frac{1}{p}} - \left(\sum_{i=1}^{\infty} |b_n|^p \right)^{\frac{1}{p}} \right| \leq \inf_{\|(c_n)\|_{L^p} = B} \sum_{i=1}^{\infty} |a_i - c_i|.$$

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An optimal quadrature formulas for numerical integration of Riemann-Liouville fractional integral

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In the present article, the problem of construction of the optimal quadrature formula is discussed for numerical integration of the right Riemann-Liouville integral in the Hilbert space $W_2^{(m,m-1)}[t,1]$ of real-valued functions. Initially, the norm of the error functional is found using the extremal function of the error functional of the quadrature formula. Since the error functional is defined on the Hilbert space, the quadrature formula that we are constructing is exact for zeros of this space, that is, we have the conditions that the influence of the error functional on these functions is equal to zero. Then, the Lagrange function is constructed to find the conditional extremum of the error functional. Thereby, a system of linear equations is obtained for the coefficients of the optimal quadrature formula. The existence and uniqueness of the solution of the obtained system are studied. This system of linear equations is solved by the Sobolev method. And the analytical form of the coefficients is obtained.

SRBerta - BERT transformer language model for Serbian legal texts

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The proofreading of formal language in official documents is a specific challenge that requires domain knowledge regarding grammatical, lexical, and orthographic, but also formal rules used within a domain specific language. A separate part of the previous problem refers to checking the correct use of formal language when writing legislative texts. Such tasks are delegated to specialists, and domain experts, whose daily work could be facilitated by the development of software tools for these specific purposes. The ultimate goal, which is also the biggest challenge, is for a machine to understand a language. For a machine to learn a particular language, it must understand, not only the words and rules used in a particular language, but also the context of sentences and the meaning that words take on in a particular context. In the experimental development that was carried out, the goal of the language model we developer - SRBerta, was to understand the formal language of Serbian legal documents. In 2018, the Google Research AI team presented the BERT (Bidirectional Encoder Representation from Transformers) artificial neural network architecture [1], setting 2 goals: masked language modeling and next sentence prediction. Starting from the BERT architecture, in 2019 the Facebook AI team presented RoBERTa (Robustly optimized BERT pretraining approach), a network optimized for the task of masked language modeling [2]. SRBerta was created on the basis of the RoBERTa architecture, whereby the training of the SRBerta network for the task of understanding the formal language of Serbian legislation was carried out in two phases. In the first phase, the OSCAR dataset was used to train the SRBerta network. OSCAR is a large set of open data created using linguistic classification over data from the Common Crawl corpus [3]. The dataset we used consisted of 645,747 texts. The evaluation of the SRBert network was performed using 10of which consists of 60,000 input sequences, i.e., small texts in the Serbian language. A random masking of 15first stage show that the SRBerta model converges around an accuracy value of 73increases to a value of 73.7Serbian whose words are masked in 15(token) hidden behind it in 73.7In the second phase, SRBerta was fine-tuned using a larger number of available legal texts. This data was gathered from the Legal Information System of the Republic of Serbia. These legislative texts, each of which is between 12 and 15 MB of data, had to be prepared, that is, preprocessed, in a slightly more specific way, with the aim of generating as many input sequences as possible. At the end of the preprocessing process and after the creation of input sequences (tensors), created in a

Similar way as in the process of initial training of the network over the Serbian language, masked input sequences for training were created, with a total size of 10,266. Four fine-tuning epochs were performed, with the best-measured value for the accuracy metric being 84.8task of masked language modeling of legislative texts and thus proved the feasibility of creating such a tool based on the previously defined principles of natural language processing. Based on all of the above, the development and testing conclude that it is possible to achieve a high level of accuracy (industrially acceptable of over 90having a sufficiently high-quality and large set of data and an appropriate physical architecture of the system on which we perform the training process.

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Boubaker collocation method to solve the single degree of freedom system

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In many mechanical vibration situations, a complex system can be idealized as a single degree of freedom spring-mass system [1]. This research offers a new and straightforward method for single degree of freedom system in terms of Boubaker polynomials. On the basis of this method, the solution of the problem is approximated by truncated Boubaker series [2]. The assumed solution and its derivatives are written in the matrix form and then they are substituted in the equation. By utilizing the collocation points, the equation is transformed into a system of linear algebraic equations. This system is expressed in the matrix form. Finally, a new system is obtained by using this last system and the conditions. The solution of this system determines coefficients of the assumed solution. The effectiveness of the method is mentioned as a result of comparing the obtained results with the exact solution.

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Fuzzy and rough approximations

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The world we live in is pervaded with uncertainty and imprecision, which are incorporated into every information system that attempts to provide a complete and accurate model of the real world. Until recently, almost all aspects of imperfect data were modeled by probability theory but in the last decades, many new models have been developed to represent uncertain, imprecise or incomplete data, and provide reasoning that is approximate rather than precise. Fuzzy sets and rough sets take a prominent place among those models. The rapid development of these two approaches provided a basis for soft computing, an area of computing that, along with fuzzy sets and rough sets, also includes neural networks, probabilistic reasoning, belief networks, machine learning, evolutionary computing, and chaos theory.

Fuzzy sets and rough sets are complementary generalizations of classical sets. Capability of fuzzy sets to cope with uncertainty is based on the concepts of graded truth and graded set membership, while rough sets are a formal approximation of conventional sets in terms of a pair of sets which give the lower and the upper approximation of the original set. In the standard version of rough set theory, the lower- and upper-approximation sets are conventional sets, but nowadays a specific combination of rough and fuzzy approaches is increasingly used, where the approximating sets are fuzzy sets. Our research is based on just such an approach.

