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# SOME $L_1$-BICONSERVATIVE LORENTZIAN HYPERSURFACES IN THE LORENTZ-MINKOWSKI SPACES

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Abstract. The biconservative hypersurfaces of Euclidean spaces have conservative stress-energy with respect to the bienergy functional. We study Lorentzian hypersurfaces of Minkowski spaces, satisfying an extended condition (namely, $L_1$-biconservativity condition), where $L_1$ (as an extension of the Laplace operator $\Delta = L_0$) is the *linearized operator* arisen from the first normal variation of 2nd mean curvature vector field. A Lorentzian hypersurface $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$ is said to be $L_1$-biconservative if the tangent component of vector field $L_1^2 x$ is identically zero. The geometric motivation of this subject is a well-known conjecture of Bang-Yen Chen saying that the only biharmonic submanifolds (i.e., satisfying condition $L_0^2 x = 0$) of Euclidean spaces are the minimal ones. We discuss on $L_1$-biconservative Lorentzian hypersurfaces of the Lorentz-Minkowski space $\mathbb{L}^{n+1}$. After illustrating some examples, we prove that these hypersurfaces, with at most two distinct principal curvatures and constant ordinary mean curvature, have constant 2nd mean curvature.

## 1. Introduction

The main geometric motivation of the subject of biconservative hypersurfaces is a well-known conjecture of Bang-Yen Chen (in 1987) which states that every biharmonic submanifold of a Euclidean space is harmonic. Further, Chen proved that his conjecture is true for biharmonic surfaces in $\mathbb{E}^3$. In 1992, Dimitrić proved that any biharmonic hypersurface in $\mathbb{E}^m$ with at most two distinct principal curvatures is minimal ([10]). Let $\mathbf{x} : M^n \to \mathbb{E}^{n+1}$ denotes an isometric immersion of a hypersurface $M^n$ into the $(n+1)$-dimensional Euclidean space with the Laplace operator $\Delta$, the shape operator $A$ associated to a unit normal vector field $\mathbf{n}$ and the ordinary mean curvature

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$H$ on $M^n$. The hypersurface $M^n$ is said to be harmonic if $\mathbf{x}$ satisfies condition $\Delta\mathbf{x} = 0$. It is said to be biharmonic if $\mathbf{x}$ satisfies condition $\Delta^2\mathbf{x} = 0$. Also, $M^n$ is said to be biconservative if the tangential part of $\Delta^2\mathbf{x}$ vanishes identically. A famous law due to Beltrami says that $\Delta\mathbf{x} = -nH\mathbf{n}$, so the condition $\Delta\mathbf{x} = 0$ is equivalent to $H \equiv 0$ and the condition $\Delta^2\mathbf{x} = 0$ is equivalent to $\Delta(H\mathbf{n}) = 0$. In 1995, Hasanis and Vlachos proved an extension of Chen's result to the hypersurfaces in Euclidean 4-space ([11]). As an extended case, a hypersurface $\mathbf{x} : M_p^3 \to \mathbb{E}_s^4$, whose mean curvature vector field is an eigenvector of the Laplace operator $\Delta$, has been studied, for instance, in [8, 9] for the Euclidean case (where $p = s = 0$), and for the Lorentz case in [4, 5] (for $s = 1$ and $p = 0, 1$). On the other hand, Chen himself had found a nice relation between the finite type hypersurfaces and biharmonic ones. The theory of finite type hypersurfaces is a well-known subject initiated by Chen (for instance, in [6, 7]) and also studied by L. J. Alias, S. M. B. Kashani and others. In [12], Kashani has studied the notion of $L_1$-finite type Euclidean hypersurfaces as an extension of finite type ones. One can see main results in Chapter 11 of Chen's book ([6]).

The map $L_1$ is an extension of the Laplace operator $L_0 = \Delta$, which stands for the linearized operator of the first variation of the 2th mean curvature of the hypersurface (see, for instance, [1, 17, 20]). This operator is defined by $L_1(f) = tr(P_1 \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where $P_1 = nHI - A$ denotes the first Newton transformation associated to the second fundamental from of the hypersurface and $\nabla^2 f$ is the hessian of $f$. It is interesting to generalize the definition of biharmonic hypersurface by replacing $\Delta$ by $L_1$. Recently, in [15], we have studied the $L_1$-biharmonic spacelike hypersurfaces in 4-dimentional Minkowski space $\mathbb{L}^4$. In this paper, we show that every $L_1$-biconservative Lorentzian hypersurfaces in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$, with constant mean curvature and at most two distinct principal curvatures, has constant 2nd mean curvature.

We present the organization of paper. In Section 2, we remember some preliminaries which will be needed in paper. In Section 3, we present some examples of $L_1$-biconservative Lorentzian hypersurfaces in $\mathbb{L}^{n+1}$. Section 4 is dedicated to $L_1$-biconservative Lorentzian hypersurfaces of $\mathbb{L}^{n+1}$. First, in Theorem 4.1, 4.2 and 4.3 we discuss on $L_1$-biconservative Lorentzian hypersurfaces of $\mathbb{L}^{n+1}$ with diagonalizable shape operator. The other cases that the shape operator of hypersurface is non-diagonalizable will be seen in Theorem 4.4, 4.5 and 4.6.

## 2. Preliminaries

In this section, we recall preliminaries from [1, 13, 14] and [16–19]. The $m$-dimensional Lorentz-Minkowski space $\mathbb{L}^m$ means the pseudo-Euclidean space with index 1, $\mathbb{E}_1^m$, which is the real vector space $\mathbb{R}^m$ endowed with the scalar product defined by $\langle x, y\rangle := -x_1y_1 + \Sigma_{i=2}^m x_iy_i$ for every $x, y \in \mathbb{R}^m$. Throughout the paper, we study on every Lorentzian hypersurface of $\mathbb{L}^{n+1}$, defined by an isometric immersion $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$. The symbols $\tilde{\nabla}$ and $\bar{\nabla}$ stand for the Levi-Civita connection on $M_1^n$ and $\mathbb{L}^{n+1}$, respectively. For every tangent vector fields $X$ and $Y$ on $M$, the Gauss formula is given by $\bar{\nabla}_X Y = \tilde{\nabla}_X Y + \langle AX, Y\rangle \mathbf{n}$ for every $X, Y \in \chi(M)$, where $\mathbf{n}$ is a (locally) unit normal vector field on $M$ and $A$ is the shape operator of $M$ relative to $\mathbf{n}$. For each non-zero vector $X \in \mathbb{L}^{n+1}$, the real value $\langle X, X\rangle$ may be a negative, zero or positive number and then, the vector $X$ is said to be time-like, light-like or space-like, respectively.

