Kragujevac Journal of Mathematics
Volume 50(5) (2026), Pages 683–724.

EXISTENCE, UNIQUENESS AND CONTROLLABILITY RESULTS FOR FRACTIONAL NEUTRAL INTEGRO-DIFFERENTIAL EQUATIONS WITH NON-INSTANTANEOUS IMPULSES AND DELAY

K. MALAR[1] AND R. ILAVARASI[1]

Abstract. In this paper, we prove the existence, uniqueness and controllability results for fractional neutral integro-differential equation and non-instantaneous impulses in Banach spaces. To obtain the existence and controllability results, we have enforced the concepts of fractional calculus and fixed point theorems. Examples are also given to illustrate the results.

1. Introduction

The theory of fractional differential and integral equations have been demonstrated to be important apparatuses and successful within the modeling of numerous marvels in different areas of building and logical disciplines such as material science, chemistry, science, control theory, flag and picture preparing, blood stream wonders, optimal design and so on. Fractional derivatives give an fabulous instrument for the portrayal of memory and innate properties of different materials and processes. The investigation of both qualitative and quantitative properties of solutions to fractional differential equations is an active and ongoing area of research. For more information on the theory of fractional calculus, one can refer to the monographs of Kilbas et al. [23], Lakshmikanthan et al. [25], Miller and Rose [27] and Podlubny [34], Baleanu et al. [4], as well as the papers by [6, 7, 9, 18–21, 38, 39] along with the reference cited therein.

Key words and phrases. Fractional differential equations, non-instantaneous impulses, measure of noncompactness, fixed point theorem, infinite delay.

2010*Mathematical Subject Classification*: Primary: 26A33, 34A08, 34K37, 47H08.

DOI

Received: August 27, 2023.

Accepted: April 31, 2024.

Impulsive functional differential and integro-differential systems are particularly advantageous in modeling processes and phenomena that undergo short-time disturbances at the time their evolution. Impulsive differential equations of integer order have been broadly applications in reasonable scientific modeling for a wide extend of commonsense circumstances, counting organic wonders with edges, bursting beat models in medication and science, ideal control models in financial matters, and recurrence tweaked frameworks. For a comprehensive understanding of impulsive differential equations, including relevent developments, [26] and the references therein. Ordinarily, the impulses within the advancement handle depicted by impulsive differential equations are expected to be unexpected and instantaneous. In other words, the annoyances (impulses) begin suddenly and their term is irrelevant in compared to the overall term of the method. However, Hernández et al. [18] introduced the concept of non-instantaneous impulses where the impulses start abruptly at the points t_k and their action continues over a finite time interval $[t_k, s_k]$. This speaks to a circumstance impulsive action that begin suddenly and remains dynamic for a limited period of time. Pierri et al. [35] examined the existence of solutions for a class of first order semilinear abstract impulsive differential equations with non-instantaneous impulses utilizing the hypothesis of analytic semigroup and fractional power of closed operators. Gabeleh et al. [16] investigating a new survey of the theory of measure of noncompactness and their applications. Wang et al. [42] studied the concept of a PC-mild solution to a general new class of noninstantaneous impulsive fractional differential inclusions involving the generalized Caputo derivative with the lower bound at zero in infinite dimensional Banach spaces. One can refer to that references for further details [1–3, 5, 40].

Neutral differential equations arise in many areas of applied mathematics and have received significant attention in recent decades. Good references for ordinary neutral functional differential equations include the books by Graef et al. [15], Benchohra et al. [5], Lakshmikantham et al. [26] and the reference cited therein. Integro-differential equations are imperative for examining issues emerging from common wonders and have been examined from different points of view. In later a long time, this hypothesis has been connected to a wide course of non-linear differential equations in Banach spaces. For more data, see the references therein [11, 12].

Meraj and Pandey [30] examined the existence of mild solutions for fractional non-instantaneous impulsive integro-differential equations with nonlocal conditions by utilizing noncompact semigroup hypothesis and fixed point theorem.

In recent years, fractional calculus has brought almost modern viewpoints within the field of control theory. The primary challenges in control theory, such as post task, stabilization, and ideal control, can be tended to by accepting that the framework is controllable. The concept of controllability was first introduced by Kalmen in 1960 and has been extensively studied. Controllability could be a significant characteristic, both in terms of quantity and quality, in control systems and plays a pivotal role in various control problems, including those in finite and infinite-dimensional spaces.

In recent times, the controllability of fractional dynamical systems has risen as a profoundly dynamic range inside this field. The controllability of linear systems in finite dimensional spaces has been broadly inspected in [24] and controllability of fractional evolution dynamical systems in a dimensional space has been talked about in works such as, [6, 7, 10, 14, 36, 41].

Ji et al. [22] considered the controllability of impulsive differential systems with nonlocal conditions by using Mönch's fixed point technique. Wang et al. [42] established the adequate conditions for nonlocal controllability for fractional evolution systems and the comes about were gotten by utilizing fractional calculus and Mönch's fixed point theorem. Meraj and Pandey [31] considered the existence of mild solutions and approximate controllability for a class of fractional semilinear integrodifferential equations with nonlocal and impulsive conditions where the impulses are not instantaneous. They employed semigroup theory and fixed point theorems to analyze this problem.

This paper deals with the Fractional Neutral Integro-Differential Equations and Non-Instantaneous impulses with infinite delay

$$ {}^{C}D_t^r[u(t) - \mathcal{G}(t, u_t)] = \mathfrak{A}u(t) + \mathscr{F}\Big(t, u_t, \int_0^t \mathcal{H}(t, s, u_s)ds\Big), \quad t \in (s_k, t_{k+1}], \quad k = 0, 1, 2, \dots, m, \tag{1.1} $$

$$ u(t) = \mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(t, u_t), \quad t \in (t_k, s_k], \quad k = 1, 2, \dots, m, \tag{1.2} $$

$$ u_0 = \Phi \in \mathscr{B}_h, \quad t \in (-\infty, 0]. \tag{1.3} $$

where ${}^{C}D_t^r$ is the Caputo fractional derivative of order $r \in (0, 1)$ and $\mathcal{J} = [0, \mathcal{T}]$. The operator $\mathfrak{A}$ denotes the infinitesimal generator of an analytic semigroup $\{\mathcal{Q}(t)\}_{t\geq 0}$ in a Banach space $\mathcal{X}$ having norm $\|\cdot\|$, this suggests that we can find $\mathscr{M}_A \geq 1$ to ensure that $\|\mathcal{Q}(t)\| \leq \mathscr{M}_A$, $\mathscr{F} : \mathcal{J} \times \mathscr{B}_h \times \mathcal{X} \to \mathcal{X}$, $\mathcal{G} : \mathcal{J} \times \mathscr{B}_h \to \mathcal{X}$, $\mathcal{H} : \mathfrak{D} \times \mathscr{B}_h \to \mathcal{X}$ are given functions satisfying certain assumptions, $\mathscr{G}_k : (t_k, s_k] \times \mathcal{X} \to \mathcal{X}$, $\mathfrak{I}_k : \mathcal{X} \to \mathcal{X}$ for $k = 1, 2, \dots, m$. $\mathscr{B}_h$ is a phase space characterised in preliminaries. Here $\mathfrak{D} = \{(t, s) \in \mathcal{J} \times \mathcal{J} : 0 \leq s \leq t \leq \mathcal{T}\}$, $0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < \cdots < t_m \leq s_m < t_{m+1} = \mathcal{T}$ are fixed numbers.

The impulses in problem (1.1)–(1.3) start abruptly at the points t_k and their action continues on the interval $[t_k, s_k]$. To be precise, the function u takes an abrupt impulse at t_k and follows different rules in the two subintervals $(t_k, s_k]$ and $(s_k, t_{k+1}]$ of the interval $(t_k, t_{k+1}]$. At the point s_k, the function u is continuous. The term $\mathfrak{I}_k(u(t_k))$ means that the impulses are also related to the value of $u(t_k) = u(t_k^-)$.

We remark that if $t_k = s_k$ and the second equation of (1.1)–(1.3) takes the form of $\Delta u(t_k) = \mathfrak{I}_k(u(t_k)) = u(t_k^+) - u(t_k^-)$ with $u(t_k^+) = \lim_{\varepsilon\to 0^+} u(t_k + \varepsilon)$, $u(t_k^-) = \lim_{\varepsilon\to 0^-} u(t_k - \varepsilon)$ representing the right and left limits of $u(t)$ at $t = t_k$.

For almost every continuous function u defined on $(-\infty, \mathcal{T}]$ and for almost every $t \geq 0$, we designate by u_t the part of $\mathscr{B}_h$ characterized by $u_t(\theta) = u(t + \theta)$ for $\theta \leq 0$. Now $u_t(\cdot)$ refers to the historical backdrop of the state from every $\theta \in (-\infty, 0]$ like the current time t.

Motivated by the above mentioned works, the main aim of this paper is to establish the existence and controllability of impulsive fractional neutral integro-differential system and non-instantaneous impulse with infinite delay using the new definition of the phase space and fixed point theorem of Mӧnch's and the technique of Hausdorff measure of noncompactness. To best of our knowledge there is some new results in this paper.

The paper is organized as follows. In Section 2, we recall some basic definitions, notations and preliminary facts. In Section 3, the existence and uniqueness results for equation (1.1)–(1.3) using fixed point theorems. In Section 4, the controllability results for fractional neutral integro-differential equation and non-intantaneous impulses with delay. In Section 5, we have examples to demonstrate the obtained results.

2. Preliminaries

In this section, we mention notations, definitions, lemmas and preliminary facts needed to establish our main results.

Let $\mathcal{L}(\mathfrak{X}) : \mathfrak{X} \to \mathfrak{X}$ represents the Banach space of all bounded linear operators, obtain its norm recognized as $\| \cdot \|_{\mathcal{L}(\mathfrak{X})}$.

Let $C(\mathfrak{I}, \mathfrak{X})$ symbolize the space of all continuous functions from $\mathfrak{I}$ into $\mathfrak{X}$, having norm $\| \cdot \|_{C(\mathfrak{I},\mathfrak{X})}$. Moreover, $B_r(u, \mathfrak{X})$ represents the closed ball in $\mathfrak{X}$ with the middle at u and the distance r.

We recall that a measurable function $u : \mathfrak{I} \to \mathfrak{X}$ is Bochner integrable if and only if $\|u\|$ is Lebesgue integrable. To get extra insights as regards the Bochner integral, refer to the treatise of Yosida [45].