The basic characteristic of our approach to fuzzy rough approximations is dealing with fuzzy approximation spaces defined by a family of fuzzy relations, rather than by a single fuzzy relation, which is common with other authors. This is very important due to applications in various fuzzy (multi)relational systems such as fuzzy transition systems, fuzzy networks and others [13, 9, 15, 12]. We consider fuzzy rough approximation operators both on one and two universes of discourse, and we relate them with particular systems of fuzzy relation equations and inequations with one and two unknown fuzzy sets. We describe various algebraic properties of the solution sets of these systems, and expanding the methodology developed in our previous papers dealing with fuzzy relation equations and inequations [2, 3, 4, 5, 11] we provide efficient algorithms for computing the greatest and the least solutions.

This research was supported by the Science Fund of the Republic of Serbia, Grant no 7750185, Quantitative Automata Models: Fundamental Problems and Applications - QUAM.

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Linguistic representation models

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Fuzzy logic presents many potential applications for modelling and simulation. In particular, this paper analyses fuzzy linguistic representation model with one of the most popular fuzzy logic techniques for computing with words [4, 2]. We will formalize the way people make conclusions on basis of linguistic description which is a set of fuzzy *if–then* rules understood as natural language expressions [1]. A model presented here is based on automated method given in [3].

The research of the first author is supported by the Science Fund of the Republic of Serbia, Grant no 7750185, Quantitative Automata Models: Fundamental Problems and Applications - QUAM.

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Parameter estimation of the combined $SDLINAR$ model of order p and further sideways

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The combined $SDLINAR$ model of order p is a model of time series that is suitable for modeling phenomena whose values change over time through integer increments or decrements, and which, with certain probabilities, can be the result of the near or distant past. Under the spotlight of this work are results of parameter estimations of such models, as well as the problems that have arisen and possible directions for solving those problems.

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On perfectness of systems of weights satisfying Pearson's equation with nonstandard parameters

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Measures generating classical orthogonal polynomials are determined by Pearson's equation, whose parameters usually provide the positivity of the measures. The case of general complex parameters (nonstandard) is also of interest; the non-Hermitian orthogonality with respect to (now complex-valued) measures is considered on curves in $\mathbb{C}$.

Some applications lead to multiple orthogonality with respect to a number of such measures. In this talk, we will introduce a unified approach allowing to prove the perfectness of the systems of complex measures satisfying Pearson's equation with nonstandard parameters. For a system of r orthogonality measures, the perfectness is an important property implying, in particular, the uniqueness for the whole family of corresponding multiple orthogonal polynomials, and for their $(r+2)$-term recurrence relations.

We will also consider the polynomials satisfying multiple orthogonality relations with respect to a system of classical discrete (Charlier, Krawtchouk, Meixner or Hahn) measures. The corresponding measures solve the difference counterpart of Pearson's equation. Using the same approach, we verify the perfectness of such systems for general parameters. For some values of the parameters, discrete measures should be replaced with the related continuous measures with non-real supports.

The work is supported by Moscow Center of Fundamental and Applied Mathematics, agreement with Ministry of Science and Higher Education of Russia No. 075-15-2022-283.

On optimal quadrature formulas for approximation of Fourier integrals and their application to CT image reconstruction

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This paper is devoted to construction of optimal quadrature formulas in the Hilbert space $\widetilde{W_2}^{(m,m-1)}$ of complex-valued, periodic functions for the numerical calculation of the integral $\int_0^1 e^{2\pi i\omega x}\varphi(x)dx$ for $\omega \in \mathbb{Z}$. In the cases $m = 1$ and $m = 2$, the exponentially weighted integrals of some functions at the values of some N and ω are approximated using the constructed optimal quadrature formulas, and it is shown in numerical results that the orders of convergence of this formulas are $O(h + |\omega|^{-1})$ and $O((h + |\omega|^{-1})^2)$, respectively. Also, in the space $\widetilde{W_2}^{(m,m-1)}$, the sharp upper bound of the error for the optimal quadrature formulas is obtained, and it is shown analytically that the order of convergence of the optimal quadrature formula is $O((h+|\omega|^{-1})^m)$. Furthermore, in the case $\omega \in \mathbb{R}$, effective quadrature formulas for the approximate calculation of Fourier integrals are obtained and they are used in the reconstruction of CT images.

Idempotent-aided factorizations of matrices over a field

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A factorization of a given matrix is its representation in the form of a product of two or more matrices and, as such, entails data analysis. Although matrix factorization typically gives a more compact representation than learning the full matrix, it is a simple embedding model. In the language of computer science, the expression of a matrix D as a product amounts to pre-processing of the data in D and organizing that data into two or more parts structured to be of better use in whichever way necessary, and more accessible for computation. There are many different matrix (factorisations) decompositions over fields such as rank factorization, LU factorization, QR factorisation, Cholesky factorization, singular value decomposition, spectral factorisation etc. and each finds use among a particular class of problems, such as SVD-like machine learning model, mathematical problems in social network analysis and real-world recommendation systems, airflow problems and so on... The concept of an idempotent-aided factorization, presented here, can be viewed as a semigroup theoretical generalization of the full rank factorization of matrices (cf. [1, 13]). Namely, Green's $\mathcal{D}$-classes of the semigroup of matrices, consist of all matrices of the same rank. If D is considered to be a matrix over a field, a factorization of the matrix D with respect to the idempotent matrix E (with the same rank as D) represents the decomposition of the matrix D into the product $D = UV$ of the matrices U and V such that U has the same null space (kernel) with E and the same range (image, column space) as D, while V has the same range as E and the same null space as D. This factorization is a full-rank factorization of D and, moreover, U is a left invertible and V is a right invertible matrix. Several effective algorithms related to idempotent-aided factorizations of matrices have been provided and the correctness of those algorithms has been proven.