**Definition 2.1.** For a $n$-dimensional Lorentzian vector space $V_1^n$, a basis $\mathcal{B} := \{e_1, \ldots, e_n\}$ is said to be *orthonormal* if it satisfies $\langle e_i, e_j\rangle = \epsilon_i \delta_i^j$ for $i, j = 1, \ldots, n$, where $\epsilon_1 = -1$ and $\epsilon_i = 1$ for $i = 2, \ldots, n$. As usual, $\delta_i^j$ stands for the Kronecker delta. $\mathcal{B}$ is called *pseudo-orthonormal* if it satisfies $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0$, $\langle e_1, e_2\rangle = -1$ and $\langle e_i, e_j\rangle = \delta_i^j$ for $i = 1, \ldots, n$ and $j = 3, \ldots, n$.

As well-known, the shape operator $A$ of the Lorentzian hypersurface $M_1^n$ in $\mathbb{L}^{n+1}$, as a self-adjoint linear map on the tangent bundle of $M_1^n$, locally can be put into one of four possible canonical matrix forms, usually denoted by $I$, $II$, $III$ and $IV$. Where in cases $I$ and $IV$, with respect to an orthonormal basis of the tangent space of $M_1^n$, the matrix representation of the induced metric on $M_1^n$ is $G_1 = \mathrm{diag}_n[-1, 1, \ldots, 1]$ and the shape operator of $M_1^n$ can be put into matrix forms $B_1 = \mathrm{diag}[\lambda_1, \ldots, \lambda_n]$ and

$$B_4 = \mathrm{diag}\left[\begin{bmatrix} \kappa & \lambda \\ -\lambda & \kappa \end{bmatrix}, \eta_1, \ldots, \eta_{n-2}\right],$$

where $\lambda \neq 0$, respectively. For cases $II$ and $III$, using a pseudo-orthonormal basis of the tangent space of $M_1^n$, the induced metric on which has matrix form $G_2 = \mathrm{diag}_n[\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right], 1, \ldots, 1]$ and the shape operator of $M_1^n$ can be put into matrix forms

$$B_2 = \mathrm{diag}_n\left[\begin{bmatrix} \kappa & 0 \\ 1 & \kappa \end{bmatrix}, \lambda_1, \ldots, \lambda_{n-2}\right]$$

and

$$B_3 = \mathrm{diag}_n\left[\begin{bmatrix} \kappa & 0 & 0 \\ 0 & \kappa & 1 \\ -1 & 0 & \kappa \end{bmatrix}, \lambda_1, \ldots, \lambda_{n-3}\right],$$

respectively. In case $IV$, the matrix $B_4$ has two conjugate complex eigenvalues $\kappa \pm i\lambda$, but in other cases the eigenvalues of the shape operator are real numbers.

*Remark* 2.1. In two cases $II$ and $III$, one can substitute the pseudo-orthonormal basis $\mathcal{B} := \{e_1, e_2, \ldots, e_n\}$ by a new orthonormal basis $\tilde{\mathcal{B}} := \{\tilde{e_1}, \tilde{e_2}, e_3, \ldots, e_n\}$, where $\tilde{e_1} := \frac{1}{2}(e_1 + e_2)$ and $\tilde{e_2} := \frac{1}{2}(e_1 - e_2)$. Therefore, we obtain new matrices $\tilde{B}_2$ and $\tilde{B}_3$ (instead of $B_2$ and $B_3$, respectively) as

$$\tilde{B}_2 = \mathrm{diag}_n\left[\begin{bmatrix} \kappa + \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \kappa - \frac{1}{2} \end{bmatrix}, \lambda_1, \ldots, \lambda_{n-2}\right]$$

and

$$\tilde{B}_3 = \mathrm{diag}_n\left[\begin{bmatrix} \kappa & 0 & \frac{\sqrt{2}}{2} \\ 0 & \kappa & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \kappa \end{bmatrix}, \lambda_1, \ldots, \lambda_{n-3}\right].$$

After this changes, to unify the notations we denote the orthonormal basis by $\mathcal{B}$ in all cases.

**Notation.** According to four possible matrix representations of the shape operator of $M_1^n$, we define its principal curvatures, denoted by unified notations $\kappa_i$ for $i = 1,\ldots,n$, as follow. In case $I$, we put $\kappa_i := \lambda_i$ for $i = 1,\ldots,n$, where $\lambda_i$'s are the eigenvalues of $B_1$. In cases $II$, where the matrix representation of $A$ is $\tilde{B_2}$, we take $\kappa_i := \kappa$ for $i = 1,2$, and $\kappa_i := \lambda_{i-2}$ for $i = 3,\ldots,n$. In case $III$, where the shape operator has matrix representation $\tilde{B_3}$, we take $\kappa_i := \kappa$ for $i = 1,2,3$ and $\kappa_i := \lambda_{i-3}$ for $i = 4,\ldots,n$. Finally, in the case $IV$, where the shape operator has matrix representation $\tilde{B_4}$, we put $\kappa_1 = \kappa + i\lambda$, $\kappa_2 = \kappa - i\lambda$ and $\kappa_i := \eta_{i-2}$ for $i = 3,\ldots,n$.

The characteristic polynomial of $A$ on $M_1^n$ is of the form $Q(t) = \prod_{i=1}^n (t-\kappa_i) = \sum_{j=0}^n (-1)^j s_j t^{n-j}$, where $s_0 := 1$, $s_i := \sum_{1\le j_1<\cdots<j_i\le n} \kappa_{j_1}\cdots\kappa_{j_i}$ for $i = 1,2,\ldots,n$.

For $j = 1,\ldots,n$, the *jth mean curvature* $H_j$ of $M_1^n$ is defined by $H_j = \frac{1}{\binom{n}{j}} s_j$. When $H_j$ is identically null, $M_1^n$ is said to be $(j-1)$-*minimal*.

**Definition 2.2.** $(i)$ A Lorentzian hypersurface $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$, with diagonalizable shape operator, is said to be *isoparametric* if all of it's principal curvatures are constant.

$(ii)$ A Lorentzian hypersurface $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$, with non-diagonalizable shape operator, is said to be *isoparametric* if the minimal polynomial of it's shape operator is constant.

*Remark* 2.2. Here we remember Theorem 4.10 from [14], which assures us that there is no isoparametric Lorentzian hypersurface of $\mathbb{L}^{n+1}$ with complex principal curvatures.

The well-known Newton transformations $P_j : \chi(M) \to \chi(M)$ on $M_1^n$, is defined by

$$P_0 = I, \quad P_j = s_j I - A \circ P_{j-1}, \quad j = 1,2,\ldots,n,$$

where $I$ is the identity map. Using its explicit formula, $P_j = \sum_{i=0}^j (-1)^i s_{j-i} A^i$, where $A^0 = I$, which gives, by the Cayley-Hamilton theorem (stating that any operator is annihilated by its characteristic polynomial), that $P_n = 0$. It can be seen that, $P_j$ is self-adjoint and commutative with $A$ (see [1, 17]).