Permit $\mathcal{L}^1(\mathfrak{I}, \mathfrak{X})$ signifies the Banach space of all measurable functions $u : \mathfrak{I} \to \mathfrak{X}$ which are Bochner integrable and have the norm

$$\|u\|_{L^1} = \int_0^{\mathfrak{T}} \|u(t)\| dt, \quad \text{for all } u \in L^1(\mathfrak{I}, \mathfrak{X}).$$

Definition 2.1 ([37])**.** Let $\mathfrak{A} : \mathfrak{D} \subseteq \mathfrak{X} \to \mathfrak{X}$ be a closed linear operator. The operator $\mathfrak{A}$ is considered to be sectorial if we can find $0 < \theta < \frac{\pi}{2}$, $\mathscr{M} > 0$, and $\mu \in \mathbb{R}$ in such a way that the $\rho(\mathfrak{A})$ exists exterior of the segment

$$\mu + S_\theta = \Big\{\mu + \lambda : \lambda \in \tilde{C}, \| \arg(-\lambda)\| < \theta\Big\},$$

$$\Big\|(\lambda I - \mathfrak{A})^{-1}\Big\| \leq \frac{\mathscr{M}}{\lambda - \mu}, \quad \lambda \notin \mu + S_\theta.$$

For short, we say that $\mathfrak{A}$ is sectorial of type $(\mathscr{M}, \theta, \mu)$.

Let $\mathfrak{A}$ defines the infinitesimal generator of an anlyatic semigroup in a Banach space and $0 \in \rho(\mathfrak{A})$, where $\rho(\mathfrak{A})$ is the resolvent set of $\mathfrak{A}$. We characterize the fractional power $\mathfrak{A}^q$ for $0 < q \leq 1$, as a closed linear operator on its domain $\mathfrak{D}(\mathfrak{A}^q)$ with inverse $\mathfrak{A}^{-q}$ (see [33]). The following are basic properties of $\mathfrak{A}^q$.

(i) $\mathfrak{D}(\mathfrak{A}^q)$ is a Banach space with the norm $\|u\|_q = \|\mathfrak{A}^q u\|$ for $u \in \mathfrak{D}(\mathfrak{A}^q)$.

(ii) $\mathcal{Q}(t) : \mathcal{X} \to \mathcal{X}_q$ for $t \geq 0$.
(iii) $\mathfrak{A}^q \mathcal{Q}(t)u = \mathcal{Q}(t)\mathfrak{A}^q u$ for each $u \in \mathfrak{D}(\mathfrak{A}^q)$ and $t \geq 0$.
(iv) For every $t > 0$, $\mathfrak{A}^q \mathcal{Q}(t)$ is bounded on $\mathcal{X}$ and there exists $\mathscr{M}_q > 0$ such that $\|\mathfrak{A}^q \mathcal{Q}(t)\| \leq \mathscr{M}_q t^q e^{-\delta t}$.
(v) For $0 < q \leq 1$ and $u \in \mathfrak{D}(\mathfrak{A}^q)$, we obtain $\|\mathcal{Q}(t)u - u\| \leq C_q t^q \|\mathfrak{A}^q u\|$.

We denote by $PC([0, \mathcal{T}], \mathcal{X})$ the space of piecewise continuous function from $[0, \mathcal{T}]$ into $\mathcal{X}$. $g : PC(\mathcal{J}, \mathcal{X}) \to \mathcal{X}$ are given functions satisfying certain assumptions. In particular, we introduce the space PC formed by all piecewise continuous function $u : [0, \mathcal{T}] \to \mathcal{X}$ such that $u(\cdot)$ is continuous at $t \neq t_i$, $u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists for $i = 1, 2, \ldots, m$. We assume that PC is a Banach space, endowed with the norm $\|u\|_{PC} = \sup_{s\in[0,\mathcal{T}]} \|u(s)\|_{PC}$. It is clear that $(PC, \|\cdot\|_{PC})$ is a Banach space. $PC((0, \mathcal{T}], \mathcal{X}) = \{u : (0, \mathcal{T}] \to \mathcal{X}$ such that $u_i \in C((t_i, t_{i+1}], \mathcal{X})$, $i = 0, 1, 2, \ldots, m$ and there exist $u(t_i^+)$ and $u(t_i^-)$ with norm $u(t_i) = u(t_i^-)$, $i = 0, 1, 2, \ldots, m\}$. We define $C_L(\mathcal{J}, \mathcal{X}) = \{v \in PC((0, \mathcal{T}], \mathcal{X}) : \|v(t) - v(s)\| \leq \tilde{I}|t - s|$ for all $t, s \in [0, \mathcal{T}]\}$, where $\tilde{I}$ is some positive constant, is a Banach space endowed with piecewise norm. It should be fixed that, once the delay is infinite, then we need to discuss about the theoretical phase space $\mathscr{B}_h$ in a useful way. In this we consider the phase spaces $\mathscr{B}_h, \mathscr{B}_h'$ which are same as described in [13].

We present the abstract phase space $\mathscr{B}_h$. Suppose $\mathcal{H} : (-\infty, 0] \to (0, +\infty)$ is a continuous function with $l = \int_{-\infty}^0 \mathcal{H}(t)dt < +\infty$ and for any $a > 0$, we define $\mathscr{B} = \{\psi : [-a, 0] \to \mathcal{X}$ such that $\psi(t)$ is bounded and measurable$\}$ and equip the space $\mathscr{B}$ with the norm $\|\psi\|_{[-a,0]} = \sup_{s\in[-a,0]} \|\psi(s)\|$ and $\psi \in \mathscr{B}$. Let us define $\mathscr{B}_h = \{\psi : (-\infty, 0] \to \mathcal{X}$ for any $c > 0$, $\psi|_{[-c,0]} \in \mathscr{B}$ and $\int_{-\infty}^0 \mathcal{H}(s)\|\psi\|_{[s,0]}ds < +\infty\}$. If $\mathscr{B}_h$ is endowed with the norm $\|\psi\|_{\mathscr{B}_h} = \int_{-\infty}^0 \mathcal{H}(s)\|\psi\|_{[s,0]}ds$, for every $\psi \in \mathscr{B}_h$, then it is clear that $(\mathscr{B}_h, \|\cdot\|_{\mathscr{B}_h})$ is a Banach space. Now we consider the space $\mathscr{B}_h' = PC((-\infty, \mathcal{T}], \mathcal{X}) = \{u : (-\infty, \mathcal{T}] \to \mathcal{X}$ such that $u|_{\mathcal{J}} \in C(\mathcal{J}_i, \mathcal{X})$ and there exist $u(t_i^+)$ and $u(t_i^-)$ with $u(t_i) = u(t_i^-)$, $u_0 = \phi \in \mathscr{B}_h$, $i = 0, 1, 2, \ldots, m\}$, where u_i is the restriction of u to $\mathcal{J}_i = (t_i, t_{i+1}]$, set $\|\cdot\|_{\mathscr{B}_h'}$ be the seminorm in $\mathscr{B}_h'$ defined by $\|u\|_{\mathscr{B}_h'} = \|\phi\|_{\mathscr{B}_h} + \sup\{\|u(s)\| : s \in [0, \mathcal{T}]\}$, $u \in \mathscr{B}_h'$.

We count on that the phase space $(\mathscr{B}_h, \|\cdot\|_{\mathscr{B}_h})$ could be a semi-normed linear area of function mapping $(-\infty, 0]$ into $\mathcal{X}$, and enjoyable the next elementary adages as a results of Hale and Kato (see case in purpose in [17, 20]).

If u is continuous function from $(-\infty, \mathcal{T}]$, $\mathcal{T} > 0$, into $\mathcal{X}$, defined on $\mathcal{J}$ and $u_0 \in \mathscr{B}_h$, then for every $t \in \mathcal{J}$ the following situations preserve.

(J1) u_t is in $\mathscr{B}_h$.
(J2) $\|u(t)\|_{\mathcal{X}} \leq \tilde{\mathscr{H}} \|u_t\|_{\mathscr{B}_h}$.
(J3) $\|u_t\|_{\mathscr{B}_h} \leq \mathcal{D}_1(t) \sup\{\|u(s)\|_{\mathcal{X}} : 0 \leq s \leq t\} + \mathcal{D}_2(t)\|u_0\|_{\mathscr{B}_h}$, where $\tilde{\mathscr{H}} > 0$ is a constant and $\mathcal{D}_1(\cdot) : [0, +\infty) \to [0, +\infty)$ is continuous, $\mathcal{D}_2(\cdot) : [0, +\infty) \to [0, +\infty)$ is locally bounded, and $\mathcal{D}_1$, $\mathcal{D}_2$ are independent of $u(\cdot)$. For our convenience, denote $\mathcal{D}_1^* = \sup_{s\in\mathcal{J}} \mathcal{D}_1(s)$, $\mathcal{D}_2^* = \sup_{s\in\mathcal{J}} \mathcal{D}_2(s)$.

Let us recall the following known definitions. For more details see [4, 23, 27].

Definition 2.2 ([23])**.** The fractional integral of order r with the lower limit zero for a function f is defined as

$$\mathscr{I}^r f(t) = \frac{1}{\Gamma(r)} \int_0^t \frac{f(s)}{(t-s)^{1-r}} ds, \quad t > 0, r > 0,$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.3 ([23])**.** The Riemann-Liouville derivative of order r with the lower limit zero for a function $f : [0, +\infty) \to \mathbb{R}$ can be written as

$${}^L D^r f(t) = \frac{1}{\Gamma(n-r)} \left(\frac{d^n}{dt^n}\right) \int_0^t (t-s)^{n-r-1} f(s) ds, \quad n-1 < r < n, t > 0.$$

Definition 2.4 ([23, 34])**.** The Caputo derivative of order r for a function $f : [0, +\infty) \to \mathbb{R}$ can be written as

$${}^C D^r f(t) = {}^L D^r \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, n-1 < r < n.$$

Remark 2.1 ([23]). (1) If $f(t) \in C^n[0, +\infty)$, then

$${}^C D^r f(t) = \frac{1}{\Gamma(n-r)} \int_0^t (t-s)^{n-r-1} f^{(n)} ds = \mathscr{I}^{n-r} f^{(n)}(t), \quad t > 0.$$

(2) The Caputo derivative of a constant is equal to zero.

(3) If f is an abstract function with values in $\mathcal{X}$, then integrals which appear in Definitions 2.2 and 2.3 are taken in Bochner's sense.

Definition 2.5. A function $u \in C(\mathcal{I}, \mathcal{X})$ is said to be a mild solution of the following problem:

$$\begin{cases} {}^C D^r u(t) = \mathfrak{A} u(t) + v(t), & t \in (0, \mathcal{T}], \\ u(0) = u_0, \end{cases}$$

if it satisfies the integral equation

$$u(t) = \mathcal{Q}_r(t) u_0 + \int_0^t (t-s)^{r-1} \mathcal{P}_r(t-s) v(s) ds.$$

Here

$$\mathcal{Q}_r(t) = \int_0^{+\infty} \wp_r(\theta) T(t^r \theta) d\theta, \quad \mathcal{P}_r(t) = r \int_0^{+\infty} \theta \wp_r(\theta) T(t^r \theta) d\theta,$$

$$\wp_r(\theta) = \frac{1}{r} \theta^{-1-\frac{1}{r}} \sigma_r\left(\theta^{-\frac{1}{r}}\right) \geq 0,$$

$$\sigma_r(\theta) = \frac{1}{\pi} \sum_{n=1}^{+\infty} (-1)^{n-1} \theta^{-nr-1} \frac{\Gamma(nr+1)}{n!} \sin(n\pi r), \quad \theta \in (0, +\infty),$$

and $\wp_r$ is a probability density function defined on $(0, +\infty)$, that is,

$$\wp_r(\theta) \geq 0, \quad \theta \in (0, +\infty), \quad \int_0^{+\infty} \wp_r(\theta) d\theta = 1.$$

It is not difficult to verify that

$$\int_0^{+\infty} \theta \wp_r(\theta) d\theta = \frac{1}{\Gamma(1+r)}.$$

We make the following assumption H(A1) in the whole paper.

