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ON LAPLACIAN ESTRADA INDEX OF UNION AND CARTESIAN PRODUCT OF GRAPHS

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Abstract. The Estrada index EE of a graph G of order n is defined as the sum of the terms e^{λ_i}, $i = 1, 2, \ldots, n$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are its adjacency eigenvalues. The Laplacian Estrada index LEE of a graph G is defined as the sum of the terms e^{μ_i}, $i = 1, 2, \ldots, n$, where $\mu_1, \mu_2, \ldots, \mu_n$ are the Laplacian eigenvalues of G. In this paper we have obtained the upper bounds for the Laplacian Estrada index of union of graphs and computed Laplacian Estrada index of Cartesian product of some graphs.

1. Introduction

Throughout this paper we are concerned with simple graphs, that is, the graphs having no loops or multiple edges or directed edges. Let G be such a graph with n vertices $v_1, v_2, \ldots, v_n$ and m edges. In what follows we say that G is an (n, m)-graph.

Let $D(G)$ be the diagonal matrix of order n whose (i, i)-th entry is the degree of a vertex v_i. The adjacency matrix of a graph G, denoted by $A(G)$, is the square matrix of order n whose (i, j)-th entry is equal to the number of edges between the vertices v_i and v_j. The eigenvalues of $A(G)$ denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called the *adjacency eigenvalues* of G [4]. The matrix $C(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G. The eigenvalues of $C(G)$ denoted by $\mu_i = \mu_i(G)$, $i = 1, 2, \ldots, n$, are called the *Laplacian eigenvalues* of G and their collection is called the *Laplacian spectrum* of G [21].

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The *Laplacian energy* of a graph was introduced by Gutman and Zhou [18] and is defined as

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|.$$

The *Estrada index* of a graph G is defined as

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i}.$$

This graph invariant appeared for the first time in year 2000, in a paper by Ernesto Estrada, dealing with the folding of protein molecules [6–8]. A large number of recent works devoted to the study of its mathematical properties can be found in [5, 10–17, 23, 25]. The *Laplacian Estrada index* of G was defined in [9] as

$$LEE(G) = \sum_{i=1}^{n} e^{\mu_i}.$$

Independent of [9], another varient of the Laplacian Estrada index was put forward in [20], as

$$LEE_{LSC}(G) = \sum_{i=1}^{n} e^{\mu_i - (2m/n)}.$$

Evidently, $LEE_{LSC}(G) = e^{-2m/n} LEE(G)$, and therefore results obtained for LEE can be immediately re-stated for LEE_{LSC} and vice-versa.

Some basic properties of LEE were determined in the papers [3, 9, 20, 26, 27]. At the outset we note that

$$LEE(G) = \sum_{k \geq 0} \frac{1}{k!} \sum_{i=1}^{n} \mu_i^k,$$

where the standard notational convention that $0^0 = 1$ is used.

Let K_n be the complete graph on n vertices and $\overline{K_n}$ be its complement. In [9] the following bound for $LEE(G)$ was obtained.

$$LEE(G) \leq e^{2m/n}(n - 1 + e^{LE(G)}),$$

with equality if and only if $G \cong \overline{K_n}$.

In [27], the authors have obtained the bound for $LEE(G)$ as

$$LEE(G) \leq e^{2m/n}(n - 1 - LE(G) + e^{LE(G)}),$$

with equality if and only if $G \cong \overline{K_n}$.

In this paper we obtain the upper bounds for the Laplacian Estrada index of union of graphs. Further we obtain the Laplacian Estrada index of Cartesian product of some graphs.

2. Laplacian Estrada Index of Union of Graphs

Let G_1 be a graph with vertex set V_1 and edge set E_1 and G_2 be another graph with vertex set V_2 and edge set E_2. The *union* of G_1 and G_2 is a graph $G_1 \cup G_2$ with vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$. If G_1 is an (n_1, m_1)-graph and G_2 is an (n_2, m_2)-graph then $G_1 \cup G_2$ has $n_1 + n_2$ vertices and $m_1 + m_2$ edges. The Laplacian spectrum of $G_1 \cup G_2$ is the union of the Laplacian spectra of G_1 and G_2.