This research was supported by the Science Fund of the Republic of Serbia, Grant no 7750185, Quantitative Automata Models: Fundamental Problems and Applications - QUAM.

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Comparison of the tail thickness estimators under GARCH using the extremal exchange rate changes

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This research study aims to compare and evaluate the performance of Hill's tail index estimator and the estimator proposed in Bacro and Brito (1993). The analysis was performed on GARCH (1,1) data which are widely used for modelling processes with time varying volatility. These include financial time series, which can be particularly heavy tailed. The tail index is a key parameter for quantifying the extreme tail behavior of financial time series, which is crucial for the risk management and decision-making. The work is the empirical continuation of Ilic et al. (2019) and it tracks the behavior of the tail index estimators in the simulated GARCH sample and also in the case of the GBP/CAD exchange rate between 1st May 2007 and 18th October 2010. The accuracy and precision of the estimators are also compared in the case when certain portion of the sample is missing. The results highlight the strengths and limitations of each estimator and thus provide the possibility of certain improvement of the risk assessment and decision-making processes in various financial applications.

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Error estimates for Gaussian quadrature formulae

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We studied the error bound of Gaussian quadrature for analytic functions. The basic idea is to express the remainder of Gaussian quadrature as a contour integral, then the error bound is reduced to find the maximum of the kernel function:

$$(7) \quad K_n(z;\omega) = \frac{\varrho_n(z;\omega)}{\pi_n(z)}, \quad \varrho_n(z;\omega) = \int_{-1}^{1} \frac{\pi_n(t)}{z-t}\,dt, \quad z \in \mathbb{C} \setminus [-1,1].$$

The integral representation of the error term leads directly to the error bound

$$(8) \qquad |R_n(f)| \leq \frac{l(\Gamma)}{2\pi} \left(\max_{z\in\Gamma} |K_n(z)| \right) \left(\max_{z\in\Gamma} |f(z)| \right),$$

where $l(\Gamma)$ is the length of the choosen contour Γ.

A common choice for the contour Γ is one of the confocal ellipses with foci at the points ∓ 1, also known as the Bernstein ellipses, and the sum of semi-axes $\rho > 1$,

$$(9) \qquad \mathcal{E}_\rho = \left\{ z \in \mathbb{C} : \ z = \frac{1}{2}\left(u + u^{-1}\right), u = \rho e^{i\theta}, \ 0 \leq \theta < 2\pi \right\}.$$

For such Γ we studied the estimates (8) when w is one of the four generalized Chebyshev weight functions.

Optimal design of Kirchhoff-Love plates under the low-contrast assumption

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Homogenization theory is one of the most successful approaches for dealing with optimal design problems, that consists in arranging given materials such that obtained body satisfies some optimality criteria, which is mathematically usually expressed as minimization of some (integral) functional under some (PDE) constrains.

We consider optimal design problems in the setting of the Kirchhoff-Love equation describing an elastic, thin, symmetric plate, which is a fourth order elliptic equation, and we restrict ourselves to domains filled with two isotropic elastic materials. The optimization method is based on the small amplitude, or small contrast, approximation for homogenization. Since the classical solution usually does not exist, we use relaxation by the homogenization method in order to get a proper relaxation of the original problem.

Numerical results are presented for compliance minimization, i.e., maximizing the global stiffness of the plate, and for minimizing the integral of the square of the deflection in a subset of the plate.

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Summation of certain series containing the digamma function

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Series containing the digamma function arise when calculating the parametric derivatives of the hypergeometric functions and play a role in mathemtical physics. As these series are typically non-hypergeometric, a few instances when they are summable in terms of hypergeometric functions are of importance. In this talk, we will discuss some known and new examples of this type. In particular, we will present several identities that can be viewed as hypergeometric expressions for the 1-norm of the gradient of the generalized hypergeometric function with respect to all its parameters.

The talk is based on a joint work with Asena Çetinkaya.

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Landau and Grüss type inequalities for inner product type integral transformers in norm ideals

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For a probability measure μ and for square integrable fields $(\mathscr{A}_t)$ and $(\mathscr{B}_t)$ $(t \in \Omega)$ of commuting normal operators we prove Landau type inequality

$$\begin{aligned}&\left|\!\left|\!\left| \int_\Omega \mathscr{A}_t X \mathscr{B}_t d\mu(t) - \int_\Omega \mathscr{A}_t d\mu(t)\, X \int_\Omega \mathscr{B}_t d\mu(t) \right|\!\right|\!\right| \\ &\leqslant \left|\!\left|\!\left| \sqrt{\int_\Omega |\mathscr{A}_t|^2 d\mu(t) - \left|\int_\Omega \mathscr{A}_t d\mu(t)\right|^2}\, X \sqrt{\int_\Omega |\mathscr{B}_t|^2 d\mu(t) - \left|\int_\Omega \mathscr{B}_t d\mu(t)\right|^2} \right|\!\right|\!\right|\end{aligned}$$

for all $X \in B(\mathcal{H})$ and for all unitarily invariant norms $|\!|\!|\cdot|\!|\!|$.