H(A1): The operator $\mathfrak{A}$ generators a strongly continuous semigroup $\{\mathcal{Q}_r(t) : t \geq 0\}$ in $\mathcal{X}$, and there is a constant, $\mathscr{M}_A \geq 1$ such that $\sup_{t\in[0,\infty)} \|\mathcal{Q}_r(t)\|_{L(\mathcal{X})} \leq \mathscr{M}_A$. For any $t > 0$, $\mathcal{Q}_r(t)$ is compact.

Lemma 2.1 ([41, 46]). *Let* H(A1) *hold. Then, the operators* $\mathcal{Q}_r$ *and* $\mathcal{P}_r$ *have the following properties.*

(1) *For any fixed* $t \geq 0$, $\mathcal{Q}_r(t)$ *and* $\mathcal{P}_r(t)$ *are linear and bounded operators, and for any* $u \in \mathcal{X}$,

$$\left\|\mathcal{Q}_r(t)u\right\| \leq \mathscr{M}_A\|u\|, \quad \left\|\mathcal{P}_r(t)u\right\| \leq \frac{r\mathscr{M}_A}{\Gamma(1+r)}\|u\|.$$

(2) $\{\mathcal{Q}_r(t) : t \geq 0\}$ *and* $\{\mathcal{P}_r(t) : t \geq 0\}$ *are strongly continuous.*
(3) *For every* $t > 0$, $\mathcal{Q}_r(t)$ *and* $\mathcal{P}_r(t)$ *are compact operators.*

We define the following definiton of the mild solution for the problem (1.1)–(1.3).

Definition 2.6. A function $u \in PC(\mathcal{J}, \mathcal{X})$ is said to be a PC mild solution of problem (1.1)–(1.3) if it satisfies the following equation:

$$u(t) = \begin{cases} \mathcal{Q}_r(t)[\Phi(0) + \mathcal{G}(0, \Phi(0))] - \mathcal{G}(t, u(t)) + \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, u_s)ds \\ + \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\left(s, u_s, \int_0^s \mathcal{H}(s,\tau,u_\tau)d\tau\right) ds, \quad t \in [0, t_1], \\ k = 0, 1, \ldots, m, \\ \mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(t, u_t), \quad t \in (t_k, s_k], \quad k = 1, 2, \ldots, m, \\ \mathcal{Q}_r(t-s_k)\mathscr{D}_k - \mathcal{G}(t, u(t)) + \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, u_s)ds \\ + \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\left(s, u_s, \int_0^s \mathcal{H}(s,\tau,u_\tau)d\tau\right) ds, \quad t \in (s_k, t_{k+1}], \\ k = 1, 2, \ldots, m, \end{cases}$$

where

$$\begin{aligned} \mathscr{D}_k =& \mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(s_k, u(s_k)) + \mathcal{G}(s_k, u(s_k)) \\ &- \int_0^{s_k} (s_k - s)^{r-1}\mathfrak{A}\mathcal{P}_r(s_k - s)\mathcal{G}(s, u(s_k))ds \\ &- \int_0^{s_k} (s_k - s)^{r-1}\mathcal{P}_r(s_k - s)\mathscr{F}\Big(s, u_s, \int_0^s \mathcal{H}(s,\tau,u_\tau)d\tau\Big)ds, \quad k = 1, 2, \ldots, m. \end{aligned} \tag{2.1}$$

Now, we introduce the Hausdorff measure of noncompactness $\hbar_{\mathcal{Y}}$ defined by

$$\hbar_{\mathcal{Y}}(\mathcal{B}) = \inf\{\epsilon > 0 : \mathcal{B} \text{ has a finite } \epsilon\text{-net in } \mathcal{X}\},$$

for a bounded set $\mathcal{B}$ in any Banach space $\mathcal{Y}$. Some basic properties of $\hbar_{\mathcal{Y}}(\cdot)$ are given in the following definition and lemmas.

Definition 2.7 ([8])**.** Let $\mathcal{Y}$ be a real Banach space and $\mathcal{B}, C \subseteq \mathcal{Y}$ be bounded, and the following properties are satisfied:

(1) $\mathcal{B}$ is pre-compact if and only if $\hbar_{\mathcal{Y}}(\mathcal{B}) = 0$;
(2) $\hbar_{\mathcal{Y}}(\mathcal{B}) = \hbar_{\mathcal{Y}}(\bar{\mathcal{B}}) = \hbar\mathcal{Y}(\operatorname{conv}\mathcal{B})$, where $\bar{\mathcal{B}}$ and $\operatorname{conv}\mathcal{B}$ are the closure and the convex hull of $\mathcal{B}$, respectively;
(3) $\hbar_{\mathcal{Y}}(\mathcal{B}) \leq \hbar_{\mathcal{Y}}(C)$, when $\mathcal{B} \subseteq C$;
(4) $\hbar_{\mathcal{Y}}(\mathcal{B} + C) \leq \hbar_{\mathcal{Y}}(\mathcal{B}) + \hbar_{\mathcal{Y}} C$, where $\mathcal{B} + C = \{x + y : x \in \mathcal{B}, y \in C\}$;
(5) $\hbar_{\mathcal{Y}}(\mathcal{B} \cup C) = \max\{\hbar_{\mathcal{Y}}(\mathcal{B}), \hbar_{\mathcal{Y}}(C)\}$;
(6) $\hbar_{\mathcal{Y}}(\lambda\mathcal{B}) \leq |\lambda|_{\hbar_{\mathcal{Y}}}(\mathcal{B})$ for any $\lambda \in \mathbb{R}$;
(7) if the map $\Phi : \mathcal{D}(\Phi) \subseteq \mathcal{Y} \to \mathcal{Z}$ is Lipschitz continuous with constant κ then $\hbar_{\mathcal{Y}}(\Phi\mathcal{B}) \leq \kappa_{\hbar_{\mathcal{Y}}}(\mathcal{B})$ for any bounded subset $\mathcal{B} \subseteq \mathcal{D}(\Phi)$, where $\mathcal{Z}$ is a Banach space;
(8) if $\{\mathcal{W}_n\}_{n=1}^{+\infty}$ is a decreasing sequence of bounded closed nonempty subset of $\mathcal{Y}$ and $\lim_{n\to+\infty\hbar_{\mathcal{Y}}}(\mathcal{W}_n) = 0$, then $\cap_{n=1}^{+\infty}\mathcal{W}_n$ is nonempty and compact in $\mathcal{Y}$.

Lemma 2.2 ([8])**.** *If $\mathcal{W} \subset C([a,b],\mathcal{X})$ is bounded and equicontinuous, then $\hbar(\mathcal{W}(t))$ is continuous for $t \in [a,b]$ and $\hbar(\mathcal{W}) = \sup\{\hbar(\mathcal{W}(t)) : t \in [a,b]\}$, where $\mathcal{W}(t) = \{u(t) : u \in \mathcal{W}\} \subseteq \mathcal{X}$.*

Theorem 2.1 ([32, 41])**.** *If $\{x_n\}_{n=1}^{+\infty}$ is a sequence of Bochner integrable functions from $\mathcal{I}$ into $\mathcal{X}$ with the estimation $\|x_n(t)\| \leq \mu(t)$ for almost all $t \in \mathcal{I}$ and every $n \geq 1$, where $\mu \in L^1(\mathcal{I},\mathbb{R})$, then the function $\chi(t) = \hbar(\{x_n(t) : n \geq 1\})$ belongs to $L^1(\mathcal{I},\mathbb{R})$ and satisfies $\hbar(\{\int_0^t \chi(s)ds : n \geq 1\} \leq 2\int_0^t \chi(s)ds$.*

Lemma 2.3 ([8] Darbo-Sadovskii)**.** *If $\mathcal{W} \subseteq \mathcal{Y}$ is bounded, closed and convex, the continuous map $\mathcal{F} : \mathcal{W} \to \mathcal{W}$ is an $\hbar$-contraction, then the map $\mathcal{F}$ has at least one fixed point in $\mathcal{W}$.*

The following fixed point thoerem, a nonlinear alternative of Mönch's type, plays a key role in our proof of system (1.1)–(1.3).

Lemma 2.4 ([28], Theorem 2.2)**.** *Let $\mathcal{D}$ be a closed convex subset of a Banach space $\mathcal{X}$ and $0 \in \mathcal{D}$. Assume that $\mathcal{F} : \mathcal{D} \to \mathcal{X}$ is a continuous map which satisfies Mönch's condition, that is ($M \subseteq \mathcal{D}$ is countable, $M \subseteq \bar{co}(\{0\} \cup \mathcal{F}(M))$ implies $\bar{M}$ is compact). Then, $\mathcal{F}$ has a fixed point in $\mathcal{D}$.*

3. Main Results

In this section, we present and prove the existence results for problem (1.1)–(1.3). In order to prove the main theorem of this section, we assume the following hypotheses.

H(A2): (i) A genertates a strongly continuous semigroup $\{\mathcal{Q}_r(t) : t \geq 0\}$ in $\mathcal{X}$.
(ii) For all bounded subsets $\mathcal{D} \subset \mathcal{X}$ and $u \in \mathcal{D}$, $\|\mathcal{Q}_r(t_2^r\theta)u - \mathcal{Q}_r(t_1^r\theta)u\| \to 0$ as $t_1 \to t_2$ for each fixed $\theta \in (0,+\infty)$.

H(A3): The function $\mathcal{G} : \mathcal{I}\times\mathscr{B}_h \to \mathcal{X}$ is continuous and we can find constants $\beta \in (0,1)$. $\mathscr{C}_1 > 0$, and $\mathscr{C}_2 > 0$ in such a way that $\mathcal{G}$ is $\mathcal{X}_\beta$-valued and fulfills the subsequent and assumptions:

$$\Big\|\mathfrak{A}^\beta \mathcal{G}(t,u_1) - \mathfrak{A}^\beta \mathscr{G}(t,u_2)\Big\|_{\mathcal{X}} \le \mathscr{C}_1\Big\|u_1-u_2\Big\|_{\mathscr{B}_h}, \quad t\in\mathcal{I}, u_1,u_2\in\mathscr{B}_h,$$

$$\Big\|\mathfrak{A}^\beta \mathcal{G}(t,u)\Big\|_{\mathcal{X}} \le \mathscr{C}_1\|u\|_{\mathscr{B}_h} + \mathscr{C}_2, \quad t\in\mathcal{I}, u\in\mathscr{B}_h.$$

H(A4): The function $\mathcal{H} : \mathcal{D}\times\mathscr{B}_h \to \mathcal{X}$ satisfies the following.