Now, we define a notation as

$$\mu_{i_1,i_2,\cdots i_t;k} = \sum_{1\le j_1<\cdots<j_k\le n; j_l\notin\{i_1,i_2,\cdots i_t\}} \kappa_{j_1}\cdots\kappa_{j_k}, \quad i = 1,\ldots,n,\ 1\le k\le n-1,$$

$\mu_{i_1,i_2,\cdots i_t;0} := 1$ and $\mu_{i_1,i_2,\cdots i_t;s} := 0$ for $s < 0$. Corresponding to four possible forms $\tilde{B_i}$ for $1 \le i \le 4$ of $A$, the Newton transformation $P_j$ has different representations. In the case $I$, where $A = \tilde{B_1}$, we have $P_j = \mathrm{diag}[\mu_{1;j},\ldots,\mu_{n;j}]$ for $j = 1,2,\ldots,n-1$.

When $A = B_2$ (in the case $II$), we have

$$P_j = \mathrm{diag}\left[\begin{bmatrix} \mu_{1,2;j} + (\kappa - \frac{1}{2})\mu_{1,2;j-1} & -\frac{1}{2}\mu_{1,2;j-1} \\ \frac{1}{2}\mu_{1,2;j-1} & \mu_{1,2;j} + (\kappa + \frac{1}{2})\mu_{1,2;j-1} \end{bmatrix}, \mu_{3;j},\ldots,\mu_{n;j}\right]$$

and $s_j = \mu_{1,2;j} + 2\kappa\mu_{1,2;j-1} + \kappa^2\mu_{1,2;j-2}$ for $j = 1,\ldots,n-1$.

In the case $III$, we have $A = B_3$ and putting

$$\Lambda_j := \begin{bmatrix} u_j + 2\kappa u_{j-1} + (\kappa^2 - \frac{1}{2})u_{j-2} & -\frac{1}{2}u_{j-2} & -\frac{\sqrt{2}}{2}(u_{j-1} + \kappa u_{j-2}) \\ \frac{1}{2}u_{j-2} & u_j + 2\kappa u_{j-1} + (\kappa^2 + \frac{1}{2})u_{j-2} & \frac{\sqrt{2}}{2}(u_{j-1} + \kappa u_{j-2}) \\ \frac{\sqrt{2}}{2}(u_{j-1} + \kappa u_{j-2}) & \frac{\sqrt{2}}{2}(u_{j-1} + \kappa u_{j-2}) & u_j + 2\kappa u_{j-1} + \kappa^2 u_{j-2} \end{bmatrix},$$

we have $P_j = \mathrm{diag}[\Lambda_j, \mu_{4;j}, \dots, \mu_{n;j}]$, where $u_l := \mu_{1,2,3;l}$ and

$$s_j = u_j + 3\kappa u_{j-1} + 3\kappa^2 u_{j-2} + \kappa^3 u_{j-3}, \quad \text{for } j = 1, \dots, n-1.$$

In the case $IV$, we have $A = B_4$,

$$P_j = \mathrm{diag}\left[\begin{bmatrix} \kappa\mu_{1,2;j-1} + \mu_{1,2;j} & -\lambda\mu_{1,2;j-1} \\ \lambda\mu_{1,2;j-1} & \kappa\mu_{1,2;j-1} + \mu_{1,2;j} \end{bmatrix}, \mu_{3;j}, \dots, \mu_{n;j}\right]$$

and $s_j = \mu_{1,2;j} + 2\kappa\mu_{1,2;j-1} + (\kappa^2 + \lambda^2)\mu_{1,2;j-2}$ for $j = 1, \dots, n-1$.

In all cases, the following important identities occur for $j = 1, \dots, n-1$, similar to those in [1–3, 17, 18]:

$$\begin{aligned} s_{j+1} =& \kappa_i\mu_{i;j} + \mu_{i;j+1}, \quad 1 \le i \le n, \\ \mu_{i;j+1} =& \kappa_l\mu_{i,l;j} + \mu_{i,l;j+1}, \quad 1 \le i, l \le n, \ i \ne l, \\ tr(P_j) =& (n-j)s_j = c_j H_j, \\ tr(P_j \circ A) =& (n - (n-j-1))s_{j+1} = (j+1)s_{j+1} = c_j H_{j+1}, \\ tr(P_j \circ A^2) =& \tbinom{n}{j+1}[nH_1H_{j+1} - (n-j-1)H_{j+2}], \end{aligned}$$

where $c_j = (n-j)\binom{n}{j} = (j+1)\binom{n}{j+1}$.

The *linearized operator* of the $(j+1)$th mean curvature of $M$, $L_j : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is defined by the formula $L_j(f) := tr(P_j \circ \nabla^2 f)$, where $\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X \nabla f, Y\rangle$ for every $X, Y \in \chi(M)$.

Associated to the orthonormal frame $\{e_1, \dots, e_n\}$ of tangent space on a local coordinate system in the hypersurface $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$ , $L_1(f)$ has an explicit expression as $L_1(f) = \sum_{i=1}^n \epsilon_i\mu_{i,1}(e_ie_if - \nabla_{e_i}e_if)$. For a Lorentzian hypersurface $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$, with a chosen (local) unit normal vector field $\mathbf{n}$, for an arbitrary vector $\mathbf{a} \in \mathbb{E}_1^{n+1}$ we use the decomposition $\mathbf{a} = \mathbf{a}^T + \mathbf{a}^N$, where $\mathbf{a}^T \in TM$ is the tangential component of $\mathbf{a}$, $\mathbf{a}^N \perp TM$, and we have the following formulae from [1, 17]:

$$\begin{aligned} \nabla\langle \mathbf{x}, \mathbf{a}\rangle =& \mathbf{a}^T, \quad \nabla\langle \mathbf{n}, \mathbf{a}\rangle = -A\mathbf{a}^T, \\ L_1\mathbf{x} =& n(n-1)H_2\mathbf{n}, \quad L_1\mathbf{n} = -\frac{n(n-1)}{2}\left(\nabla(H_2) + (nH_1H_2 - (n-2)H_3)\mathbf{n}\right), \end{aligned}$$

and finally, we have

$$\begin{aligned} L_1^2\mathbf{x} =& n(n-1)\left(2P_2\nabla H_2 - \frac{3}{2}n(n-1)H_2\nabla H_2\right) \\ &+ n(n-1)\left(L_1H_2 - \frac{n(n-1)}{2}H_2(nH_1H_2 - (n-2)H_3)\right)\mathbf{n}. \end{aligned}$$

Assume that a hypersurface $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$ satisfies the condition $L_1^2\mathbf{x} = 0$, then it is said to be $L_1$-*biharmonic*. By the last equalities, from the condition $L_1(H_2\mathbf{n}) = 0$

(which is equivalent to $L_1$-biharmonicity) we obtain simpler conditions on $M_1^n$ to be a $L_1$-biharmonic hypersurface in $\mathbb{L}^{n+1}$, as:

$$L_1 H_2 = \frac{n(n-1)}{2} H_2(nH_1H_2 - (n-2)H_3), \quad P_2 \nabla H_2 = \frac{3}{4} n(n-1) H_2 \nabla H_2. \tag{2.1}$$

A Lorentzian hypersurface $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$ is said to be $L_1$-*bicoservative*, if its 2th mean curvature satisfies the second condition in (2.1).