For Schatten p-norms similar inequalities are given for arbitrary double square integrable fields, i.e. in that case normality and commutativity conditions can be dropped. Also, for all bounded self-adjoint fields satisfying $C \leqslant \mathscr{A}_t \leqslant D$ and $E \leqslant \mathscr{B}_t \leqslant F$ for all $t \in \Omega$ and some bounded self-adjoint operators C, D, E and F, and for all $X \in C_{|\!|\!|\cdot|\!|\!|}(\mathcal{H})$ we prove Grüss type inequality

$$\begin{aligned}&\left|\!\left|\!\left| \int_\Omega \mathscr{A}_t X \mathscr{B}_t d\mu(t) - \int_\Omega \mathscr{A}_t d\mu(t)\, X \int_\Omega \mathscr{B}_t d\mu(t) \right|\!\right|\!\right| \\ &\leqslant \frac{\|D-C\| \cdot \|F-E\|}{4} \cdot |\!|\!|X|\!|\!|.\end{aligned}$$

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On the Christoffel–Darboux formula for multilevel interpolations

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Let $\vec{\mu} = (\mu_1, \dots, \mu_d)$ be a vector of positive Borel measures on $\mathbb{R}$. We denote by $\widehat{\mu}_j(z) := \int (z-x)^{-1}\, d\mu_j(x)$ their Cauchy transforms. We start from the type I and type II Hermite–Padé interpolation problems. For an arbitrary multi-index $\vec{n} \in \mathbb{N}^d$ the problem is to find polynomials $q_{\vec{n},0}, q_{\vec{n},1}, \dots, q_{\vec{n},d}$ and $p_{\vec{n}}, p_{\vec{n},1}, \dots, p_{\vec{n},d}$ with $\deg p_{\vec{n}} = |\vec{n}| := n_1 + \cdots + n_d$, such that the following interpolation conditions are satisfied for $j = 1, \dots, d$ and $z \to \infty$:

$$q_{\vec{n}} := q_{\vec{n},0} + \sum_{k=1}^{d} q_{\vec{n},k}\widehat{\mu}_k = z^{-|\vec{n}|}(1 + o(1)), \qquad \deg q_{\vec{n},j} < n_j, \tag{10}$$

$$p_{\vec{n}} = z^{|\vec{n}|}(1 + o(1)), \qquad r_{\vec{n},j} := p_{\vec{n}}\widehat{\mu}_j + p_{\vec{n},j} = O(z^{-n_j-1}). \tag{11}$$

If for each $\vec{n} \in \mathbb{N}^d$ the solution of this problem exists and is unique, then the system of measures $\vec{\mu}$ is called *perfect*. We have

$$\alpha_{\vec{n},j} := \lim_{z\to\infty} z^{n_j+1} r_{\vec{n},j}(z) \neq 0$$

for perfect systems. In this case let us define the following function:

$$F_{\vec{n}}(x, y) := p_{\vec{n}}(x)q_{\vec{n}}(y) - \sum_{j=1}^{d} a_{\vec{n},j} p_{\vec{n}-\vec{e}_j}(x) q_{\vec{n}+\vec{e}_j}(y), \tag{12}$$

where $a_{\vec{n},j} := \alpha_{\vec{n},j}/\alpha_{\vec{n}-\vec{e}_j,j}$ and $E := \{\vec{e}_1, \dots, \vec{e}_d\}$ is the standard basis in $\mathbb{R}^d$.

The Christoffel–Darboux formula for Hermite–Padé interpolations was obtained in [1]. Here we reformulate it in the following way. For two multi-indices $\vec{n}_0, \vec{n} \in \mathbb{N}^d$ we consider a *path* $\{\vec{n}_m\}_{m=0}^{M} \subset \mathbb{N}^d$ connecting $\vec{n}_0$ to $\vec{n}$, that is

$$\vec{n}_M = \vec{n}, \qquad \vec{n}_{m+1} - \vec{n}_m \in \pm E, \quad m = 0, \dots, M-1.$$

Let

$$p_m := \begin{cases} p_{\vec{n}_m}, & \text{if } \vec{n}_{m+1} - \vec{n}_m \in E, \\ -p_{\vec{n}_{m+1}}, & \text{if } \vec{n}_{m+1} - \vec{n}_m \in -E, \end{cases}$$

$$q_m := \begin{cases} q_{\vec{n}_{m+1}}, & \text{if } \vec{n}_{m+1} - \vec{n}_m \in E, \\ q_{\vec{n}_m}, & \text{if } \vec{n}_{m+1} - \vec{n}_m \in -E. \end{cases}$$

Then we have the identity:

$$(x-y)\sum_{m=0}^{M-1} p_m(x)q_m(y) = F_{\vec{n}}(x,y) - F_{\vec{n}_0}(x,y). \tag{13}$$

The right-hand side does not depend on the path but only on its ends. In particular, it is equal to zero for a closed path, then $\vec{n}_0 = \vec{n}$.

Let us consider one important class of perfect systems, namely the Nikishin systems. The Nikishin system [2] is based on a set of generating measures $(\sigma_1, \ldots, \sigma_d)$ supported on segments $\operatorname{supp} \sigma_j \subset \Delta_j$, $\Delta_j \cap \Delta_{j+1} = \emptyset$. More specifically, we put $s_{j,j} := \sigma_j$, and then by induction on $|k-j|$ we define $ds_{j,k} := \widehat{s}_{j+1,k}\, d\sigma_j$ for $k > j$ and $ds_{j,k} := \widehat{s}_{j-1,k}\, d\sigma_j$ for $k < j$. The vector of measures $(s_{1,1}, \ldots, s_{1,d})$ is perfect, see [3].