(i) For every $(t,s)\in\mathcal{D}$, the function $\mathcal{H}(t,s,\cdot) : \mathscr{B}_h \to \mathcal{X}$ is continuous and for each $u\in\mathscr{B}_h$, the function $\mathcal{H}(\cdot,\cdot,u) : \mathcal{D}\to\mathcal{X}$ is strongly measurable.

(ii) There exist a function $\nu \in L^1(\mathcal{I},\mathbb{R}^+)$ and a continuous non-decreasing function $\omega : \mathbb{R}^+ \to (0,\infty)$ to ensure that

$$\|\mathcal{H}(t,s,u)\|_{\mathcal{X}} \le \nu(s)\omega(\|u\|_{\mathscr{B}_h}), \quad \text{for a.e. } t,s\in\mathcal{I}, u\in\mathscr{B}_h.$$

(iii) There exists $\Theta\in L^1(\mathcal{I}\times\mathcal{I},\mathbb{R}^+)$ to ensure that

$$\hbar\left(\mathcal{H}(t,s,D)\right) \le \zeta(t,s)\Big[\sup_{-\infty<\theta\le 0} \hbar(D(\theta))\Big], \quad \text{for a.e. } t,s\in\mathcal{I},$$

where $D(\theta) = \{x(\theta) : x\in\mathcal{X}\}$ and $\hbar$ is the Hausdorff measures of non-compactness.

H(A5): The function $\mathscr{F} : \mathcal{I}\times\mathscr{B}_h\times\mathcal{X}\to\mathcal{X}$ satisfies the following.

(i) For a.e. $t\in\mathcal{X}$, $(\phi,u)\mapsto\mathscr{F}(t,\phi,u)$ is continuous and for all $(\phi,u)\in\mathscr{B}_h\times\mathcal{X}$, $t\mapsto\mathscr{F}(t,\phi,u)$ is strongly measurable.

(ii) There exists a function $m\in L^1(\mathcal{I},\mathbb{R}^+)$ and a continuous non-decreasing function $\Omega : \mathbb{R}^+\to(0,+\infty)$ to ensure that

$$\Big\|\mathscr{F}(t,\phi,u)\Big\|_{\mathcal{X}} \le m(t)\Omega(\|\phi\|_{\mathscr{B}_h} + \|u\|), \quad (t,\phi,u)\in\mathcal{X}\times\mathscr{B}_h\times\mathcal{X}.$$

(iii) For every bounded sets $D\subset\mathscr{B}_h$, $F^*\subset\mathcal{X}$, there exists a positive function $\eta\in L^1(\mathcal{X},\mathbb{R}^+)$ is such a way that

$$\hbar\Big(\mathscr{F}(t,D,F^*)\Big) \le \eta(t)\Big[\sup_{-\infty<\theta\le 0}\hbar(D(\theta)) + \hbar(F^*)\Big], \quad \text{for a.e. } t\in\mathcal{I},$$

where $D(\theta) = \{v(\theta) : v\in D\}$.

H(A6): The function $\mathscr{G}_k : (t_k,s_k]\times\mathscr{B}_h\to\mathcal{X}$, $k=1,2,\dots,m$ are continuous, and satisfies the following conditions.

(i) There exist constants $\mathcal{C}_i,\bar{\mathcal{C}}_i > 0$, $i=1,2,\dots,m$, in such a way that

$$\Big\|\mathscr{G}_k(t,\phi)\Big\|_{\mathcal{X}} \le \mathcal{C}_i\|\phi\|_{\mathscr{B}_h} + \bar{\mathcal{C}}_i, \quad t\in(t_k,s_k], \phi\in\mathscr{B}_h.$$

(ii) There exists constants $\tilde{\nu}_i > 0$ such that, for each bounded $D\subset\mathscr{B}_h$.

$$\hbar(\mathscr{G}_k(t,D)) \le \tilde{\nu}_i\Big[\sup_{-\infty<\theta\le 0}\hbar(D(\theta))\Big], \quad \text{for a.e. } t\in(t_k,s_k], k=1,2,\dots,m,$$

where $D(\theta) = \{x(\theta) : x\in D\}$.

H(A7): For $k = 1, 2, \ldots, m$, $\mathfrak{I}_K \in C(\mathfrak{X}, \mathfrak{X})$ and there is a constant $\mathcal{L}_f > 0$ such that

$$\left\|\mathfrak{I}_k(u) - \mathfrak{I}_k(v)\right\| \leq \mathcal{L}_f \|u - v\|, \quad \text{for all } u, v \in \mathfrak{X}.$$

H(A8): For $k = 1, 2, \ldots, m$, the functions $\mathscr{G}_k \in C([t_k, s_k] \times \mathfrak{X}; \mathfrak{X})$ and there exists $\mathcal{L}_g \in C(\mathfrak{I}, \mathbb{R}^+)$ such that

$$\left\|\mathscr{G}_k(t, u) - \mathscr{G}_k(t, v)\right\| \leq \mathcal{L}_g(t) \|u - v\|, \quad \text{for all } u, v \in \mathfrak{X} \text{ and } t \in [t_k, s_k].$$

H(A9): For every bounded set $\chi \subset \mathscr{B}_h$, the set $\{t \mapsto \mathscr{G}_k(t, u_t) : u_t \in \chi\}$, $k = 1, 2, \ldots, m$, is equicontinuous in $\mathscr{B}_h$.

H(A9*): The following inequalities hold:

$$\left[\mathscr{M}_A(\mathcal{L}_k + \tilde{\nu}_i) + \frac{2\mathscr{M}_A^2 \mathscr{M}_6 r \mathfrak{T}^r}{\Gamma(1 + r)} \int_0^t \eta(s) ds + \frac{2\mathscr{M}_A \mathscr{M}_6 r \mathfrak{T}^r}{\Gamma(1 + r)} \int_0^t \eta(s) ds\right] \hbar_{PC}(\tilde{\mathcal{W}}(\tau)) < 1,$$

$$\tilde{\mathscr{L}} = (\mathscr{M}_A + 1)\Big[\Big(\mathscr{M}_0 + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta + 1)}{\Gamma(r\beta + 1)} \cdot \frac{T^{r\beta}}{\beta}\Big)\mathscr{C}_1 \mathcal{D}_1\Big] < 1,$$

and

$$\iota = \max_{1 \leq k \leq m} (\mathscr{M}_A + 1)\Big\{(\mathcal{L}_k + \tilde{\nu}_i) + \frac{\mathscr{M}_A(1 + \zeta^*)T^r}{\Gamma(r + 1)} \int_0^T \tilde{\eta}(s) ds\Big\} < 1.$$

For our convenience, let us take

$$\mathcal{K}_i := \left[\left(\frac{1 - q_i}{q - q_i}\right) b^{\frac{q - q_i}{1 - q_i}}\right]^{1 - q_i}, \quad i = 0, 1, 2, \quad \mathscr{M}_4 := \mathcal{K}_1 \|m\|_{L^{\frac{1}{q_1}}(\mathscr{I}, R^+)}$$

and $\mathscr{M}_6 := \mathcal{K}_2 \|\eta\|_{L^{\frac{1}{q_2}}(\mathscr{I}, R^+)}$.

Theorem 3.1. *Assume the hypotheses* H(A1)-H(A9*) *are satisfied, then the problem* (1.1)–(1.3) *has at least one mild solution on* $[0, \mathfrak{T}]$ *provided that*

$$\max_{1 \leq k \leq m} \mathcal{D}_1 \left[(\mathscr{M}_A(\mathcal{L}_k + \tilde{\nu}_i)) + \mathscr{M}_0 \mathscr{C}_1 + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta + 1)}{\Gamma(r + 1)} \mathscr{C}_1 \frac{T^{r\beta}}{\beta}\right] < 1. \tag{3.1}$$

Proof. We will transform the system (1.1)–(1.3) into a fixed point problem. Let the operator $\Upsilon : \mathscr{B}_h' \to \mathscr{B}_h'$ be defined by

$$(\Upsilon u)(t) = \begin{cases} \Phi(t), \quad t \in (-\infty, 0], \\ \mathcal{Q}_r(t)[\Phi(0) + \mathcal{G}(0, \Phi(0))] - \mathcal{G}(t, u_t) + \int_0^t (t - s)^{r-1} \mathfrak{A} \mathcal{P}_r(t - s) \mathcal{G}(s, u_s) ds \\ + \int_0^t (t - s)^{r-1} \mathcal{P}_r(t - s) \mathscr{F}\left(s, u_s, \int_0^s \mathcal{H}(s, \tau, u_\tau) d\tau\right) ds, \quad t \in [0, t_1], \\ k = 0, 1, 2, \ldots, m, \\ \mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(t, u_t), \quad t \in (t_k, s_k], \quad k = 1, 2, \ldots, m, \\ \mathcal{Q}_r(t - s_k)\mathscr{D}_k - \mathcal{G}(t, u_t) + \int_0^t (t - s)^{r-1} \mathfrak{A} \mathcal{P}_r(t - s) \mathcal{G}(s, u_s) ds \\ + \int_0^t (t - s)^{r-1} \mathcal{P}_r(t - s) \mathscr{F}\left(s, u_s, \int_0^s \mathcal{H}(s, \tau, u_\tau) d\tau\right) ds, \quad t \in (s_k, t_{k+1}], \\ k = 1, 2, \ldots, m, \end{cases}$$

with $\mathscr{D}_k$, $k = 1, 2, \ldots, m$, defined by (2.1).