The well-known structure equations on $\mathbb{L}^{n+1}$ are given by $d\omega_i = \sum_{j=1}^{n+1} \omega_{ij} \wedge \omega_j$, $\omega_{ij} + \omega_{ji} = 0$ and $d\omega_{ij} = \sum_{l=1}^{n+1} \omega_{il} \wedge \omega_{lj}$. Restricted on $M$, we have $\omega_{n+1} = 0$ and then, $0 = d\omega_{n+1} = \sum_{i=1}^{n} \omega_{n+1,i} \wedge \omega_i$. So, by Cartan's lemma, there exist functions $h_{ij}$ such that $\omega_{n+1,i} = \sum_{j=1}^{n} h_{ij}\omega_j$ and $h_{ij} = h_{ji}$, which give the second fundamental form of $M$, as $B = \sum_{i,j} h_{ij}\omega_i\omega_j e_{n+1}$. The mean curvature $H$ is given by $H = \frac{1}{n}\sum_{i=1}^{n} h_{ii}$. Therefore, we obtain the structure equations on $M$ as $d\omega_i = \sum_{j=1}^{n} \omega_{ij} \wedge \omega_j$, $\omega_{ij} + \omega_{ji} = 0$ and $d\omega_{ij} = \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l=1}^{n} R_{ijkl}\omega_k \wedge \omega_l$ for $i, j = 1, 2, \ldots, n-1$, and the Gauss equations $R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk})$, where $R_{ijkl}$ denotes the components of the Riemannian curvature tensor of $M$. Denoting the covariant derivative of $h_{ij}$ by $h_{ijk}$, we have $dh_{ij} = \sum_{k=1}^{n} h_{ijk}\omega_k + \sum_{k=1}^{n} h_{kj}\omega_{ik} + \sum_{k=1}^{n} h_{ik}\omega_{jk}$ and by the Codazzi equation we get $h_{ijk} = h_{ikj}$.

Finally, we recall the definition of an $L_1$-finite type hypersurface from [12], which is the basic notion of the paper.

**Definition 2.3.** An isometrically immersed hypersurface $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$ is said to be of $L_1$-finite type if $\mathbf{x}$ has a finite decomposition $\mathbf{x} = \sum_{i=0}^{m} \mathbf{x}_i$, for some positive integer $m$, satisfying the condition $L_1\mathbf{x}_i = \tau_i\mathbf{x}_i$, where $\tau_i \in \mathbb{R}$ and $\mathbf{x}_i : M_1^n \to \mathbb{L}^{n+1}$ is smooth maps, for $i = 1, 2, \ldots, m$, and $\mathbf{x}_0$ is constant. If all $\tau_i$'s are mutually different, $M_1^n$ is said to be of $L_1$-$m$-type. An $L_1$-$m$-type hypersurface is said to be null if for at least one $i$, $1 \le i \le m$, we have $\tau_i = 0$.

## 3. Examples

Now, we provide two families of examples of $L_1$-biconservative Lorentzian hypersurfaces in $\mathbb{L}^{n+1}$, some of them are not $L_1$-biharmonic.

*Example* 3.1. Consider the subset $\{(y_1, \ldots, y_{n+1}) \in \mathbb{L}^{n+1} \mid -y_1^2 + \cdots + y_{l+1}^2 = r^2\}$ representing the cylindrical hypersurface $\mathbb{S}_1^l(r) \times \mathbb{E}^{n-l} \subset \mathbb{L}^{n+1}$ for $r > 0$ and $l = 1, 2, \ldots, n-1$, with the Gauss map $\mathbf{n}(y) = -\frac{1}{r}(y_1, \ldots, y_{n-l+1}, 0, \ldots, 0)$. Clearly, it has two distinct constant principal curvatures $\kappa_1 = \cdots = \kappa_l = \frac{1}{r}$ and $\kappa_{l+1} = \cdots = \kappa_n = 0$ and constant higher order mean curvatures $H_1 = \frac{l}{n}r^{-1}$ and $H_2 = \frac{l(l-1)}{n(n-1)}r^{-2}$. One can see that $\mathbb{S}_1^1(r) \times \mathbb{E}^{n-1}$ is $L_1$-biharmonic, but $\mathbb{S}_1^l(r) \times \mathbb{E}^{n-l}$ is not $L_1$-biharmonic for $l = 2, \ldots, n-1$.

*Example* 3.2. Consider the subset $\{(y_1, \ldots, y_{n+1}) \in \mathbb{L}^{n+1} \mid y_{l+1}^2 + \cdots + y_{n+1}^2 = r^2\}$ denoting the hypersurface $\mathbb{L}^l \times \mathbb{S}^{n-l}(r) \subset \mathbb{L}^{n+1}$ with $\mathbf{n}(y) = -\frac{1}{r}(0, \ldots, , 0, y_{l+1}, \ldots, y_{n+1})$ as the Gauss map for $r > 0$ and $l = 1, 2, \ldots, n-1$. It has two distinct principal curvatures $\kappa_1 = \cdots = \kappa_l = 0$ and $\kappa_{l+1} = \cdots = \kappa_n = \frac{1}{r}$ and constant higher order mean curvatures $H_1 = \frac{n-l}{n} r^{-1}$, and $H_2 = \frac{(n-l)(n-l-1)}{n(n-1)} r^{-2}$. One can see that $\mathbb{L}^l \times \mathbb{S}^{n-l}(r)$ is not $L_1$-biharmonic for $l = 1, 2, \ldots, n-2$, but $\mathbb{L}^{n-1} \times \mathbb{S}^1(r)$ is $L_1$-biharmonic.

## 4. $L_1$-Biconservative Lorentzian Hypersurfeces in $\mathbb{L}^{n+1}$

In this section, we give six theorems on the $L_1$-biconservative connected orientable timelike hypersurface in $\mathbb{L}^{n+1}$ with constant ordinary mean curvature. Theorem 4.1, 4.2 and 4.3 are appropriated to the case that the shape operator on hypersurface is diagonalizable. Theorem 4.4, 4.5 and 4.6 are related to the cases that the shape operator on hypersurface is of type $II$, $III$ and $IV$, respectively.

### 4.1. **Hypersurfaces with diagonalizable shape operator.**

**Theorem 4.1.** *Every $L_1$-biconservative Lorentzian hypersurface of $\mathbb{L}^{n+1}$ for any natural number $n \geq 2$, having a diagonalizable shape operator with exactly one eigenvalue function of multiplicity $n$, has constant $2$nd mean curvature.*