Theorem 2.1. *Let G_1 be an (n_1, m_1)-graph and G_2 be an (n_2, m_2)-graph where $\frac{m_1}{n_1} > \frac{m_2}{n_2}$. Then*

$$LEE(G_1 \cup G_2) \leq e^Y \left\{(n_1+n_2) + e^X \left[e^{LE(G_1)} + e^{LE(G_2)}\right] - 2\right\}, \tag{2.1}$$

where $Y = \frac{2(m_1+m_2)}{n_1+n_2}$ and $X = \frac{2(m_1 n_2 - m_2 n_1)}{n_1+n_2}$.

Equality holds when $G_1 = G_2 = \overline{K_n}$.

Proof. Let $G = G_1 \cup G_2$. The number of vertices of $G_1 \cup G_2$ is $n = n_1 + n_2$ and the number of edges of $G_1 \cup G_2$ is $m = m_1 + m_2$. By the definition of Laplacian Estrada index, we get

$$\begin{aligned}
LE(G_1 \cup G_2) &= e^{2(m_1+m_2)/(n_1+n_2)} \sum_{i=1}^{n_1+n_2} e^{\mu_i(G) - 2(m_1+m_2)/(n_1+n_2)} \\
&= e^Y \sum_{i=1}^{n_1+n_2} e^{\mu_i(G) - Y} \\
&= e^Y \left[(n_1+n_2) + \sum_{i=1}^{n_1+n_2} \sum_{k\geq 1} \frac{1}{k!} (\mu_i(G) - Y)^k\right] \\
&\leq e^Y \left[(n_1+n_2) + \sum_{i=1}^{n_1+n_2} \sum_{k\geq 1} \frac{1}{k!} |\mu_i(G) - Y|^k\right] \\
&= e^Y \left\{(n_1+n_2) + \sum_{k\geq 1} \frac{1}{k!} \left[\sum_{i=1}^{n_1} |\mu_i(G) - Y|^k + \sum_{i=n_1+1}^{n_1+n_2} |\mu_i(G) - Y|^k\right]\right\} \\
&= e^Y \left\{(n_1+n_2) + \sum_{k\geq 1} \frac{1}{k!} \left[\sum_{i=1}^{n_1} |\mu_i(G_1) - Y|^k + \sum_{i=1}^{n_2} |\mu_i(G_2) - Y|^k\right]\right\} \\
&\leq e^Y \left\{(n_1+n_2) + \sum_{k\geq 1} \frac{1}{k!} \left[\left[\sum_{i=1}^{n_1} |\mu_i(G_1) - Y|\right]^k + \left[\sum_{i=1}^{n_2} |\mu_i(G_2) - Y|\right]^k\right]\right\} \\
&= e^Y \left\{(n_1+n_2) + \sum_{k\geq 1} \frac{1}{k!} \left[\left(\sum_{i=1}^{n_1} \left|\mu_i(G_1) - \frac{2m_1}{n_1} + \frac{2m_1}{n_1} - Y\right|\right)^k \right.\right. \\
&\quad \left.\left. + \left(\sum_{i=1}^{n_2} \left|\mu_i(G_2) - \frac{2m_2}{n_2} + \frac{2m_2}{n_2} - Y\right|\right)^k\right]\right\}
\end{aligned}$$

$$\begin{aligned} &\leq e^Y \left\{ (n_1+n_2) + \sum_{k\geq 1} \frac{1}{k!} \left[\left(\sum_{i=1}^{n_1} \left| \mu_i(G_1) - \frac{2m_1}{n_1} \right| + n_1 \left| \frac{2m_1}{n_1} - Y \right| \right)^k \right.\right. \\ &\left.\left. + \left(\sum_{i=1}^{n_2} \left| \mu_i(G_2) - \frac{2m_2}{n_2} \right| + n_2 \left| \frac{2m_2}{n_2} - Y \right| \right)^k \right] \right\}. \end{aligned} \tag{2.2}$$