Obviously the fixed points of the operator Υ are mild solutions of the model (1.1)–(1.3). The function $\tilde{y}(\cdot):(-\infty,\mathfrak{T}]\to\mathfrak{X}$ is defined by

$$\tilde{y}(t)=\begin{cases}\Phi(t), & t\in(-\infty,0],\\ \mathfrak{Q}_r(t)\Phi(0), & t\in\mathfrak{J}.\end{cases}$$

Then, $\tilde{y}_0=\Phi$. For every function $\tilde{z}\in C(\mathfrak{J},\mathbb{R})$ with $\tilde{z}(0)=0$, we take as z is defined by

$$z(t)=\begin{cases}0, & t\le 0,\\ \tilde{z}(t), & t\in\mathfrak{J}.\end{cases}$$

If $u(\cdot)$ fulfils (2.1), we are able to split it as $u(t)=\tilde{y}(t)+\tilde{z}(t)$, $t\in\mathfrak{J}$, which suggests $u_t=\tilde{y}_t+\tilde{z}_t$, for each $t\in\mathfrak{J}$ and also the function $\tilde{z}(\cdot)$ fulfills

$$\tilde{z}(t)=\begin{cases}\mathfrak{Q}_r(t)\mathfrak{G}(0,\Phi)-\mathfrak{G}(t,\tilde{z}_t+\tilde{y}_t)+\int_0^t(t-s)^{r-1}\mathfrak{A}\mathfrak{P}_r(t-s)\mathfrak{G}(s,\tilde{z}_s+\tilde{y}_s)ds\\ +\int_0^t(t-s)^{r-1}\mathfrak{P}_r(t-s)\mathscr{F}\left(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathfrak{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\right)ds, \quad t\in[0,t_1],\\ k=0,1,2,\dots,m,\\ \mathfrak{I}_k(u(t_k))+\mathscr{G}_k(t,\tilde{z}_t+\tilde{y}_t), \quad t\in(t_k,s_k], \quad k=1,2,3,\dots,m,\\ \mathfrak{Q}_r(t-s_k)\mathscr{D}_k-\mathfrak{G}(t,\tilde{z}_t+\tilde{y}_t)+\int_0^t(t-s)^{r-1}\mathfrak{A}\mathfrak{P}_r(t-s)\mathfrak{G}(s,\tilde{z}_s+\tilde{y}_s)ds\\ +\int_0^t(t-s)^{r-1}\mathfrak{P}_r(t-s)\mathscr{F}\left(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathfrak{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\right)ds,\\ t\in(s_k,t_{k+1}], \quad k=1,2,\dots,m,\end{cases}$$

where

$$\begin{aligned}\mathscr{D}_k=&\mathfrak{I}_k(u(t_k))+\mathscr{G}_k(s_k,\tilde{z}_{s_k}+\tilde{y}_{s_k})+\mathfrak{G}(s_k,\tilde{z}_{s_k}+\tilde{y}_{s_k})\\ &-\int_0^{s_k}(s_k-s)^{r-1}\mathfrak{A}\mathfrak{P}_r(s_k-s)\mathfrak{G}(s,\tilde{z}_{s_k}+\tilde{y}_{s_k})ds\\ &-\int_0^{s_k}(s_k-s)^{r-1}\mathfrak{P}_r(s_k-s)\mathscr{F}\left(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathfrak{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\right)ds,\end{aligned}\tag{3.2}$$

$k=1,2,3,\dots,m$. Let $\mathscr{B}_h^{''}=\{\tilde{z}\in\mathscr{B}_h^{'}:\tilde{z}_0=0\in\mathscr{B}_h\}$. For any $\tilde{z}\in\mathscr{B}_h^{''}$,

$$\|\tilde{z}\|_{\mathscr{B}_h^{''}}=\sup_{t\in\mathfrak{J}}\|\tilde{z}(t)\|_{\mathfrak{X}}+\|\tilde{z}_0\|_{\mathscr{B}_h}=\sup_{t\in\mathfrak{J}}\|\tilde{z}(t)\|_{\mathfrak{X}}, \quad \tilde{z}\in\mathscr{B}_h^{''},$$

as a result $(\mathscr{B}_h^{''},\|\cdot\|_{\mathscr{B}_h^{''}})$ is a Banach space. Consider $B_q=\{\tilde{z}\in\mathscr{B}_h^{''}:\|\tilde{z}\|_{\mathfrak{X}}\le q\}$ for some $q\ge 0$. Then for each $B_q\subset\mathscr{B}_h^{''}$ is uniformly bounded, and for $\tilde{z}\in B_q$. We have the phase space axioms (J1)-(J2),

$$\begin{aligned}\|\tilde{z}_s+\tilde{y}_s\|_{\mathscr{B}_h}&\le\|\tilde{z}_s\|_{\mathscr{B}_h}+\|\tilde{y}_s\|_{\mathscr{B}_h}\\ &\le\mathfrak{D}_1\sup_{(0\le\tau\le\tilde{z}_s+\tilde{y}_s)}\|\tilde{z}(\tau)\|_{\mathfrak{X}}+\mathfrak{D}_2\|\tilde{z}_0\|_{\mathscr{B}_h}+\mathfrak{D}_1\sup_{(0\le\tau\le\tilde{z}+\tilde{y})}\|\tilde{y}(\tau)\|+\mathfrak{D}_2\|\tilde{y}_0\|_{\mathscr{B}_h}\\ &\le\mathfrak{D}_1\sup_{(0\le\tau\le s)}\|\tilde{z}(\tau)\|_{\mathfrak{X}}+\mathfrak{D}_1\|\mathfrak{Q}_r(t)\|_{\mathcal{L}(\mathfrak{X})}\|\Phi(0)\|_{\mathscr{B}_h}+\mathfrak{D}_2\|\Phi\|_{\mathscr{B}_h}\\ &\le\mathfrak{D}_1\|\tilde{z}\|_{\mathfrak{X}}+(\mathfrak{D}_1\mathscr{M}_1+\mathfrak{D}_2)\|\Phi\|_{\mathscr{B}_h}\\ &\le\mathfrak{D}_1 q+\tilde{c}_n.\end{aligned}$$

In the event that $\|\tilde{z}\|_{\mathcal{X}} < q,\ q > 0$,

$$\|\tilde{z}_s + \tilde{y}_s\|_{\mathscr{B}_h} \leq \mathcal{D}_1 q + \tilde{c}_n, \tag{3.3}$$

where $\tilde{c}_n = (\mathcal{D}_1 \mathscr{M}_1 + \mathcal{D}_2)\|\Phi\|_{\mathscr{B}_h}$. Define the operator $\tilde{\Upsilon} : \mathscr{B}_h'' \to \mathscr{B}_h''$ by

$$(\tilde{\Upsilon}\tilde{z})(t) = \begin{cases} \mathcal{Q}_r(t)\mathcal{G}(0,\Phi) - \mathcal{G}(t, \tilde{z}_t + \tilde{y}_t) \\ + \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, \tilde{z}_s + \tilde{y}_s)ds \\ + \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s) \\ \times \mathscr{F}\left(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\right)ds, \quad t \in [0, t_1], \\ k = 0, 1, 2, \ldots, m, \\ \mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(t, \tilde{z}_t + \tilde{y}_t), \quad t \in (t_k, s_k], \quad k = 1, 2, 3, \ldots, m, \\ \mathcal{Q}_r(t - s_k)\mathscr{D}_k - \mathcal{G}(t, \tilde{z}_t + \tilde{y}_t) \\ + \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\mathcal{G}(s, \tilde{z}_s + \tilde{y}_s)ds \\ + \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\left(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\right)ds, \\ t \in (s_k, t_{k+1}], \end{cases}$$

with $\mathscr{D}_k$, $k = 1, 2, 3, \ldots, m$, defined by (3.2).

Thus, the operator Υ has a fixed point if and only if $\tilde{\Upsilon}$ has a fixed point. Now, first we calculate the following estimations.

Remark 3.1. By utilizing the above equation H(A1)-H(A9), we obtain

$$P_1 = \|\mathcal{Q}_r(t)\mathcal{G}(0,\Phi)\|_{\mathcal{X}} = \mathscr{M}_A\|\mathfrak{A}^{-\beta}\|\left\|\mathfrak{A}^{-\beta}\mathcal{G}(0,\Phi)\right\|_{\mathcal{X}} \leq \mathscr{M}_A\mathscr{M}_0\left[\mathscr{C}_1\|\Phi\|_{\mathscr{B}_h} + \mathscr{C}_2\right],$$

where $\mathscr{M}_0 = \|\mathfrak{A}^{-\beta}\|$,

$$\begin{aligned} P_2 &= \|\mathcal{G}(t, \tilde{z}_t + \tilde{y}_t)\|_{\mathcal{X}} = \left\|\mathfrak{A}^{-\beta}(\mathfrak{A}^{\beta})\mathcal{G}(t, \tilde{z}_t + \tilde{y}_t)\right\|_{\mathcal{X}} \leq \mathscr{M}_0\left[\mathscr{C}_1\left\|\tilde{z}_t + \tilde{y}_t\right\|_{\mathscr{B}_h} + \mathscr{C}_2\right] \\ &\leq \mathscr{M}_0\left[\mathscr{C}_1\left(\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n\right) + \mathscr{C}_2\right] \leq \mathscr{M}_0\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n) + \mathscr{M}_0\mathscr{C}_2, \\ P_3 &= \|\mathcal{G}(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k})\|_{\mathcal{X}} = \|\mathfrak{A}^{-\beta}(\mathfrak{A}^{\beta})\mathcal{G}(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k})\|_{\mathcal{X}} \\ &\leq \mathscr{M}_0\left[\mathscr{C}_1\|\tilde{z}_{s_k} + \tilde{y}_{s_k}\|_{\mathscr{B}_h} + \mathscr{C}_2\right] \leq \mathscr{M}_0\left[\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_{s_k} + \tilde{c}_n) + \mathscr{C}_2\right] \\ &\leq \mathscr{M}_0\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_{s_k} + \tilde{c}_n) + \mathscr{M}_0\mathscr{C}_2, \quad t \in (s_k, t_{k+1}], \\ P_4 &= \left\|\int_0^t (t-s)^{r-1}\mathfrak{A}^{1-\beta}\mathcal{P}_r(t-s)\mathfrak{A}^{\beta}\mathcal{G}(s, \tilde{z}_s + \tilde{y}_s)ds\right\|_{\mathcal{X}} \\ &\leq \frac{r\mathscr{M}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+r\beta)}\int_0^t (t-s)^{r\beta-1}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_s + \tilde{c}_n) + \mathscr{C}_2)ds \\ &\leq \frac{r\mathscr{M}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+r\beta)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n) + \mathscr{C}_2)\int_0^t (t-s)^{r\beta-1}ds \\ &\leq \frac{\mathscr{M}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+r\beta)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n) + \mathscr{C}_2)\frac{(t_1)^{r\beta}}{\beta}, \quad t \in [0, t_1], \\ P_5 &= \left\|\int_0^{s_k} (s_k - s)^{r-1}\mathfrak{A}^{1-\beta}\mathcal{P}_r(s_k - s)(\mathfrak{A}^{\beta})\mathcal{G}(s, \tilde{z}_s + \tilde{y}_s)ds\right\|_{\mathcal{X}} \end{aligned}$$

$$\leq \frac{\mathscr{M}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+r\beta)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{C}_2)\frac{(s_k)^{r\beta}}{\beta}, \quad t\in(s_k,t_{k+1}],$$

$$P_6=\Big\|\int_0^t(t-s)^{r-1}\mathfrak{A}^{1-\beta}\mathcal{P}_r(t-s)\mathfrak{A}^{\beta}\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)ds\Big\|_{\mathcal{X}}$$

$$\leq \frac{\mathscr{M}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+r\beta)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{C}_2)\frac{T^{r\beta}}{\beta}, \quad t\in[0,\mathcal{T}],$$

$$P_7=\|\mathfrak{I}_k(u(t_k))\|_{\mathcal{X}}+\|\mathscr{G}_k(t,\tilde{z}_t+\tilde{y}_t)\|_{\mathcal{X}}\leq\mathcal{L}_k+\mathcal{C}_i[\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n]+\bar{\mathcal{C}}_i, \quad t\in(t_k,s_k],$$