*Proof.* Let $x : M_1^n \to \mathbb{L}^{n+1}$ be a $L_1$-biconservative Lorentzian hypersurface of $\mathbb{L}^{n+1}$ with assumed conditions. Defining the open subset $\mathcal{U}$ of $M$ as $\mathcal{U} := \{p \in M_1^n \mid \nabla H_2^2(p) \neq 0\}$, we prove that $\mathcal{U}$ is empty. Assuming $\mathcal{U} \neq \emptyset$, we consider $\{e_1, \ldots, e_n\}$ as a local orthonormal frame of principal directions of $A$ on $\mathcal{U}$ such that for $i = 1, \ldots, n$, we have $Ae_i = \lambda e_i$ and

$$\mu_{i,2} = \frac{1}{2}(n-1)(n-2)\lambda^2, \quad H_2 = \lambda^2. \tag{4.1}$$

By assumption, we have $P_2(\nabla H_2) = \frac{3}{4} n(n-1) H_2 \nabla H_2$, which using the polar decomposition $\nabla H_2 = \sum_{i=1}^n \epsilon_i \langle \nabla H_2, e_i \rangle e_i$, gives

$$\epsilon_i \langle \nabla H_2, e_i \rangle \left( \mu_{i,2} - \frac{3}{4} n(n-1) H_2 \right) = 0$$

on $\mathcal{U}$ for $i = 1, \ldots, n$. Hence, if for some $i$ we have $\langle \nabla H_2, e_i \rangle \neq 0$ on $\mathcal{U}$, then we get $\mu_{i,2} = \frac{3}{4} n(n-1) H_2$, which, using (4.1), gives $\lambda^2 = 0$ and then $H_2 = 0$ on $\mathcal{U}$, which is a contradiction. Hence, $\mathcal{U}$ is empty and $H_2$ is constant on $M$. □

**Theorem 4.2.** *Let $\boldsymbol{x} : M_1^n \to \mathbb{L}^{n+1}$ be an $L_1$-biconservative Lorentzian hypersurface of $\mathbb{L}^{n+1}$ with diagonalizable shape operator, constant ordinary mean curvature and exactly two distinct principal curvature functions $\lambda$ and $\eta$ of multiplicities $n-1$ and $1$, respectively. Then $M_1^n$ has constant $2$nd mean curvature.*

*Proof.* Taking the open subset $\mathcal{V}$ of $M_1^n$ as $\mathcal{V} := \{p \in M_1^n \mid \nabla H_2^2(p) \neq 0\}$, we prove that $\mathcal{V}$ is empty. Assuming $\mathcal{V} \neq \emptyset$, we consider $\{e_1, \ldots, e_n\}$ as a local orthonormal frame of principal directions of $A$ on $\mathcal{V}$ such that $Ae_i = \lambda e_i$ for $i = 1, \ldots, n-1$ and $Ae_n = \eta e_n$. Therefore, we obtain

$$\mu_{1,2} = \cdots = \mu_{n-1,2} = \frac{1}{2}(n-2)(n-3)\lambda^2 + (n-2)\lambda\eta, \tag{4.2}$$

$$\begin{aligned}\mu_{n,2} =&\frac{1}{2}(n-1)(n-2)\lambda^2,\\ nH_1 =&(n-1)\lambda+\eta, \quad n(n-1)H_2=(n-1)(n-2)\lambda^2+2(n-1)\lambda\eta,\\ \binom{n}{3}H_3 =&\binom{n-1}{3}\lambda^3+\binom{n-1}{2}\lambda^2\eta.\end{aligned}$$

Using the polar decomposition $\nabla H_2=\sum_{i=1}^n \epsilon_i\langle\nabla H_2,e_i\rangle e_i$, from (2.1) we have

$$\epsilon_i\langle\nabla H_2,e_i\rangle\left(\mu_{i,2}-\frac{3}{4}n(n-1)H_2\right)=0,$$

on $\mathcal{V}$ for $i=1,\dots,n$. Since, by definition of the subset $\mathcal{V}$, we have $\langle\nabla H_2,e_i\rangle\neq 0$ on $\mathcal{V}$ for some $i$, then we get

$$\mu_{i,2}=\frac{3}{4}n(n-1)H_2, \tag{4.3}$$

for some $i$ which gives one of the following states.

**State 1.** $\langle\nabla H_2,e_i\rangle\neq 0$, for some $i\in\{1,\dots,n-1\}$. Using (4.2), from (4.3) we obtain $(n-2)(n-9)\lambda^2-4(n+l)\lambda\eta=0$, which gives $\lambda=0$ or $\eta=-\frac{(n-2)(n+3)}{2(n+1)}\lambda$. If $\lambda=0$, then $H_2=0$. Otherwise, we get $\lambda=\frac{2n(n+1)}{n^2-n+4}H_1$ and $H_2=-\frac{8n(n+1)(n-2)}{(n^2-n+4)^2}H_1^2$.

**State 2.** $\langle\nabla H_2,e_i\rangle=0$ for all $i\in\{1,\dots,n-1\}$ and $\langle\nabla H_2,e_n\rangle\neq 0$. By (4.2) and (4.3), we obtain $\lambda=0$ or $\eta=\frac{2-n}{6}\lambda$. If $\lambda=0$, then $H_2=0$. Otherwise, we get $\lambda=\frac{6n}{5n-4}H_1$ and $H_2=\frac{24n(n-2)}{(5n-4)^2}H_1^2$.

Therefore, $H_2$ is constant on $M_1^n$. □

**Theorem 4.3.** *Let $\boldsymbol{x}:M_1^n\to\mathbb{L}^{n+1}$ be an $L_1$-biconservative Lorentzian hypersurface of $\mathbb{L}^{n+1}$ with diagonalizable shape operator, constant ordinary mean curvature and exactly two distinct principal curvature functions $\lambda$ and $\eta$ of multiplicities $n-k$ and $k$, respectively, where $2\leq k\leq n-2$. Then, the 2nd mean curvature of $M_1^n$ has to be constant.*

*Proof.* Defining the open subset $\mathcal{V}$ of $M_1^n$ as $\mathcal{V}:=\{p\in M_1^n\mid \nabla H_2^2(p)\neq 0\}$, we prove that $\mathcal{V}$ is empty. Assuming $\mathcal{V}\neq\emptyset$, we consider $\{e_1,\dots,e_n\}$ as a local orthonormal frame of principal directions of $A$ on $\mathcal{V}$ such that $Ae_i=\lambda e_i$ for $i=1,\dots,n-k$ and $Ae_i=\eta e_i$ for $i=n-k+1,\dots,n$. Therefore, we obtain

$$\begin{aligned}\mu_{1,2}=&\cdots=\mu_{n-k,2}\\ =&\frac{1}{2}(n-k-1)(n-k-2)\lambda^2+\frac{1}{2}k(k-1)\eta^2+(n-k-1)k\lambda\eta,\end{aligned} \tag{4.4}$$

$$\begin{aligned}\mu_{n-k+1,2}=&\cdots=\mu_{n,2}\\ =&(n-k)\left(\frac{1}{2}(n-k-1)\lambda^2+(k-1)\lambda\eta\right)+\frac{1}{2}(k-1)(k-2)\eta^2,\end{aligned} \tag{4.5}$$

$$\begin{aligned}nH_1=&(n-k)\lambda+k\eta,\\ n(n-1)H_2=&(n-k)((n-k-1)\lambda^2+2k\lambda\eta)+k(k-1)\eta^2,\end{aligned}$$

$$\binom{n}{3}H_3 = \binom{n-k}{3}\lambda^3 + k\binom{n-k}{2}\lambda^2\eta + (n-k)\binom{k}{2}\lambda\eta^2 + \binom{k}{3}\eta^3.$$

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^n \epsilon_i \langle \nabla H_2, e_i\rangle e_i$, from (2.1) we have $\epsilon_i\langle \nabla H_2, e_i\rangle(\mu_{i,2} - \frac{3}{4}n(n-1)H_2) = 0$ on $\mathcal{V}$ for $i = 1, \dots, n$. Hence, $\langle \nabla H_2, e_i\rangle \neq 0$ on $\mathcal{V}$ for some $i$ and then

$$\mu_{i,2} = \frac{3}{4}n(n-1)H_2. \tag{4.6}$$

By definition, we have $\nabla H_2 \neq 0$ on $\mathcal{U}$, which gives one or both of the following states.