Since $\frac{m_1}{n_1} > \frac{m_2}{n_2}$, the Eq. (2.2) becomes

$$\begin{aligned} LEE(G_1 \cup G_2) &\leq e^Y \left\{ (n_1+n_2) + \sum_{k\geq 1} \frac{1}{k!} \left[\left(LE(G_1) + n_1 \left(\frac{2m_1}{n_1} - Y \right) \right)^k \right.\right. \\ &\left.\left. + \left(LE(G_2) + n_2 \left(Y - \frac{2m_2}{n_2} \right) \right)^k \right] \right\} \\ &= e^Y \left\{ (n_1+n_2) + \sum_{k\geq 1} \frac{1}{k!} \left[(LE(G_1)+X)^k + (LE(G_2)+X)^k \right] \right\} \\ &= e^Y \left\{ (n_1+n_2) + e^{LE(G_1)+X} - 1 + e^{LE(G_2)+X} - 1 \right\} \\ &= e^Y \left\{ (n_1+n_2) + e^X \left[e^{LE(G_1)} + e^{LE(G_2)} \right] - 2 \right\}, \end{aligned}$$

as desired. □

Corollary 2.1. *Let G_1 be an r_1-regular graph on n_1 vertices and G_2 be an r_2-regular graph on n_2 vertices where $r_1 > r_2$. Then*

$$LEE(G_1 \cup G_2) \leq e^P \left\{ (n_1+n_2) + e^Q \left[e^{LE(G_1)} + e^{LE(G_2)} \right] - 2 \right\},$$

where $P = \frac{n_1 r_1 + n_2 r_2}{n_1+n_2}$ and $Q = \frac{n_1 n_2 (r_1 - r_2)}{n_1+n_2}$.

Proof. Result follows by putting $m_1 = n_1 r_1/2$ and $m_2 = n_2 r_2/2$ in the Theorem 2.1. □

Corollary 2.2. *Let G be an (n,m)-graph where $m > \frac{n(n-1)}{4}$ and $\overline{G}$ be the complement of G. Then*

$$LEE(G \cup \overline{G}) \leq e^{\frac{n-1}{2}} \left\{ 2n + e^{2m - \binom{n}{2}} \left[e^{LE(G)} + e^{LE(\overline{G})} \right] - 2 \right\}.$$

Proof. If G is an (n,m)-graph, then its complement $\overline{G}$ has n vertices and $\frac{n(n-1)}{2} - m$ edges. Substituting this in Eq. (2.1), the result follows. □

Corollary 2.3. *Let G be an (n,m)-graph and G' be the graph obtained from G by removing k edges, $0 \leq k \leq m$. Then*

$$LEE(G \cup G') \leq e^{(2m-k)/n} \left\{ 2n + e^k \left[e^{LE(G)} + e^{LE(G')} \right] - 2 \right\}.$$

Proof. The number of vertices and the number of edges of G' is n and $m-k$, respectively. Substituting this in Eq. (2.1), the result follows. □

3. Laplacian Estrada Index of Some Cartesian Product

Let G be a graph with vertex set $V(G)$ and H be a graph with vertex set $V(H)$. The *Cartesian product* of G and H, denoted by $G \times H$ is a graph with vertex set $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \times H$ if and only if either $u_1 = u_2$ and v_1 is adjacent to v_2 in H or $v_1 = v_2$ and u_1 is adjacent to u_2 in G [19].

Theorem 3.1. [22] *If $\mu_1, \mu_2, \ldots, \mu_n$ are the Laplacian eigenvalues of a graph G, then the Laplacian eigenvalues of $G \times K_2$ are $\mu_1, \mu_2, \ldots, \mu_n$ and $\mu_1 + 2, \mu_2 + 2, \ldots, \mu_n + 2$.*

Theorem 3.2. *The Laplacian Estrada index of $G \times K_2$ is*

$$LEE(G \times K_2) = (1 + e^2) LEE(G). \tag{3.1}$$

Proof. By Theorem 3.1, we get

$$\begin{aligned} LEE(G \times K_2) &= \sum_{i=1}^{n} e^{\mu_i} + \sum_{i=1}^{n} e^{\mu_i + 2} \\ &= (1 + e^2) LEE(G). \end{aligned}$$

□

Theorem 3.3. [21] *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of a regular graph G of order n and of degree r, then its Laplacian eigenvalues are $r - \lambda_i$, $i = 1, 2, \ldots, n$.*

By Theorem 3.3, the Laplacian Estrada index of an r-regular graph of order n is [9]

$$LEE(G) = \sum_{i=1}^{n} e^{r - \lambda_i}, \tag{3.2}$$

where λ_i, $i = 1, 2, \ldots, n$, are the adjacency eigenvalues of G.

