Relations for Bernoulli–Barnes numbers and Barnes Zeta functions

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The *Barnes ζ-function* is

$$\zeta(z, x; \mathbf{a}) := \sum_{\mathbf{m} \in \mathbf{Z}_{\geq 0}^n} \frac{1}{(x + m_1 a_1 + \cdots + m_n a_n)^z}$$

defined for $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(z) > n$ and continued meromorphically to $\mathbf{C}$. Specialized at negative integers $-k$, the Barnes ζ-function gives

$$\zeta(-k, x; \mathbf{a}) = \frac{(-1)^n k!}{(k+n)!} \, B_{k+n}(x; \mathbf{a})$$

where $B_k(x; \mathbf{a})$ is a *Bernoulli–Barnes polynomial*, which can be also defined through a generating function that has a slightly more general form than that for Bernoulli polynomials. Specializing $B_k(0; \mathbf{a})$ gives the *Bernoulli–Barnes numbers*. We exhibit relations among Barnes ζ-functions, Bernoulli–Barnes numbers and polynomials, which generalize various identities of Agoh, Apostol, Dilcher, and Euler.

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