**State 1.** $\langle \nabla H_2, e_i\rangle \neq 0$ for some $i \in \{1, \dots, n-k\}$. Using (4.4), from (4.6) we obtain $(n-k-1)(n-k+4)\lambda^2 + k(k-1)\eta^2 + 2k(n-k+2)\lambda\eta = 0$, which gives $\eta = d_0\lambda$, where

$$d_0 = -\left(\frac{n-k+2}{k-1} \pm \frac{\sqrt{kn(n-k+3) + k(5k-4)}}{k(k-1)}\right).$$

Hence, we get $\lambda = \frac{n}{n-k(1-d_0)}H_1$ and $\eta = \frac{nd_0}{n-k(1-d_0)}H_1$, which give $H_2 = d_1H_1^2$ for a fixed coefficient $d_1$ (i.e., $H_2$ is constant on $M_1^n$).

**State 2.** $\langle \nabla H_2, e_i\rangle = 0$ for all $i \in \{1, \dots, n-l\}$ and $\langle \nabla H_2, e_i\rangle \neq 0$ for some $i \in \{n-l+1, \dots, n\}$. By (4.4) and (4.6), we obtain

$$(n-l)(n-l-1)\lambda^2 + (l+4)(l-1)\eta^2 + 2(n-l)(l+2)\lambda\eta = 0,$$

which gives $(n-1)\lambda(6\eta + (n-2)\lambda) = 0$. If $\lambda = 0$, then $H_2 = 0$. Otherwise, we have $\eta = -\frac{n-2}{6}\lambda$, which gives $\lambda = \frac{6n}{(6-k)n-4k}H_1$ and $\eta = -\frac{n(n-2)}{(6-k)n-4k}H_1$ and then $H_2 = d_2H_1^2$ for a fixed coefficient $d_2$ (i.e., $H_2$ is constant on $M_1^n$). ☐

4.2. **Hypersurfaces with non-diagonalizable shape operator.** This subsection is appropriated to cases that the Lorentzian hypersurfaces of $\mathbb{L}^{n+1}$ have shape operator of type $II$, $III$ or $IV$.

**Theorem 4.4.** *Every $L_1$-biconservative Lorentzian hypersurface $M_1^n$ in $\mathbb{L}^{n+1}$, where $n \geq 3$ with shape operator of type $II$, having constant ordinary mean curvature and at most two distinct principal curvatures, has constant 2nd mean curvature.*

*Proof.* Assume that, an isometric immersion $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$ satisfies all conditions of the theorem. So, it is $L_1$-biconservative with shape operator of type $II$, constant ordinary mean curvature and two distinct principal curvatures. Taking the open subset $\mathcal{U} = \{p \in M_1^n \mid \nabla H_2^2(p) \neq 0\}$, we show that $\mathcal{U} = \emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, \dots, e_n\}$ on $M_1^n$, the shape operator $A$ has the matrix form $\tilde{B}_2$, such that $Ae_1 = (\kappa + \frac{1}{2})e_1 - \frac{1}{2}e_2$, $Ae_2 = \frac{1}{2}e_1 + (\kappa - \frac{1}{2})e_2$ and $Ae_i = \lambda e_i$ for $i = 3, \dots, n$. Then we have the following equalities:

$$\begin{aligned}
&nH_1 = 2\kappa + (n-2)\lambda,\ n(n-1)H_2 = 2\kappa^2 + (n-2)(n-3)\lambda^2 + 4(n-2)\kappa\lambda,\\
&P_2e_1 = \left(\frac{(n-2)(n-3)}{2}\lambda^2 + (n-2)(\kappa - \frac{1}{2})\lambda\right) e_1 + \frac{n-2}{2}\lambda e_2,\\
&P_2e_2 = -\frac{n-2}{2}\lambda e_1 + \left(\frac{(n-2)(n-3)}{2}\lambda^2 + (n-2)(\kappa + \frac{1}{2})\lambda\right) e_2,\\
&P_2e_i = \left(\kappa^2 + 2(n-3)\kappa\lambda + \frac{(n-3)(n-4)}{2}\lambda^2\right) e_i, \quad i = 3, \dots, n.
\end{aligned}$$

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{n} \epsilon_i e_i(H_2)e_i$, from (2.1) we get

$$\begin{aligned}
&\left((n-3)\lambda^2 + (2\kappa-1)\lambda - \frac{3n(n-1)}{2(n-2)}H_2\right) \epsilon_1 e_1(H_2) = \lambda\epsilon_2 e_2(H_2),\\
&\left((n-3)\lambda^2 + (2\kappa+1)\lambda - \frac{3n(n-1)}{2(n-2)}H_2\right) \epsilon_2 e_2(H_2) = -\lambda\epsilon_1 e_1(H_2),\\
&\left(\kappa^2 + 2(n-3)\kappa\lambda + \frac{(n-3)(n-4)}{2}\lambda^2 - \frac{3}{4}n(n-1)H_2\right) \epsilon_i e_i(H_2) = 0, \quad i = 3, \dots, n.
\end{aligned} \tag{4.7}$$

Now, we prove the main claim.

*Claim.* $e_i(H_2) = 0$ for $i = 1, \dots, n$. If $e_1(H_2) \neq 0$, then by dividing both sides of two equalities in (4.7) by $\epsilon_1 e_1(H_2)$ we get

$$\begin{aligned}
&\frac{(n-2)(n-3)}{2}\lambda^2 + (n-2)\left(\kappa - \frac{1}{2}\right)\lambda - \frac{3}{4}n(n-1)H_2 = \frac{n-2}{2}\lambda u,\\
&\left(\frac{(n-2)(n-3)}{2}\lambda^2 + (n-2)\left(\kappa + \frac{1}{2}\right)\lambda - \frac{3}{4}n(n-1)H_2\right) u = -\frac{n-2}{2}\lambda,
\end{aligned} \tag{4.8}$$

where $u := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$. From (4.8) we obtain $\lambda(1+u)^2 = 0$, then $\lambda = 0$ or $u = -1$. If $\lambda = 0$. Then we obtain $H_2 = 0$, which means $H_2$ is constant. Otherwise, we have $u = -1$, which gives $\frac{(n-2)(n-3)}{2}\lambda^2 + (n-2)\kappa\lambda = \frac{3}{4}n(n-1)H_2$, then we obtain $6\kappa^2 + (n-2)(n-3)\lambda^2 + 8(n-2)\kappa\lambda = 0$. Since $nH_1 = 2\kappa + (n-2)\lambda$ is assumed to be constant on $M$, by substituting which in the last equality, we get $(4-3n)(n-2)\lambda^2 + 2n(n-2)H_1\lambda + 3n^2H_1^2 = 0$, which means $\lambda$, $\kappa$ and the $k$th mean curvatures for $k = 2, \dots, n$, are also constant on $M_1^n$. So, we got a contradiction and therefore, the first part of the claim is proved.