$$P_8=\|\mathfrak{I}_k(u(t_k))\|_{\mathcal{X}}+\|\mathscr{G}_k(s_k,\tilde{z}_{s_k}+\tilde{y}_{s_k})\|_{\mathcal{X}}$$

$$\leq\mathcal{L}_k+\mathcal{C}_i[\mathcal{D}_1\|\tilde{z}\|_{s_k}+\tilde{c}_n]+\bar{\mathcal{C}}_i, \qquad t\in(s_k,t_{k+1}],$$

$$P_9=\Big\|\int_0^t(t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big\|_{\mathcal{X}}$$

$$\leq\frac{\mathscr{M}_A}{\Gamma(r)}\int_0^t(t-s)^{r-1}m(s)\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n)ds$$

$$\leq\frac{\mathscr{M}_A(t_1)^r}{\Gamma(r+1)}m(s)\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+b\nu(\tau)(1+\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))ds$$

$$\leq\frac{\mathscr{M}_A(t_1)^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+b\nu(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathcal{J}}m(s), \quad t\in[0,t_1],$$

$$P_{10}=\Big\|\int_0^{s_k}(s_k-s)^{r-1}\mathcal{P}_r(s_k-s)\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big\|_{\mathcal{X}}$$

$$\leq\frac{\mathscr{M}_A(s_k)^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+b\nu(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathcal{J}}m(s), \quad t\in(s_k,t_{k+1}],$$

$$P_{11}\leq\Big\|\int_0^t(t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big\|_{\mathcal{X}}$$

$$\leq\frac{\mathscr{M}_AT^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+b\nu(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathcal{J}}m(s), \quad t\in[0,\mathcal{T}],$$

$$P_{12}\leq\|\mathcal{G}(t,\tilde{z}_t^n+\tilde{y}_s)\|_{\mathcal{X}}\leq\Big\|(\mathfrak{A})^{\beta}(\mathfrak{A})^{-\beta}\left[\mathcal{G}(t,\tilde{z}_t^n+\tilde{y}_t)-\mathcal{G}(t,\tilde{z}_t+\tilde{y}_t)\right]\Big\|_{\mathcal{X}}$$

$$\leq\mathscr{M}_0\Big\|(\mathfrak{A})^{\beta}\left[\mathcal{G}(t,\tilde{z}_t^n+\tilde{y}_t)-\mathcal{G}(t,\tilde{z}_t+\tilde{y}_t)\right]\Big\|_{\mathcal{X}},$$

$$P_{13}\leq\Big\|\int_0^t(t-s)^{r-1}\mathfrak{A}^{1-\beta}\mathcal{P}_r(t-s)\mathfrak{A}^{\beta}\left[\mathcal{G}(s,\tilde{z}_s^n+\tilde{y}_s)-\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)\right]ds\Big\|_{\mathcal{X}}$$

$$\leq\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\frac{t_1^{r\beta}}{\beta}\int_0^t\Big\|\mathfrak{A}^{\beta}\left[\mathcal{G}(s,\tilde{z}_s^n+\tilde{y}_s)-\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)\right]\Big\|_{\mathcal{X}}ds, \quad t\in[0,t_1],$$

$$P_{14}\leq\Big\|\int_0^{s_k}(s_k-s)^{r-1}\mathfrak{A}^{1-\beta}\mathcal{P}_r(s_k-s)\mathfrak{A}^{\beta}\left[\mathcal{G}(s,\tilde{z}_s^n+\tilde{y}_s)-\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)\right]ds\Big\|_{\mathcal{X}}$$

$$\leq\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\frac{(s_k)^{r\beta}}{\beta}\int_0^{s_k}\Big\|\mathfrak{A}^{\beta}\left[\mathcal{G}(s,\tilde{z}_s^n+\tilde{y}_s)-\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)\right]\Big\|_{\mathcal{X}}ds,$$

$$t\in(s_k,t_{k+1}],$$

$$\begin{aligned}
P_{15} \leq & \Big\| \int_0^t (t-s)^{r-1} \mathcal{P}_r(t-s) \Big[\mathscr{F}\Big(s, \tilde{z}_s^n + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau^n + \tilde{y}_\tau) d\tau\Big) \\
& - \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau) d\tau\Big)\Big]\Big\|_{\mathcal{X}} ds \\
\leq & \frac{\mathscr{M}_A t_1^r}{\Gamma(r+1)} \int_0^t \Big\| \mathscr{F}\Big(s, \tilde{z}_s^n + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau^n + \tilde{y}_\tau) d\tau\Big) \\
& - \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau) d\tau\Big)\Big]\Big\|_{\mathcal{X}} ds, \quad t \in [0, t_1], \\
P_{16} \leq & \Big\| \int_0^{s_k} (s_k-s)^{r-1} \mathcal{P}_r(s_k-s) \Big[\mathscr{F}\Big(s, \tilde{z}_s^n + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau^n + \tilde{y}_\tau) d\tau\Big) \\
& - \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau) d\tau\Big)\Big]\Big\|_{\mathcal{X}} ds \\
\leq & \frac{\mathscr{M}_A (s_k)^r}{\Gamma(r+1)} \int_0^{s_k} \Big\| \mathscr{F}\Big(s, \tilde{z}_s^n + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau^n + \tilde{y}_\tau) d\tau\Big) \\
& - \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau) d\tau\Big)\Big]\Big\|_{\mathcal{X}} ds, \quad t \in (s_k, t_{k+1}].
\end{aligned}$$

Now, we define $(\tilde{\Upsilon}_2 \tilde{z})(t)$ as

$$(\tilde{\Upsilon}_2 \tilde{z})(t) = \begin{cases}
\mathcal{Q}_r(t)\mathcal{G}(0,\Phi) - \mathcal{G}(t, \tilde{z}_t + \tilde{y}_t) + \int_0^t (t-s)^{r-1} \mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, \tilde{z}_s + \tilde{y}_s) ds, \\
t \in [0, t_1], \quad k = 0, 1, 2, \ldots, m, \\
0, \quad t \in (t_k, s_k], \quad k = 1, 2, \ldots, m, \\
\mathcal{Q}_r(t)(t-s_k)\Big[\mathcal{G}(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k}) - \int_0^{s_k} (s_k-s)^{r-1} \mathfrak{A}\mathcal{P}_r(s_k-s) \\
\times \mathcal{G}(s, \tilde{z}_s + \tilde{y}_s) ds\Big] - \mathcal{G}(t, \tilde{z}_t + \tilde{y}_t) \\
+ \int_0^t (t-s)^{r-1} \mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, \tilde{z}_s + \tilde{y}_s) ds, \quad t \in (s_k, t_{k+1}], \\
k = 1, 2, 3, \ldots, m.
\end{cases}$$

We obtain

$$\begin{aligned}
\|(\tilde{\Upsilon}_2 \tilde{z})(t) - (\tilde{\Upsilon}_2 \tilde{\bar{z}})(t)\|_{\mathcal{X}} \leq & \|(\mathfrak{A})^\beta\| \left\| (\mathfrak{A})^\beta \mathcal{G}(t, \tilde{z}_t + \tilde{y}_t)_{\mathcal{X}} - (\mathfrak{A})^\beta \mathcal{G}(t, \tilde{\bar{z}}_t, \tilde{y}_t) \right\|_{\mathcal{X}} \\
& + \Big\| \int_0^t (t-s)^{r-1} (\mathfrak{A})^{1-\beta} \mathcal{P}_r(t-s) \\
& \times \left[(\mathfrak{A})^\beta \mathcal{G}(s, \tilde{z}_s + \tilde{y}_s) - (\mathfrak{A}^\beta) \mathcal{G}(s, \tilde{\bar{z}}_s + \tilde{y}_s) \right] ds \Big\|_{\mathcal{X}} \\
\leq & \mathscr{M}_0 \mathscr{C}_1 \left\| \tilde{z}_t - \tilde{\bar{z}}_t \right\|_{\mathscr{B}_h} + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)} \cdot \frac{t_1^{r\beta}}{\beta} \mathscr{C}_1 \left\| \tilde{z}_t - \tilde{\bar{z}}_t \right\|_{\mathscr{B}_h},
\end{aligned}$$

since

$$\begin{aligned}
\left\| \tilde{z}_t - \tilde{\bar{z}}_t \right\|_{\mathscr{B}_h} \leq & \mathcal{D}_1 \|\tilde{z}(t)\|_{\mathcal{X}} + (\mathcal{D}_2 + \mathcal{J}^\Phi)\|z_0\|_{\mathscr{B}_h} - \mathcal{D}_1 \|\tilde{\bar{z}}(t)\|_{\mathcal{X}} - (\mathcal{D}_2 + \mathcal{J}^\Phi)\|\tilde{\bar{z}}_0\|_{\mathscr{B}_h} \\
\leq & \mathcal{D}_1 \|\tilde{z}(t) - \tilde{\bar{z}}(t)\|_{\mathcal{X}} \leq \mathcal{D}_1 \|\tilde{z} - \tilde{\bar{z}}\|_{\mathscr{B}_h''},
\end{aligned}$$

we have

$$\begin{aligned}\Big\|(\tilde{\Upsilon}_2\tilde{z})(t)-(\tilde{\Upsilon}_2\tilde{\tilde{z}})(t)\Big\|_{\mathcal{X}} \leq &\mathscr{M}_0\mathscr{C}_1\mathcal{D}_1\|\tilde{z}-\tilde{\tilde{z}}\|_{\mathscr{B}''_h}+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\cdot\frac{t_1^{r\beta}}{\beta}\mathscr{C}_1\mathcal{D}_1\|\tilde{z}-\tilde{\tilde{z}}\|_{\mathscr{B}''_h}\\ \leq&\Big[\mathscr{M}_0\mathscr{C}_1\mathcal{D}_1+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\cdot\frac{t_1^{r\beta}}{\beta}\mathscr{C}_1\mathcal{D}_1\Big]\|\tilde{z}-\tilde{\tilde{z}}\|_{\mathscr{B}''_h},\\ &t\in[0,t_1],\end{aligned}$$

