Some modifications of the Chebyshev measure and the corresponding orthogonal polynomials

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Repeated modifications for distinct and the same linear divisors have been studied by Gautschi in [1] and applied to generate special Gaussian rules for dealing with nearby poles. Among interesting examples, he considered the Szegő-Bernstein measure

$$\mathrm{d}\mu_m(t) = \frac{1}{(c_1^2 - t^2)(c_2^2 - t^2)\cdots(c_m^2 - t^2)}\frac{\mathrm{d}t}{\sqrt{1-t^2}}, \quad -1 < t < 1,$$

with $c_k = 1 + 1/k > 1$ for each $k = 1, \ldots, m$, and $m \le 24$ (working in 52-digit arithmetic). This lecture is devoted to these problems by symbolic computation. In the case of identical quadratic divisors, $c^2 + t^2$ $(c > 0)$, i.e., $\mathrm{d}\mu_m(t) = \frac{1}{(c^2+t^2)^m}\frac{\mathrm{d}t}{\sqrt{1-t^2}}$ $(-1 < t < 1)$, the moments $\mu_k^{(m)} = \int_{-1}^{1} t^k \,\mathrm{d}\mu_m(t)$ $(k \ge 0)$ are $\mu_k^{(m)} = 0$ for odd k, and for even k they can be expressed in terms of the hypergeometric function

$$\mu_k^{(m)} = \pi \binom{k}{k/2} 2^{2m-k+1} \frac{X^m}{(1+X)^{2m}} \, {}_2F_1\left(\frac{1}{2}, m; \frac{k}{2}+1; \frac{4X}{(1+X)^2}\right),$$

where $c = \sinh\varphi$ and $a = 2c^2 + 1 = \cosh 2\varphi = \frac{1}{2}(\mathrm{e}^{2\varphi} + \mathrm{e}^{-2\varphi}) = \frac{1}{2}\left(X + X^{-1}\right) > 1$, $X = \mathrm{e}^{-2\varphi}$. Here, evidently $0 < X < 1$.

For the coefficients $\beta_\nu^{(m)}$ in the recurrence relation for the corresponding orthogonal polynomials, we can obtain

$$\beta_\nu^{(m)} = \frac{1}{4}, \quad \nu \ge m+2, \quad \beta_{m+1}^{(m)} = \frac{1}{4}\left(1 + X^m\right), \quad \beta_m^{(m)} = \frac{1 - X^{2m} + mX^{m-1}(1 - X^2)}{4(1+X^m)},$$

etc. For distinct quadratic divisors, $c_\nu^2 + t^2$ $(c_\nu > 0)$, $\nu = 1, \ldots, m$, the corresponding recurrence coefficients can be expressed in terms of symmetric functions of $X_\nu = \mathrm{e}^{-2\varphi_\nu}$, $\nu = 1, \ldots, m$.

References

[1] W. Gautschi, Repeated modifications of orthogonal polynomials by linear divisors, Numer. Algorithms **63** (2013), 369–383.