Now we move to the multilevel interpolation problem for the Nikishin system [4]: given $\vec{n} \in \mathbb{N}^d$ find polynomials $q_{\vec{n},0}, q_{\vec{n},1}, \ldots, q_{\vec{n},d}$ and $p_{\vec{n},0}, p_{\vec{n},1}, \ldots, p_{\vec{n},d}$ such that for $j = 1, \ldots, d$ and $z \to \infty$ the following interpolation conditions hold

$$q_{\vec{n}} := q_{\vec{n},0} + \sum_{k=1}^{d} \widehat{s}_{1,k} q_{\vec{n},k} = z^{-|\vec{n}|}(1 + o(1)), \tag{14}$$

$$q_{\vec{n},j} + \sum_{k=j+1}^{d} \widehat{s}_{j+1,k} q_{\vec{n},k} = O(z^{n_j - 1}),$$

$$p_{\vec{n}} := p_{\vec{n},0} = z^{|\vec{n}|}(1 + o(1)), \tag{15}$$

$$\sum_{k=1}^{j} p_{\vec{n},k-1}\widehat{s}_{j,k} + p_{\vec{n},j} = O(z^{-n_j - 1}).$$

For each $\vec{n} \in \mathbb{N}^d$ the solution of this problem exists and is unique [5]. The solution also satisfies [6] the Christoffel–Darboux formula (13). The proof based on recurrent relations is similar to [7]. We will discuss some applications of this result during the talk. The particular case $\vec{n} = n\vec{e}_d$ with $d = 2$ corresponds to the biorthogonal Cauchy polynomials, see [8, 9].

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Hermite interpolation at Pollaczek zeros

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The Authors consider an Hermite–type interpolation process and show its convergence in weighted function spaces, under proper necessary and sufficient conditions on the weights.

The main difficulty is that the Pollaczek zeros have not arcsin distribution. The proof is reduced to the behavior of a Lagrange polynomial.

Computing ε-weak bisimulations for fuzzy automata over truncated product structures: algorithms and complexity analysis

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Weak bisimulations have been widely studied as a generalization of bisimulations for fuzzy automata (FAs). These bisimulations maintain language equivalence and perform better in the state reduction of FAs. However, a significant limitation of weak bisimulations is their inability to be computed for all $(\vee, \cdot)$-FAs, where $\vee$ represents the maximum operation and $\cdot$ denotes the product t-norm on the real-unit interval $[0, 1]$. The reason is that weak bisimulations are solutions to specific linear systems of fuzzy relation inequalities, and such systems can consist of infinitely many inequalities when observed under such FAs.

We propose a solution to this problem by introducing a new type of weak bisimulation, named ε-weak bisimulation. By choosing a small value $\varepsilon > 0$, we define ε-weak bisimulations, which yield finite systems of fuzzy relation inequalities. These ε-weak bisimulations preserve a new form of approximation for language equivalence. Specifically, we demonstrate that two $(\vee, \cdot)$-FAs that are ε-weak bisimilar recognize each word with degrees that are either equal or both less than or equal to ε.

Furthermore, we present two algorithms developed for computing the ε-weak bisimulation. The first algorithm computes the ε-weak bisimulation for a single automaton, while the second algorithm extends the computation to two automata. We discuss the time complexity of these algorithms and analyze the complexity of storing the resulting bisimulations.

As ε-weak bisimulations possess the property of approximating equivalence for arbitrarily small values of $\varepsilon > 0$, they effectively model an "almost-equivalence" between two FAs. This means that words accepted with degrees smaller than or equal to ε can be considered irrelevant. Our proposed ε-weak bisimulations offer a practical approach to address the challenge of computing bisimulations for $(\vee, \cdot)$-FAs, expanding the scope of bisimulation techniques in the field of fuzzy automata. The presented algorithms provide efficient methods for computing the ε-weak bisimulation, and the analysis of time complexity and storage requirements further contribute to the understanding and applicability of these techniques in practice.

This research was supported by the Science Fund of the Republic of Serbia, Grant no 7750185, Quantitative Automata Models: Fundamental Problems and Applications - QUAM.

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On a linear combination of topological indices of graphs

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Let $G=(V,E)$, $V=\{v_1,v_2,\dots,v_n\}$, be a simple connected graph with n vertices, m edges and vertex–degree sequence $\Delta=d_1\geq d_2\geq \cdots\geq d_n=\delta>0$, $d_i=d(v_i)$. If vertices v_i and v_j are adjacent in G, we write $i\sim j$.

In graph theory, a graph invariant is property of the graph that is preserved by isomorphisms. Topological indices are special kinds of numerical graph invariants. The first and the second Zagreb indices, $M_1(G)$ and $M_2(G)$, are graph invariants defined in terms of vertex degrees as [4, 5]

$$M_1(G)=\sum_{i=1}^{n} d_i^2=\sum_{i\sim j}(d_i+d_j) \quad \text{and} \quad M_2(G)=\sum_{i\sim j} d_i d_j .$$

The linear combinations of these indices were considered in [1, 2, 6]. In [1] it is proven that

$$M_2(G)-M_1(G)\geq 11m-12n,$$

in [2]

$$M_1(G)+2M_2(G)\leq 4m^2,$$

and in [6]

$$\delta M_1(G)-M_2(G)\leq m\delta^2,$$

$$\Delta M_1(G)-M_2(G)\leq m\Delta^2,$$

$$\Delta M_1(G)-M_2(G)\geq m\Delta\delta.$$

Inspired by these results, here we determine bounds for linear combinations of harmonic index, $H(G)$, and the inverse sum indeg index, $ISI(G)$, which are defined as [3, 7]:

$$H(G)=\sum_{i\sim j}\frac{2}{d_i+d_j} \quad \text{and} \quad ISI(G)=\sum_{i\sim j}\frac{d_i d_j}{d_i+d_j} .$$

Here we prove the following inequalities

$$ISI(G) + \frac{\delta^2}{2} H(G) \geq m\delta,$$

$$ISI(G) + \frac{\Delta^2}{2} H(G) \geq m\Delta,$$

$$M_1(G) - 2ISI(G) + \Delta\delta H(G) \leq m(\Delta + \delta).$$

Let us note that obtained bounds are sharp since there are many classes of graphs for which equalities are attained.