If $e_2(H_2) \neq 0$, then by dividing both sides of two equalities in (4.7) by $\epsilon_2 e_2(H_2)$ we get

$$\begin{aligned}
&\left(\frac{(n-2)(n-3)}{2}\lambda^2 + (n-2)\left(\kappa - \frac{1}{2}\right)\lambda - \frac{3}{4}n(n-1)H_2\right) v = \frac{n-2}{2}\lambda,\\
&\frac{(n-2)(n-3)}{2}\lambda^2 + (n-2)\left(\kappa + \frac{1}{2}\right)\lambda - \frac{3}{4}n(n-1)H_2 = -\frac{n-2}{2}\lambda v,
\end{aligned} \tag{4.9}$$

where $v := \frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}$. From (4.9) we obtain $\lambda(1+v)^2 = 0$. If $\lambda = 0$, from (4.9) we obtain $H_2 = 0$, which means $H_2$ is constant. Otherwise, we have $v = -1$, which gives $\frac{(n-2)(n-3)}{2}\lambda^2 + (n-2)\kappa\lambda = \frac{3}{4}n(n-1)H_2$, then similar to the first part, we obtain that $\lambda$, $\kappa$ and the $k$th mean curvatures for $k = 2, \dots, n$ are also constant on $M_1^n$. So, we got a contradiction and therefore, the second part of the claim is proved.

Finally, each of assumptions $e_i(H_2) \neq 0$ for $i = 3, \dots, n$, gives the equality $\kappa^2 + \frac{(n-3)(n-4)}{2}\lambda^2 + 2(n-3)\kappa\lambda = \frac{3}{4}n(n-1)H_2$, which gives $\kappa^2 + n(n-3)\lambda^2 + 4(n-1)\kappa\lambda = 0$. Similar to two first cases, Using formula $nH_1 = 2\kappa + (n-2)\lambda$, from the last equation we obtain that $\lambda$, $\kappa$ and the $k$th mean curvatures for $k = 2, \dots, n$, are also constant on $M_1^n$. The contradiction that $H_2$ is constant on $M$. So, the claim is confirmed. □

**Theorem 4.5.** *Every $L_1$-biconservative timelike hypersurface $M_1^n$ in $\mathbb{L}^{n+1}$ with shape operator of type III, having at most two distinct principal curvatures and constant ordinary mean curvature, has constant 2nd mean curvature.*

*Proof.* Assume that, an isometric immersion $\mathbf{x} : M_1^n \to \mathbb{L}^{n+1}$ satisfies all conditions of the theorem. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, \dots, e_n\}$ on $M_1^n$, the shape operator $A$ has the matrix form $\tilde{B}_3$, such that $Ae_1 = \kappa e_1 - \frac{\sqrt{2}}{2}e_3$, $Ae_2 = \kappa e_2 - \frac{\sqrt{2}}{2}e_3$, $Ae_3 = \frac{\sqrt{2}}{2}e_1 - \frac{\sqrt{2}}{2}e_2 + \kappa e_3$ and $Ae_i = \lambda e_i$ for $i = 4, \dots, n$. Then we have

$$
\begin{aligned}
nH_1 =& 3\kappa + (n-3)\lambda, \;\; n(n-1)H_2 = 3\kappa^2 + \frac{(n-3)(n-4)}{2}\lambda^2 + 3(n-3)\kappa\lambda,\\
P_2e_1 =& \left(\frac{(n-3)(n-4)}{2}\lambda^2 + 2(n-3)\kappa\lambda + \kappa^2 - \frac{1}{2}\right)e_1 + \frac{1}{2}e_2 + \frac{\sqrt{2}}{2}\left((n-3)\lambda + \kappa\right)e_3,\\
P_2e_2 =& \frac{1}{2}e_1 + \left(\frac{(n-3)(n-4)}{2}\lambda^2 + 2(n-3)\kappa\lambda + \kappa^2 + \frac{1}{2}\right)e_2 + \frac{\sqrt{2}}{2}\left((n-3)\lambda + \kappa\right)e_3,\\
P_2e_3 =& \frac{-\sqrt{2}}{2}\left((n-3)\lambda + \kappa\right)e_1 + \frac{\sqrt{2}}{2}\left((n-3)\lambda + \kappa\right)e_2\\
&+ \left(\frac{(n-3)(n-4)}{2}\lambda^2 + 2(n-3)\kappa\lambda + \kappa^2\right)e_3,\\
P_2e_i =& \left(3\kappa^2 + 3(n-4)\kappa\lambda + \frac{(n-4)(n-5)}{2}\lambda^2\right)e_i, \quad i = 4, \dots, n.
\end{aligned}
$$

Similar to proof of Theorem 4.4, we assume that $H_2$ is non-constant and considering the open subset $\mathcal{U} = \{p \in M_1^n \mid \nabla H_2^2(p) \neq 0\}$, we prove that $\mathcal{U} = \emptyset$. Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{n} \epsilon_i e_i(H_2) e_i$, from (2.1) we get the following system of conditions:

(4.10)

$$
\left((n-3)\lambda\left(\frac{n-4}{2}\lambda + 2\kappa\right) + \kappa^2 - \frac{1}{2} - \frac{3}{4}n(n-1)H_2\right)\epsilon_1 e_1(H_2) + \frac{1}{2}\epsilon_2 e_2(H_2)
$$

$$
\begin{aligned}
=&\frac{\sqrt{2}}{2}\left((n-3)\lambda+\kappa\right)\epsilon_3 e_3(H_2),\\
&\frac{1}{2}\epsilon_1 e_1(H_2)+\left((n-3)\lambda\left(\frac{n-4}{2}\lambda+2\kappa\right)+\kappa^2+\frac{1}{2}-\frac{3}{4}n(n-1)H_2\right)\epsilon_2 e_2(H_2)\\
=&-\frac{\sqrt{2}}{2}((n-3)\lambda+\kappa)\epsilon_3 e_3(H_2),\\
&\frac{\sqrt{2}}{2}((n-3)\lambda+\kappa)(\epsilon_1 e_1(H_2)+\epsilon_2 e_2(H_2))\\
=&-\left((n-3)\lambda\left(\frac{n-4}{2}\lambda+2\kappa\right)+\kappa^2-\frac{3}{4}n(n-1)H_2\right)\epsilon_3 e_3(H_2),\\
&\left(3\kappa^2+((n-3)\lambda\left(\frac{n-4}{2}\lambda+2\kappa\right)-\frac{3}{4}n(n-1)H_2)\right)\epsilon_i e_i(H_2)=0,\quad i=4,\dots,n.
\end{aligned}
$$

Now, we prove that $H_2$ is constant.