$$\begin{aligned}&\Big\|(\tilde{\Upsilon}_2\tilde{z})(t)-(\tilde{\Upsilon}_2\tilde{\tilde{z}})(t)\Big\|_{\mathcal{X}}\\ \leq&\|\mathcal{Q}_r(t-s_k)\|_{\mathcal{L}(\mathcal{X})}\Big[\|(\mathfrak{A})^{-\beta}\|\|(\mathfrak{A})^{\beta}\mathcal{G}(s_k,\tilde{z}_{s_k}+\tilde{y}_{s_k})-(\mathfrak{A})^{\beta}\mathcal{G}(s_k,\tilde{\tilde{z}}_{s_k}+\tilde{y}_{s_k})\|_{\mathcal{X}}\\ &+\Big\|\int_0^{s_k}(s_k-s)^{r-1}\mathfrak{A}^{1-\beta}\mathcal{P}_r(s_k-s)\Big[\mathfrak{A}^{\beta}\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s-\mathfrak{A}^{\beta}\mathcal{G}(s,\tilde{\tilde{z}}_s+\tilde{y}_s)\Big]ds\Big\|_{\mathcal{X}}\Big]\\ &+\|\mathfrak{A}^{-\beta}\|\|\mathfrak{A}^{\beta}\mathcal{G}(t,\tilde{z}_t+\tilde{y}_t)-\mathfrak{A}^{\beta}\mathcal{G}(t,\tilde{\tilde{z}}_t+\tilde{y}_t)\|_{\mathcal{X}}\\ &+\Big\|\int_0^{t}(t-s)^{r-1}\mathfrak{A}^{1-\beta}\mathcal{P}_r(t-s)\Big[\mathfrak{A}^{\beta}\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)-\mathfrak{A}^{\beta}\mathcal{G}(s,\tilde{\tilde{z}}_s+\tilde{y}_s)\Big]ds\Big\|_{\mathcal{X}}\\ \leq&\mathscr{M}_A\Big[\mathscr{M}_0\mathscr{C}_1\mathcal{D}_1\|\tilde{z}-\tilde{\tilde{z}}\|_{\mathcal{B}''_h}+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\cdot\frac{(s_k)^{r\beta}}{\beta}\mathscr{C}_1\mathcal{D}_1\|\tilde{z}-\tilde{\tilde{z}}\|_{\mathscr{B}''_h}\Big]\\ &\times\mathscr{M}_0\mathscr{C}_1\mathcal{D}_1\|\tilde{z}-\tilde{\tilde{z}}\|_{\mathscr{B}''_h}\\ &+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\cdot\frac{(t_{k+1})^{r\beta}}{\beta}\mathscr{C}_1\mathcal{D}_1\|\tilde{z}-\tilde{\tilde{z}}\|_{\mathscr{B}''_h}\\ \leq&\Big[(\mathscr{M}_A+1)\mathscr{M}_0\mathscr{C}_1\mathcal{D}_1+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\mathscr{C}_1\mathcal{D}_1\Big\{\frac{(t_{k+1})^{r\beta}}{\beta}+\frac{\mathscr{M}_A(s_k)^{r\beta}}{\beta}\Big\}\Big]\|\tilde{z}-\tilde{\tilde{z}}\|_{\mathscr{B}''_h},\\ &t\in(s_k,t_{k+1}],\end{aligned}$$

and

$$\begin{aligned}&\Big\|(\tilde{\Upsilon}_2\tilde{z})(t)-(\tilde{\Upsilon}_2\tilde{\tilde{z}})(t)\Big\|_{\mathcal{X}}\\ \leq&(\mathscr{M}_A+1)\Big[\mathscr{M}_0\mathscr{C}_1\mathcal{D}_1+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\cdot\frac{T^{r\beta}}{\beta}\mathscr{C}_1\mathcal{D}_1\Big]\|\tilde{z}-\tilde{\tilde{z}}\|_{\mathscr{B}''_h},\quad t\in\mathcal{J}.\end{aligned}$$

Presently, let us demostrate that Υ has a fixed point. Subsequently we will prove that $\tilde{\Upsilon}$ has a fixed point by using Lemma 2.4.

Step 1. We show that there exist some $q>0$ such that $\tilde{\Upsilon}(\mathscr{B}_q)\subseteq\mathscr{B}_q$. If it is not true, then for each positive number q, there exists a function $\tilde{z}^q(.)\in\mathscr{B}_q$ and some $t\in\mathcal{J}$ such that $\|(\tilde{\Upsilon}\tilde{z}^q)(t)\|>q$.

On the other hand, from hypotheses the H(A2)-H(A9), Lemma 2.1 (1) and Hölder's inequality, for $t\in[0,t_1]$,

$$\begin{aligned}q<\|(\tilde{\Upsilon}\tilde{z}^q)(t)\|\leq&\|\mathcal{Q}_r(t)\|_{\mathcal{L}(\mathcal{X})}\|\mathcal{G}(0,\Phi)\|_{\mathcal{X}}+\|\mathcal{G}(t,\tilde{z}_t+\tilde{y}_t)\|_{\mathcal{X}}\\ &+\Big\|\int_0^t(t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)ds\Big\|_{\mathcal{X}}\end{aligned}$$

$$
\begin{aligned}
&+\Big\|\int_0^t(t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big\|_{\mathcal{X}}\\
\leq&\mathscr{M}_A\mathscr{M}_0[\mathscr{C}_1\|\Phi\|_{\mathscr{B}_h}+\mathscr{C}_2]+\mathscr{M}_0\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{M}_0\mathscr{C}_2\\
&+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{C}_2)\frac{t_1^{r\beta}}{\beta}\\
&+\frac{\mathscr{M}_At_1^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathcal{J}}m(s)\\
=&\mathcal{J}_1.
\end{aligned}
$$

For any $t\in(t_k,s_k]$, $k=1,2,\dots,m$, we have

$$
\begin{aligned}
q<\|(\tilde{\Upsilon}\tilde{z}^q)(t)\|&\leq\|\mathfrak{I}_k(u(t_k))\|_{\mathcal{X}}+\|\mathscr{G}_k(t,\tilde{z}_t+\tilde{y}_t)\|_{\mathcal{X}}\leq\mathcal{L}_k+\mathcal{C}_i[\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n]+\bar{\mathcal{C}}_i\\
&=\mathcal{J}_2.
\end{aligned}
$$

In the same way, for any $t\in(s_k,t_{k+1}]$, $k=1,2,\dots,m$,

$$
\begin{aligned}
q<\|(\tilde{\Upsilon}\tilde{z}^q)(t)\|\leq&\|\mathcal{Q}_r(t-s_k)\|_{\mathcal{L}(\mathcal{X})}\Big[\|\mathfrak{I}_k(u(t_k))\|+\|\mathscr{G}_k(s_k,\tilde{z}_k+\tilde{y}_{s_k})\|_{\mathcal{X}}\\
&+\|\mathcal{G}(s_k,\tilde{z}_{s_k}+\tilde{y}_{s_k})\|_{\mathcal{X}}\\
&+\Big\|\int_0^{s_k}(s_k-s)^{r-1}\mathfrak{A}\mathcal{P}_r(s_k-s)\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)ds\Big\|\\
&+\Big\|\int_0^{s_k}(s_k-s)^{r-1}\mathcal{P}_r(s_k-s)\\
&\times\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big\|_{\mathcal{X}}\Big]\\
&+\|\mathcal{G}(t,\tilde{z}_t+\tilde{y}_t)\|_{\mathcal{X}}+\Big\|\int_0^t(t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)ds\Big\|_{\mathcal{X}}\\
&+\Big\|\int_0^t(t-s)^{r-1}\mathcal{P}_r(t-s)\\
&\times\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big\|_{\mathcal{X}}\\
\leq&\mathscr{M}_A\Big[\mathcal{L}_k+\mathcal{C}_i[\mathcal{D}_1\|\tilde{z}\|_{s_k}+\tilde{c}_n]+\bar{\mathcal{C}}_i+\mathscr{M}_0\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_{s_k}+\tilde{c}_n)+\mathscr{M}_0\mathscr{C}_2\\
&+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_{s_k}+\tilde{c}_n)+\mathscr{C}_2)\frac{(s_k)^{r\beta}}{\beta}\\
&+\frac{\mathscr{M}_A(s_k)^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathcal{J}}m(s)\Big]\\
&+\mathscr{M}_0\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{M}_0\mathscr{C}_2\\
&+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{C}_2)\frac{(t_{k+1})^{r\beta}}{\beta}\\
&+\frac{\mathscr{M}_A(t_{k+1})^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathcal{J}}m(s).
\end{aligned}
$$

Then, for all $t \in \mathcal{J}$, we find that

$$\begin{aligned}\|(\tilde{\Upsilon}\tilde{z}^q)(t)\| \leq & \mathfrak{C}^* + (\mathscr{M}_A+1)\Big[(\mathcal{L}_k + \mathcal{C}_i + \mathscr{M}_0\mathscr{C}_1) + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\mathscr{C}_1\frac{T^{r\beta}}{\beta}\Big] \\ & \times (\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n) \\ & + \frac{\mathscr{M}_A(\mathscr{M}_A+1)T^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s + \tilde{c}_n + bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau + \tilde{c}_n))\sup_{t\in\mathcal{J}} m(s),\end{aligned}$$

where

$$\begin{aligned}\mathfrak{C}^* = \max_{1\leq k\leq m}\Big\{ & \mathscr{M}_A\mathscr{M}_0[\mathscr{C}_1\|\Phi\|_{\mathscr{B}_h} + \mathscr{C}_2] + (\mathscr{M}_A+1)\Big(\mathscr{M}_0\mathscr{C}_2 + \bar{\mathcal{C}}_i \\ & + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\mathscr{C}_2\frac{T^{r\beta}}{\beta}\Big)\Big\} \\ & + \frac{\mathscr{M}_A T^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s + \tilde{c}_n + bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau + \tilde{c}_n))\sup_{t\in\mathcal{J}} m(s).\end{aligned}$$

Combining the above equations

$$\begin{aligned}q < \|(\tilde{\Upsilon}\tilde{z}^q)(t)\| \leq & \mathfrak{C}^* + (\mathscr{M}_A+1)\Big[(\mathcal{L}_k + \mathcal{C}_i + \mathscr{M}_0\mathscr{C}_1) + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\mathscr{C}_1\frac{T^{r\beta}}{\beta}\Big] \\ & \times (\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n) \\ & + \frac{\mathscr{M}_A(\mathscr{M}_A+1)T^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s + \tilde{c}_n + bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau + \tilde{c}_n))\sup_{t\in\mathcal{J}} m(s).\end{aligned}$$

Dividing both sides by q and letting $q \to +\infty$, we obtain

$$\begin{aligned}1 \leq \|(\tilde{\Upsilon}\tilde{z}^q)(t)\| \leq & \mathfrak{C}^* + (\mathscr{M}_A+1)\Big[(\mathcal{L}_k + \mathcal{C}_i + \mathscr{M}_0\mathscr{C}_1) + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\mathscr{C}_1\frac{T^{r\beta}}{\beta}\Big] \\ & \times (\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n) \\ & + \frac{\mathscr{M}_A(\mathscr{M}_A+1)T^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s + \tilde{c}_n + bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau + \tilde{c}_n))\sup_{t\in\mathcal{J}} m(s).\end{aligned}$$

Then, by hypothesis, we get $1 \leq 0$. This is a contradiction.

Hence, for some positive integer $\tilde{\Upsilon}(\mathscr{B}_q) \subseteq \mathscr{B}_q$.