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Partitions and automorphisms of finite Kurepa trees

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Đuro Kurepa introduced in [4] a finite partially ordered set $\mathrm{T}_n = (T_n, \leq, 0)$, a rooted tree of height n with root 0 in which each node of height $k < n$ splits into $k+1$ successors. This tree is related to the Kurepa left factorial hypothesis KHp which states that for no prime $p > 2$, $p \mid\, !p$, where $!n \equiv |T_{n-1}| = 0! + 1! + \cdots + (n-1)!$, $n \geq 1$. Kurepa defined $!n$ in [3] and posed there KHp. The status of KHp is not yet resolved, but it is numerically checked by many authors, with the last achievement for $p \leq 2^{40}$, see [1]. History of the problem and related results are presented in [2] and [5]. We consider here KHp combinatorially studying partitions of T_{n-1} into n components having the same size. We proved that there are no such partitions $\mathcal{P}$ if all components in $\mathcal{P}$ are connected subsets of T_{n-1}, or there is a component invariant under all automorphisms of T_{n-1}. Along the way, we found that $\mathrm{Aut}(\mathrm{T}_n)$ is the wreath product of factorial powers of permutation groups S_k, $k \leq n$. We also found an approximation of Kurepa function $K(z)$, an extension of $!n$ to complex domain.

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Anti-Gaussian quadrature rules related to orthogonality on the semicircle

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Let Γ be a unit semicircle $\Gamma = \{z = \mathrm{e}^{\mathrm{i}\theta} : 0 \leq \theta \leq \pi\}$. Orthogonal polynomials on the semicircle with respect to the complex-valued inner product

$$\langle f, g \rangle = \int_{\Gamma} f(z) g(z) (\mathrm{i}z)^{-1} \mathrm{d}z = \int_{0}^{\pi} f(\mathrm{e}^{\mathrm{i}\theta}) g(\mathrm{e}^{\mathrm{i}\theta}) \mathrm{d}\theta$$

was introduced by Gautschi and Milovanović in [1], were the certain basic properties were proved. Such orthogonality as well as the applications involving Gauss-Christoffel quadrature rules were further studied in [2] and [4]. In this article we introduce anti-Gaussian quadrature rules related to the orthogonality on the semicircle (see [1]) and present stable numerical method for their construction. Also, some numerical examples are included.

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Numerical analysis of hyperbolic transmission problem on disjoint intervals

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In this paper we investigate initial boundary value problems for one dimensional hyperbolic equations in disjoint domains. As a model example we take an area consisting of two non-adjacent intervals. In each interval an Robin's initial-boundary value problem is given. The interaction between their solutions is described using nonlocal integral conjugation conditions Robin-Dirichlet type on the boundaries of the observed subareas. For the model problem the existence and uniqueness of its weak solution in appropriate Sobolev-like space is proved. A finite difference scheme approximating this problem is proposed and analyzed. An estimate of the convergence rate, compatible with the smoothness of the input data is obtained.

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On internality of generalized averaged Gaussian quadrature rules and their truncations

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Generalized (also called optimal) averaged Gauss quadrature formulas may yield higher accuracy than Gauss quadrature rules that use the same moment information. This is illustrated in [3]. They therefore may be attractive to use when moments or modified moments are cumbersome to evaluate. However, generalized averaged Gauss quadrature formulas may be not internal, i. e., they may have nodes outside the convex hull of the support of the measure that defines the associated Gauss rules; see, e.g., [2, 4, 1] for examples and analyses. It may therefore not be possible to use generalized averaged Gauss quadrature formulas with integrands that only are defined on the convex hull of the support of the measure. A survey of our results on internality of generalized averaged Gaussian quadrature rules and their truncations will be presented.

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Integer-valued autoregressive process in random environment

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An r-states random environment integer-valued autoregressive process is presented. This RrINAR process is based on a random environment process, defined as a Markov chain, which by taking its different values represents a selection mechanism of process marginals from the family of differently parameterized geometric distributions. This implies the non-stationarity of the finally introduced RrNGINAR model based on the negative binomial thinning. Here we present some essential properties of this process focusing on estimation procedures of model parameters. Also, the model motivation and interpretation is given by its application to specific real-life counting data, where it is compared to some other possible and competitive time series models.

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Repair time threshold for maintenance activities planning

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This work introduces a novel approach for estimating the repair rate of an observed system comprised of multiple components. The presented approach is based on the calculation of the probability density function (PDF) of a systemвЃЄs repair time by observing the probability that repair rates of its components surpass a determined threshold. Based on the obtained method, it can be concluded at what interval maintenance, repair, or replacement should be performed in order to achieve the desired level of availability. The method can be further used for planning maintenance activities and a dynamic forecast of system characteristics.

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Gradient neural dynamics for solving system of matrix equations and their applications

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We consider the gradient neural network models for solving several different systems of linear matrix equations in areal time. The convergence properties are investigated in details using Lyapunov method and it is shown that models are globally convergent to the general solution, determined by the initial values choice. A few applications are given, including the computation of matrix generalized inverses. Several numerical examples are shown to illustrate the theoretical results.

