On a linear combination of topological indices of graphs

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Let $G = (V, E)$, $V = \{v_1, v_2, \dots, v_n\}$, be a simple connected graph with n vertices, m edges and vertex–degree sequence $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$, $d_i = d(v_i)$. If vertices v_i and v_j are adjacent in G, we write $i \sim j$.

In graph theory, a graph invariant is property of the graph that is preserved by isomorphisms. Topological indices are special kinds of numerical graph invariants. The first and the second Zagreb indices, $M_1(G)$ and $M_2(G)$, are graph invariants defined in terms of vertex degrees as [4, 5]

$$M_1(G) = \sum_{i=1}^{n} d_i^2 = \sum_{i \sim j} (d_i + d_j) \quad \text{and} \quad M_2(G) = \sum_{i \sim j} d_i d_j .$$

The linear combinations of these indices were considered in [1, 2, 6]. In [1] it is proven that

$$M_2(G) - M_1(G) \geq 11m - 12n,$$

in [2]

$$M_1(G) + 2M_2(G) \leq 4m^2,$$

and in [6]

$$\delta M_1(G) - M_2(G) \leq m\delta^2,$$
$$\Delta M_1(G) - M_2(G) \leq m\Delta^2,$$
$$\Delta M_1(G) - M_2(G) \geq m\Delta\delta.$$

Inspired by these results, here we determine bounds for linear combinations of harmonic index, $H(G)$, and the inverse sum indeg index, $ISI(G)$, which are defined as [3, 7]:

$$H(G) = \sum_{i \sim j} \frac{2}{d_i + d_j} \quad \text{and} \quad ISI(G) = \sum_{i \sim j} \frac{d_i d_j}{d_i + d_j} .$$

Here we prove the following inequalities

$$ISI(G) + \frac{\delta^2}{2}H(G) \geq m\delta,$$

$$ISI(G) + \frac{\Delta^2}{2}H(G) \geq m\Delta,$$

$$M_1(G) - 2ISI(G) + \Delta\delta H(G) \leq m(\Delta + \delta).$$

Let us note that obtained bounds are sharp since there are many classes of graphs for which equalities are attained.

References

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