*Claim.* $e_i(H_2) = 0$ for $i = 1, \dots, n$.

If $e_1(H_2) \neq 0$, then by dividing both sides of three first equalities in (4.10) by $\epsilon_1 e_1(H_2)$, and using the notations $u_1 := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$ and $u_2 := \frac{\epsilon_3 e_3(H_2)}{\epsilon_1 e_1(H_2)}$, we get

$$
\begin{aligned}
\frac{1}{4}(\alpha-2)+\frac{1}{2}u_1-\beta u_2 =&0,\\
\frac{1}{2}+\frac{1}{4}(\alpha+2)u_1+\beta u_2 =&0,\\
\beta(1+u_1)+\frac{1}{4}\alpha u_2 =&0,
\end{aligned}
\tag{4.11}
$$

where $\alpha := (n-3)\lambda\left(\frac{n-4}{2}\lambda-\kappa\right)-5\kappa^2$ and $\beta := \frac{\sqrt{2}}{2}((n-3)\lambda+\kappa)$. From (4.11) we obtain

$$
\beta u_2(1+u_1)=\frac{1}{2}(u_1^2-1)-u_1,\quad \frac{1}{4}\alpha(1+u_1)=-u_1. \tag{4.12}
$$

On the other hand, since $nH_1 = 3\kappa + (n-3)\lambda$ is assumed to be constant, we can restate $\alpha$ and $\beta$ in terms of $\kappa$ as:

$$
\begin{aligned}
\alpha =&\frac{1}{2(n-3)}\left((5n-24)\kappa^2-(8n^2-30n)H\kappa+n^2(n-4)H_1^2\right),\\
\beta =&\frac{\sqrt{2}}{2}(nH_1+2\kappa).
\end{aligned}
\tag{4.13}
$$

Now, using (4.12), from (4.11) we get a polynomial equation in terms of $\kappa$ as $64\beta^2 + \alpha^3 - 8\alpha = 0$. This result says that $\kappa$ and then $\lambda$ and $H_2$ have constant values on $\mathfrak{U}$. This is a contradiction and implies that, the first claim $e_1(H_2) \equiv 0$ is proved.

If $e_2(H_2) \neq 0$, then by dividing both sides of three first equalities in (4.10) by $\epsilon_2 e_2(H_2)$ and using the identities recalled in the first paragraph of the proof and notations $v_1 := \frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}$ and $v_3 := \frac{\epsilon_3 e_3(H_2)}{\epsilon_2 e_2(H_2)}$, we get

$$\begin{aligned} \frac{1}{4}(\alpha-2)v_1+\frac{1}{2}-\beta v_3 =&0,\\ \frac{1}{2}v_1+\frac{1}{4}(\alpha+2)+\beta v_3 =&0,\\ \beta(v_1+1)+\frac{1}{4}\alpha v_3 =&0, \end{aligned} \tag{4.14}$$

where $\alpha$ and $\beta$ are as the first case. From (4.14) we obtain

$$\beta v_3(1+v_1)=\frac{1}{2}(1-v_1^2)-v_1, \quad \frac{1}{4}\alpha(1+v_1)=-1. \tag{4.15}$$

Now, using (4.13) and (4.15), from the third equation in (4.14) we get a polynomial equation in terms of $\kappa$ as $64\beta^2+\alpha^2\beta-8\alpha=0$. This result says that $\kappa$, $\lambda$ and $H_2$ have constant values on $\mathcal{U}$. This is a contradiction and implies that, the first claim $e_2(H_2)\equiv 0$ is proved.

If $e_3(H_2)\neq 0$, then by dividing both sides of equalities in (4.10) by $\epsilon_3 e_3(H_2)$, and using notations $w_1 := \frac{\epsilon_1 e_1(H_2)}{\epsilon_3 e_3(H_2)}$ and $w_2 := \frac{\epsilon_2 e_2(H_2)}{\epsilon_3 e_3(H_2)}$, we get

$$\begin{aligned} \frac{1}{4}(\alpha-2)w_1+\frac{1}{2}w_2 =&\beta,\\ \frac{1}{2}w_1+\frac{1}{4}(\alpha+2)w_2 =&-\beta,\\ \beta(w_1+w_2) =&-\frac{1}{4}\alpha, \end{aligned} \tag{4.16}$$

where $\alpha$ and $\beta$ are as the first case. From (4.16) we obtain

$$\beta(w_1+w_2)=-\frac{1}{2}(w_1+w_2)^2, \quad \frac{1}{4}\alpha(w_1+w_2)=-w_2. \tag{4.17}$$

Using (4.13) and (4.17), From (4.16) we get a polynomial equation in terms of $\kappa$ as $\alpha-8\beta^2=0$. This result says that $\kappa$ and then $\lambda$ and $H_2$ have constant value on $\mathcal{U}$. This is a contradiction and implies that, the first claim $e_3(H_2)\equiv 0$ is proved.

The forth stage is assumption $e_i(H_2)\neq 0$ for some $i\geq 4$. By the same manner, from (4.10) we get $\alpha+8\kappa^2=0$, which by using (4.13) gives a polynomial equation in terms of $\kappa$. This result says that $\kappa$ and then $\lambda$ and $H_2$ have constant value on $\mathcal{U}$. This is a contradiction and implies that $e_i(H_2)\equiv 0$ for $i=4,5,\ldots,n$. □

**Theorem 4.6.** *Every $L_1$-biconservative connected orientable Lorentzian hypersurface $M_1^n$ with shape operator of type IV in $\mathbb{L}^{n+1}$, having at most two distinct principal curvatures, has constant 2nd mean curvature.*

*Proof.* Suppose that, $H_2$ be non-constant. Considering the open subset $\mathcal{U}=\{p\in M \mid \nabla H_2^2(p)\neq 0\}$, we try to show $\mathcal{U}=\emptyset$. By assumption, the shape operator $A$ of $M_1^4$ is of type $IV$ with at most two distinct nonzero eigenvalue functions, then, with respect to a suitable (local) orthonormal tangent frame $\{e_1,\ldots,e_n\}$ on $M_1^n$, the shape operator $A$ has the matrix form $B_4$, such that $Ae_1 = -\lambda e_2$, $Ae_2 = \lambda e_1$, $Ae_i = 0$ for $i = 3, \ldots, n$. Then we have $P_2 e_1 = P_2 e_2 = 0$, $P_2 e_i = \lambda^2 e_i$ for $i = 3, \ldots, n$. Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i e_i(H_2) e_i$, from (2.1) we get

$$\begin{aligned} \frac{3}{4} n(n-1) H_2 \epsilon_i e_i(H_2) &= 0, \quad i = 1, 2, \\ \left(\lambda^2 - \frac{3}{4} n(n-1) H_2\right) \epsilon_i e_i(H_2) &= 0, \quad i = 3, \ldots, n, \end{aligned}$$

which clearly gives $e_i(H_2) = 0$ for $i = 1, \ldots, n$. Then $H_2$ is constant on $M_1^n$. □

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