The set of anti-Gaussian quadrature rules and corresponding multiple orthogonal polynomials

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Laurie in ([1]) introduced anti-Gaussian quadrature rule, that gives an error equal in magnitude but of opposite sign to that of the corresponding Gaussian quadrature rule. Here, we consider a set of anti-Gaussian quadrature rules for the optimal set of quadrature rules in Borges' sense, with respect to the set of r different weight functions, as well as the corresponding class of multiple orthogonal polynomials. Also, we define the set of averaged quadrature formulas and give some numerical examples.

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Gauss-like quadrature through manipulations with Jacobi matrices

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We discuss a construction of several types of Gauss and *rational* Gauss quadrature rules through manipulations with Jacobi or Jacobi-like matrices. We pay particular attention to the rational anti-Gauss, simplified Gauss and Gauss-Radau rules. The interest in these rules stems from the need to approximate the matrix functionals of the form $v^T f(A)v$ arising in many applications, where v is a vector, A is a large symmetric positive matrix, and f is a function defined on the spectrum of A. Although a distribution function in the functional $v^T f(A)v$ is not explicitly known, we can compute the corresponding Jacobi or Jacobi-like matrix (and therefore the Gauss or rational Gauss quadrature for the mentioned functional) by several steps of Lanczos or rational Lanczos method with input v and A. By modifying such a matrix we can compute some other quadrature that enables us to estimate the error in the Gauss or rational Gauss quadrature.

This is a joint work with Lothar Reichel and Jihan Alahmadi.

Computation of the complex zeros of the parabolic cylinder function $U(a, w)$

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We provide a Matlab algorithm to compute the complex zeros of the parabolic cylinder function $U(a, w)$. Through the combination of a fourth order root finding method described in [1] and the numerical methods studied in [2] together with a Liouville-Green expansion from [3] to evaluate the parabolic cylinder function, we find the roots of the function for a large region of its parameters.

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Optimization of composite cubature formulas on a lattice

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Quadrature and cubature formulas are widely used in various branches of mathematics and its applications. When obtaining a discrete approximation, i.e. cubature formula, it is important that the cubature formula approximates the given definite integrals as best as possible. Such cubature formulas can be obtained, for example, using variational principles. Therefore, when constructing lattice optimal cubature formulas in the Sobolev space by the variational method, this is one of the urgent problems of computational mathematics.

In the present paper, composite lattice optimal cubature formulas are constructed by the variational method in the Sobolev space. In addition, the square of the norm of the error functional of the constructed lattice optimal cubature formulas in the dual Sobolev space is explicitly calculated.

Formulas for sum of powers of consecutive integers derive from certain family of finite sums involving higher powers of (inverse)binomial coefficients

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One of the main purposes of this presentation, which contains very interesting and applicable new results and will also be converted into an article, is to surval generating functions in terms of hypergeometric function and logarithm function for finite sums involving higher powers of inverse binomial coefficients, denoted by $\mathcal{S}_v(n;\lambda,p)$ (cf. for detal see [1]). $\mathcal{S}_v(n;\lambda,p)$ is a polynomial with respect to λ. Its degree is n. The second main purposes of this presentation is to give some applications of the polynomials $\mathcal{S}_v(n;\lambda,p)$. By using these applications, we derive many formulas for sums of powers of consecutive integers involving the Bernoulli and Euler-type numbers and polynomials, the Stirling numbers, the (alternating) harmonic numbers, the Apostol-type numbers and polynomials. In addition, we also give some remarks and observations on these results.

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Numerical approximation of the third initial-boundary-value parabolic problem

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In this paper a finite difference scheme approximating the third initial-boundary-value problem for the parabolic equation with time-dependent coefficients is derived. In a special $W_2^{1,\,1/2}$ Sobolev norm ,a convergence rate estimate, compatible with the smoothness of the solution of the problem, is obtained.

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Compound step-size methods in unconstrained nonlinear optimization

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Various improvements of gradient-descent and conjugate-gradient methods for solving nonlinear the unconstrained optimization problem are investigated. More specifically, we investigate possibility of using composite step size in gradient-descent and conjugate-gradient algorithms. The composite step size is generated as a function of different parameters. The first modification suggests a slight increase in the step size involved in gradient-descent methods. Another class of methods utilizes additional parameters which are defined using appropriate scalar approximations of the Hessian. One possibility to define additional tuning in optimization methods occurs by hybridizing gradient descent methods with the Picard-Mann-Ishikawa iterative process. An additional approach to defining acceleration parameters is based on the application of the neutrosophic logic and three membership functions in determining appropriate step size for a class of descent direction methods. It is proved that the proposed methods are linearly convergent for uniformly convex functions as well as under some standard conditions. Numerical tests and comparisons are presented.

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Gauss-type quadrature rules for variable-sign weight functions

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When the Gauss quadrature formula G_n is applied, it is often assumed that the weight function (or the measure) is non-negative on the integration interval $[a, b]$. In the present paper, we introduce a Gauss-type quadrature formula Q_n for weight functions that change the sign in the interior of $[a, b]$. Construction of Q_n is based on the idea to transform the given integral into a sum of one integral which doesn't cause a quadrature error and the other integral with a property that the points from the interior of $[a, b]$ at which the weight function changes sign are the zeros of its integrand. It proves that all nodes of Q_n are pairwise distinct and contained in the interior of $[a, b]$. Moreover, G_n (with a non-negative weight function) turns out to be a special case of Q_n. Obtained results on the remainder term of Q_n suggest that the application of Q_n makes sense both when the points from the interior of $[a, b]$ at which the weight function changes sign are known exactly, as well as when those points are known approximately. The accuracy of Q_n is confirmed by numerical examples.

