On the Christoffel–Darboux formula for multilevel interpolations

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Let $\vec{\mu} = (\mu_1, \dots, \mu_d)$ be a vector of positive Borel measures on $\mathbb{R}$. We denote by $\widehat{\mu}_j(z) := \int (z-x)^{-1}\, d\mu_j(x)$ their Cauchy transforms. We start from the type I and type II Hermite–Padé interpolation problems. For an arbitrary multi-index $\vec{n} \in \mathbb{N}^d$ the problem is to find polynomials $q_{\vec{n},0}, q_{\vec{n},1}, \dots, q_{\vec{n},d}$ and $p_{\vec{n}}, p_{\vec{n},1}, \dots, p_{\vec{n},d}$ with $\deg p_{\vec{n}} = |\vec{n}| := n_1 + \dots + n_d$, such that the following interpolation conditions are satisfied for $j = 1, \dots, d$ and $z \to \infty$:

$$(1) \qquad q_{\vec{n}} := q_{\vec{n},0} + \sum_{k=1}^{d} q_{\vec{n},k} \widehat{\mu}_k = z^{-|\vec{n}|}(1 + o(1)), \qquad \deg q_{\vec{n},j} < n_j,$$

$$(2) \qquad p_{\vec{n}} = z^{|\vec{n}|}(1 + o(1)), \qquad r_{\vec{n},j} := p_{\vec{n}} \widehat{\mu}_j + p_{\vec{n},j} = O(z^{-n_j - 1}).$$

If for each $\vec{n} \in \mathbb{N}^d$ the solution of this problem exists and is unique, then the system of measures $\vec{\mu}$ is called *perfect*. We have $\alpha_{\vec{n},j} := \lim_{z\to\infty} z^{n_j+1} r_{\vec{n},j}(z) \neq 0$ for perfect systems. In this case let us define the following function:

$$(3) \qquad F_{\vec{n}}(x,y) := p_{\vec{n}}(x) q_{\vec{n}}(y) - \sum_{j=1}^{d} a_{\vec{n},j} p_{\vec{n}-\vec{e}_j}(x) q_{\vec{n}+\vec{e}_j}(y),$$

where $a_{\vec{n},j} := \alpha_{\vec{n},j} / \alpha_{\vec{n}-\vec{e}_j,j}$ and $E := \{\vec{e}_1, \dots, \vec{e}_d\}$ is the standard basis in $\mathbb{R}^d$.

The Christoffel–Darboux formula for Hermite–Padé interpolations was obtained in [1]. Here we reformulate it in the following way. For two multi-indices $\vec{n}_0, \vec{n} \in \mathbb{N}^d$ we consider a *path* $\{\vec{n}_m\}_{m=0}^{M} \subset \mathbb{N}^d$ connecting $\vec{n}_0$ to $\vec{n}$, that is

$$\vec{n}_M = \vec{n}, \qquad \vec{n}_{m+1} - \vec{n}_m \in \pm E, \quad m = 0, \dots, M-1.$$

Let

$$p_m := \begin{cases} p_{\vec{n}_m}, & \text{if } \vec{n}_{m+1} - \vec{n}_m \in E, \\ -p_{\vec{n}_{m+1}}, & \text{if } \vec{n}_{m+1} - \vec{n}_m \in -E, \end{cases} \qquad q_m := \begin{cases} q_{\vec{n}_{m+1}}, & \text{if } \vec{n}_{m+1} - \vec{n}_m \in E, \\ q_{\vec{n}_m}, & \text{if } \vec{n}_{m+1} - \vec{n}_m \in -E. \end{cases}$$

Then we have the identity:

$$(x-y)\sum_{m=0}^{M-1} p_m(x)q_m(y) = F_{\vec{n}}(x,y) - F_{\vec{n}_0}(x,y). \tag{4}$$

The right-hand side does not depend on the path but only on its ends. In particular, it is equal to zero for a closed path, then $\vec{n}_0 = \vec{n}$.

Let us consider one important class of perfect systems, namely the Nikishin systems. The Nikishin system [2] is based on a set of generating measures $(\sigma_1, \dots, \sigma_d)$ supported on segments $\operatorname{supp} \sigma_j \subset \Delta_j$, $\Delta_j \cap \Delta_{j+1} = \emptyset$. More specifically, we put $s_{j,j} := \sigma_j$, and then by induction on $|k-j|$ we define $ds_{j,k} := \widehat{s}_{j+1,k}\, d\sigma_j$ for $k > j$ and $ds_{j,k} := \widehat{s}_{j-1,k}\, d\sigma_j$ for $k < j$. The vector of measures $(s_{1,1}, \dots, s_{1,d})$ is perfect, see [3].

Now we move to the multilevel interpolation problem for the Nikishin system [4]: given $\vec{n} \in \mathbb{N}^d$ find polynomials $q_{\vec{n},0}, q_{\vec{n},1}, \dots, q_{\vec{n},d}$ and $p_{\vec{n},0}, p_{\vec{n},1}, \dots, p_{\vec{n},d}$ such that for $j = 1, \dots, d$ and $z \to \infty$ the following interpolation conditions hold

$$q_{\vec{n}} := q_{\vec{n},0} + \sum_{k=1}^{d} \widehat{s}_{1,k} q_{\vec{n},k} = z^{-|\vec{n}|}(1+o(1)), \quad q_{\vec{n},j} + \sum_{k=j+1}^{d} \widehat{s}_{j+1,k} q_{\vec{n},k} = O(z^{n_j - 1}), \tag{5}$$

$$p_{\vec{n}} := p_{\vec{n},0} = z^{|\vec{n}|}(1+o(1)), \quad \sum_{k=1}^{j} p_{\vec{n},k-1} \widehat{s}_{j,k} + p_{\vec{n},j} = O(z^{-n_j - 1}). \tag{6}$$

For each $\vec{n} \in \mathbb{N}^d$ the solution of this problem exists and is unique [5]. The solution also satisfies [6] the Christoffel–Darboux formula (4). The proof based on recurrent relations is similar to [7]. We will discuss some applications of this result during the talk. The particular case $\vec{n} = n\vec{e}_d$ with $d = 2$ corresponds to the biorthogonal Cauchy polynomials, see [8, 9].

References

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