The *line graph* of G, denoted by $L(G)$, is the graph whose vertices corresponds to the edges of G and two vertices in $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G [19]. The k-th line graph of G is defined as $L^k(G) = L(L^{k-1}(G))$ where $L^0(G) \equiv G$ and $L^1(G) \equiv L(G)$. If G is a regular graph of order n_0 and of degree r_0, then $L(G)$ is a regular graph of order $n_1 = n_0 r_0 / 2$ and of degree $r_1 = 2r_0 - 2$. Consequently, the order and degree of $L^k(G)$ are [1,2]

$$n_k = \frac{1}{2} n_{k-1} r_{k-1} = \frac{n_0}{2^k} \prod_{i=0}^{k-1} r_i = \frac{n_0}{2^k} \prod_{i=0}^{k-1} (2^i r_0 - 2^{i+1} + 2)$$

and $r_k = 2r_{k-1} - 2 = 2^k r_0 - 2^{k+1} + 2$ respectively, where n_i and r_i stand for the order and degree of $L^i(G)$, $i = 0, 1, 2, \ldots$, respectively.

Theorem 3.4. [24] *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of a regular graph G of order n and of degree r, then the adjacency eigenvalues of $L(G)$ are*

$$\lambda_i + r - 2, \qquad i = 1, 2, \ldots, n \qquad \text{and}$$

$$-2, \qquad n(r-2)/2 \text{ times.}$$

Theorem 3.5. *If G is an r-regular graph of order n and of degree $r \geq 3$, then*

$$LEE(L(G) \times K_2) = (1 + e^2) \left[LEE(G) + \frac{n(r-2)e^{2r}}{2} \right].$$

Proof. If G is an r-regular graph, then $L(G)$ is a regular graph of degree $2r - 2$. Therefore by Theorems 3.3 and 3.4, the Laplacian eigenvalues of $L(G)$ are

$$r - \lambda_i, \qquad i = 1, 2, \ldots, n \qquad \text{and}$$

$$2r, \qquad n(r-2)/2 \text{ times}.$$

Therefore

$$LEE(L(G)) = \sum_{i=1}^{n} e^{r-\lambda_i} + \frac{n(r-2)e^{2r}}{2} = LEE(G) + \frac{n(r-2)e^{2r}}{2}. \tag{3.3}$$

Therefore by Theorem 3.2 and Eq. (3.3)

$$\begin{aligned} LEE(L(G) \times K_2) &= (1 + e^2) LEE(L(G)) \\ &= (1 + e^2) \left[LEE(G) + \frac{n(r-2)e^{2r}}{2} \right]. \end{aligned}$$

□

In [9], the following result was reported.

Theorem 3.6. [9] *If G is an r-regular graph on n vertices, then for $k = 0, 1, \ldots$*

$$LEE(L^{k+1}(G)) = LEE(L^k(G)) + \frac{n_k(r_k - 2)e^{2r_k}}{2},$$

where

$$r_k = (r-2)2^k + 2 \quad \text{and} \quad n_k = \frac{n}{2^k} \prod_{i=0}^{k-1} (2^i r - 2^{i-1} + 2).$$

Using Thereoms 3.2 and 3.6 we have following result.

Theorem 3.7. *If G is an r-regular graph on n vertices, then for $k = 0, 1, \ldots$*

$$LEE(L^{k+1}(G) \times K_2) = (1 + e^2) \left[LEE(L^k(G)) + \frac{n_k(r_k - 2)e^{2r_k}}{2} \right],$$

where

$$r_k = (r-2)2^k + 2 \quad \text{and} \quad n_k = \frac{n}{2^k} \prod_{i=0}^{k-1} (2^i r - 2^{i-1} + 2).$$

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