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Optimal set of quadrature rules for trigonometric polynomials with preassigned nodes

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In the case when we have numerically evaluating of a set of definite integrals taken with respect to distinct weight functions, but related to a common integrand and interval of integration it is not efficient to use a set of Gauss–Christoffel quadrature rules, because valuable information is wasted. Borges has introduced the optimal set of quadrature rules for algebraic polynomials [1].

Quadrature rules of Gaussian type for trigonometric polynomials are generalization of the classical Gaussian quadrature rules for algebraic polynomials. We introduced multiple orthogonal trigonometric polynomials of semi-integer degree and the corresponding the optimal set of quadrature rules for trigonometric polynomials [2, 3]. Also, we investigated such quadrature rules with preassigned nodes.

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Binomial orthogonal polynomials

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Given a symmetric positive measure σ on an interval $[-a, a]$, $0 < a \leq +\infty$, one can construct two one-parametric families of orthogonal polynomials by real and pure imaginary one-dimensional perturbations of the tridiagonal matrix corresponding to the measure σ. In the present talk, we consider an example of such perturbations of a given finite discrete measure.

Namely, we construct an interesting example of a discrete finite positive strongly related to both Chebyshev and discrete Chebyshev polynomials. We find the corresponding moments, Hankel minors, and describe the measure. We also construct the pure imaginary one-dimensional perturbation of the tridiagonal matrix corresponding to the considered measure. This gives us an (explicit) example of a of non-positive moment functional $\mathcal{L}_N$ on the real line. The limit cases connected with Bessel polynomials are also discussed.

Some modifications of the Chebyshev measure and the corresponding orthogonal polynomials

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Repeated modifications for distinct and the same linear divisors have been studied by Gautschi in [1] and applied to generate special Gaussian rules for dealing with nearby poles. Among interesting examples, he considered the Szegő-Bernstein measure

$$\mathrm{d}\mu_m(t) = \frac{1}{(c_1^2 - t^2)(c_2^2 - t^2)\cdots(c_m^2 - t^2)}\frac{\mathrm{d}t}{\sqrt{1-t^2}}, \quad -1 < t < 1,$$

with $c_k = 1 + 1/k > 1$ for each $k = 1,\ldots,m$, and $m \leq 24$ (working in 52-digit arithmetic). This lecture is devoted to these problems by symbolic computation. In the case of identical quadratic divisors, c^2+t^2 ($c > 0$), i.e., $\mathrm{d}\mu_m(t) = \frac{1}{(c^2+t^2)^m}\frac{\mathrm{d}t}{\sqrt{1-t^2}}$ $(-1 < t < 1)$, the moments $\mu_k^{(m)} = \int_{-1}^{1} t^k \, \mathrm{d}\mu_m(t)$ $(k \geq 0)$ are $\mu_k^{(m)} = 0$ for odd k, and for even k they can be expressed in terms of the hypergeometric function

$$\mu_k^{(m)} = \pi\binom{k}{k/2}2^{2m-k+1}\frac{X^m}{(1+X)^{2m}}\,{}_2F_1\left(\frac{1}{2}, m; \frac{k}{2}+1; \frac{4X}{(1+X)^2}\right),$$

where $c = \sinh\varphi$ and $a = 2c^2+1 = \cosh 2\varphi = \frac{1}{2}(\mathrm{e}^{2\varphi} + \mathrm{e}^{-2\varphi}) = \frac{1}{2}\left(X + X^{-1}\right) > 1$, $X = \mathrm{e}^{-2\varphi}$. Here, evidently $0 < X < 1$.

For the coefficients $\beta_\nu^{(m)}$ in the recurrence relation for the corresponding orthogonal polynomials, we can obtain

$$\beta_\nu^{(m)} = \frac{1}{4}, \quad \nu \geq m+2, \quad \beta_{m+1}^{(m)} = \frac{1}{4}\left(1 + X^m\right),$$

$$\beta_m^{(m)} = \frac{1 - X^{2m} + mX^{m-1}(1-X^2)}{4(1+X^m)},$$

etc. For distinct quadratic divisors, $c_\nu^2 + t^2$ $(c_\nu > 0)$, $\nu = 1,\ldots,m$, the corresponding recurrence coefficients can be expressed in terms of symmetric functions of $X_\nu = \mathrm{e}^{-2\varphi_\nu}$, $\nu = 1,\ldots,m$.

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Barycentric interpolation based on equilibrium potential

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A novel barycentric interpolation algorithm with specific exponential convergence rate is designed for analytic functions defined on the complex plane, with singularities located near the interpolation region, where the region is compact and can be disconnected or multiconnected. The core of the method is the efficient computation of the interpolation nodes and poles using discrete distributions that approximate the equilibrium logarithmic potential, achieved by solving a Symm's integral equation. It takes different strategies to distribute the poles for isolated singularities and branch points, respectively. In particular, if poles are not considered, it derives a polynomial interpolation with exponential convergence. Numerical experiments illustrate the superior performance of the proposed method.

This work is partially supported by the NSF of China (No.12271528).

Two multiplicative methods of multicriteria analysis

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The paper presents two multiplicative methods in multicriteria analysis. It is characteristic of these methods that the evaluation of alternatives in multi-criteria models is based on the product of the characteristic values of the alternatives for each criterion. Both methods have an important feature that the introduction and evaluation of possible new alternatives in the multi-criteria model has no effect on the evaluations of previously evaluated alternatives. This means that there is no possibility of favoring one of the alternatives over the other alternatives, which is a characteristic of many methods of multicriteria analysis.

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