Step 2. $\tilde{\Upsilon} : \mathscr{B}_h'' \to \mathscr{B}_h''$ is continuous. For this purpose let $\{\tilde{z}^{(n)}\}_{n=0}^{+\infty} \subseteq \mathscr{B}_h''$ with $\tilde{z}^{(n)} \to \tilde{z}$ in $\mathscr{B}_h''$. Then there is a number $c' > 0$ such that $\|\tilde{z}^{(n)}(t)\| \leq c'$ for all n and a.e. $t \in \mathcal{J}$, so $\tilde{z}^{(n)} \in \mathscr{B}_{c'} = \{\tilde{z} \in \mathscr{B}_h'' : \|\tilde{z}\|_{\mathscr{B}_h''} \leq c'\} \subseteq \mathscr{B}_h''$ and $\tilde{z} \in \mathscr{B}_{c'}$. From remark, we have $\|\tilde{z}_t + \tilde{y}_t\|_{\mathscr{B}_h} \leq c''$, $t \in \mathcal{J}$.

By H(A4), H(A5), Remark $P_{12}, P_{13}, P_{14}, P_{15}$, and Lebesgue's dominated convergence theroem, we obtain, for $t \in [0, t_1]$,

$$\begin{aligned}& \|(\tilde{\Upsilon}\tilde{z}^n)(t) - (\tilde{\Upsilon}\tilde{z})(t)\|_{\mathcal{X}} \\ \leq & \mathscr{M}_0\Big\|(\mathfrak{A})^\beta\Big[\mathscr{G}(t, \tilde{z}_t^n + \tilde{y}_t) - \mathscr{G}(t, \tilde{z}_t + \tilde{y}_t)\Big]\Big\|_{\mathcal{X}} + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)} \cdot \frac{(t_1)^{r\beta}}{\beta}\end{aligned}$$

$$\times \int_0^t \|(\mathfrak{A})^{\beta}[\mathcal{G}(s, \tilde{z}_s^n + \tilde{y}_s) - \mathcal{G}(s, \tilde{z}_s + \tilde{y}_s)]\|_{\mathcal{X}} ds$$
$$+ \frac{\mathscr{M}_A(t_1)^r}{\Gamma(r+1)} \int_0^t \Big\| \mathscr{F}\Big(s, \tilde{z}_s^n + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau^n + \tilde{y}_\tau)d\tau\Big)$$
$$- \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big)\Big\|_{\mathcal{X}} ds \to 0, \quad \text{as } n \to +\infty.$$

For any $t \in (t_k, s_k]$, $k = 1, 2, \ldots, m$, we obtain $\|(\tilde{\Upsilon}\tilde{z}^n)(t) - (\tilde{\Upsilon}\tilde{z})(t)\|_{\mathcal{X}} = 0$. In the same way, for any $t \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, we have

$$\begin{aligned}
&\|(\tilde{\Upsilon}\tilde{z}^n)(t) - (\tilde{\Upsilon}\tilde{z})(t)\|_{\mathcal{X}} \\
\leq &\|\mathcal{Q}_r(t - s_k)\|_{\mathcal{L}(\mathcal{X})}\Big[\|\mathfrak{I}_k(u(t_k))\|_{\mathcal{X}} + \|\mathscr{G}_k(s_k, \tilde{z}_{s_k}^n + \tilde{y}_{s_k}) \\
&- \mathscr{G}_k(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k})\|_{\mathcal{X}} + \Big\|(\mathfrak{A})^{-\beta}\Big\| \cdot \Big\|(\mathfrak{A})^{\beta}\mathcal{G}(s_k, \tilde{z}_{s_k}^n + \tilde{y}_{s_k}) \\
&- (\mathfrak{A})^{\beta}\mathcal{G}(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k})\Big\|_{\mathcal{X}} + \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)} \cdot \frac{(s_k)^{r\beta}}{\beta} \\
&\times \int_0^{s_k} \Big\|(\mathfrak{A})^{\beta}\Big[\mathcal{G}(s_k, \tilde{z}_{s_k}^n + \tilde{y}_{s_k}) - \mathcal{G}(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k})\Big]\Big\|_{\mathcal{X}} ds \\
&+ \frac{\mathscr{M}_A(s_k)^r}{\Gamma(r+1)} \int_0^{s_k} \Big\|\mathscr{F}\Big(s, \tilde{z}_{s_k}^n + \tilde{y}_{s_k}, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau^n + \tilde{y}_\tau)d\tau\Big) \\
&- \mathscr{F}\Big(s, \tilde{z}_{s_k} + \tilde{y}_{s_k}, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big)\Big\|_{\mathcal{X}} ds\Big] \\
&+ \Big\|(\mathfrak{A})^{-\beta}\Big\| \cdot \Big\|(\mathfrak{A})^{\beta}\mathcal{G}(t, \tilde{z}_t^n + \tilde{y}_t) - (\mathfrak{A})^{\beta}\mathcal{G}(t, \tilde{z}_t + \tilde{y}_t)\Big\|_{\mathcal{X}} \\
&+ \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)} \cdot \frac{(t_{k+1})^{r\beta}}{\beta} \int_0^t \Big\|(\mathfrak{A})^{\beta}\Big[\mathcal{G}(s, \tilde{z}_s^n + \tilde{y}_s) - (\mathfrak{A})^{\beta}\mathcal{G}(s, \tilde{z}_s + \tilde{y}_s)\Big]\Big\|_{\mathcal{X}} ds \\
&+ \frac{\mathscr{M}_A(t_{k+1})^r}{\Gamma(r+1)} \int_0^t \Big\|\mathscr{F}\Big(s, \tilde{z}_s^n + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau^n + \tilde{y}_\tau)d\tau\Big) \\
&- \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big)\Big\|_{\mathcal{X}} ds \to 0, \quad \text{as } n \to +\infty.
\end{aligned}$$

It is simple to see that $\lim_{n\to+\infty} \Big\|(\tilde{\Upsilon}\tilde{z}^n) - (\tilde{\Upsilon}\tilde{z})\Big\|_{\mathscr{B}_h''} = 0$. Thus, $\tilde{\Upsilon}$ is continuous.

Step 3. $\tilde{\Upsilon}$ is $\hbar$ - contraction.

To demonstrate this we separate $\tilde{\Upsilon}$ as $\tilde{\Upsilon}_2+\tilde{\Upsilon}_3$ for $t\in\mathcal{J}$, where $(\tilde{\Upsilon}_2\tilde{z})(t)$ is defined in axioms and

$$(\tilde{\Upsilon}_3\tilde{z})(t)=\begin{cases}\int_0^t(t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds,\\ t\in[0,t_1],\quad k=0,1,2,\dots,m,\\ \mathfrak{I}_k(u(t_k))+\mathscr{G}_k(t,\tilde{z}_t+\tilde{y}_t),\quad t\in(t_k,s_k],\quad k=1,2,3,\dots,m,\\ \mathcal{Q}_r(t-s_k)\Big[\mathfrak{I}_k(u(t_k))+\mathscr{G}_k(s_k,\tilde{z}_{s_k}+\tilde{y}_{s_k})\\ -\int_0^{s_k}(s_k-s)^{r-1}\mathcal{P}_r(s_k-s)\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big]\\ +\int_0^t(t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds,\\ t\in(s_k,t_{k+1}],\quad k=1,2,3,\dots,m.\end{cases}$$

First, we show that $\tilde{\Upsilon}_2$ is Lipschitz continuous on $\mathscr{B}_h''$. In fact $\tilde{z},\tilde{\tilde{z}}\in\mathscr{B}_h''$ then from axioms, we have, for all $t\in[0,t_1]$,

$$\|(\tilde{\Upsilon}_2\tilde{z})(t)-(\tilde{\Upsilon}_2\tilde{\tilde{z}})(t)\|_{\mathcal{X}}\leq\Big[\Big(\mathscr{M}_0+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\cdot\frac{t_1^{r\beta}}{\beta}\Big)\mathscr{C}_1\mathcal{D}_1\Big]\|\tilde{z}-\tilde{\tilde{z}}\|_{\mathscr{B}_h''}.$$

In the same way, for any $t\in(s_k,t_{k+1}]$, $k=1,2,\dots,m$, we obtain

$$\|(\tilde{\Upsilon}_2\tilde{z})(t)-(\tilde{\Upsilon}_2\tilde{\tilde{z}})(t)\|_{\mathcal{X}}\leq(\mathscr{M}_A+1)\Big[\Big(\mathscr{M}_0+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\cdot\frac{(s_k)^{r\beta}}{\beta}\Big)\mathscr{C}_1\mathcal{D}_1\Big]\|\tilde{z}-\tilde{\tilde{z}}\|_{\mathscr{B}_h''}.$$

Then, for all $t\in\mathcal{J}$, we get

$$\begin{aligned}\|(\tilde{\Upsilon}_2\tilde{z})(t)-(\tilde{\Upsilon}_2\tilde{\tilde{z}})(t)\|&\leq(\mathscr{M}_A+1)\Big[\Big(\mathscr{M}_0+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\cdot\frac{T^{r\beta}}{\beta}\Big)\mathscr{C}_1\mathcal{D}_1\Big]\|\tilde{z}-\tilde{\tilde{z}}\|_{\mathscr{B}_h''}\\&\leq\tilde{\mathscr{L}}\|\tilde{z}-\tilde{\tilde{z}}\|_{\mathscr{B}_h''}.\end{aligned}$$

From the assupmtion H(A9*), we observe that $\tilde{\mathscr{L}}<1$. Hence, $\tilde{\Upsilon}_2$ is Lipschitz continuous.

Next, we prove that $\tilde{\Upsilon}_3$ maps bounded sets into equicontinuous sets of $\mathscr{B}_h''$.

Let $0<\tau_1<\tau_2\leq t_1$. For each $u\in\mathscr{B}_{c'}$, we have

$$\begin{aligned}&\Big\|(\tilde{\Upsilon}_3\tilde{z})(\tau_2)-(\tilde{\Upsilon}_3\tilde{z})(\tau_1)\Big\|_{\mathcal{X}}\\ \leq&\Big\|\int_0^{\tau_2}(\tau_2-s)^{r-1}\mathcal{P}_r(\tau_2-s)\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\\&-\int_0^{\tau_1}(\tau_1-s)^{r-1}\mathcal{P}_r(\tau_1-s)\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_\tau,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big\|_{\mathcal{X}}\\ \leq&\Big\|\int_0^{\tau_1}[(\tau_2-s)^{r-1}\mathcal{P}_r(\tau_2-s)-\mathcal{P}_r(\tau_1-s)]\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big\|_{\mathcal{X}}\\&+\Big\|\int_{\tau_1}^{\tau_2}(\tau_2-s)^{r-1}\mathcal{P}_r(\tau_2-s)\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big\|_{\mathcal{X}}\\ \leq&\int_0^{\tau_1}\Big\|(\tau_2-s)^{r-1}\mathcal{P}_r(\tau_2-s)-(\tau_1-s)^{r-1}\mathcal{P}_r(\tau_1-s)\Big\|_{\mathcal{L}(\mathcal{X})}m(s)\Omega(\tilde{z}_s+\tilde{y}_s)ds\end{aligned}$$

$$+\frac{\mathscr{M}_A}{\Gamma(r+1)}\int_{\tau_1}^{\tau_2}(\tau_2-1)^{r-1}m(s)(1+b(\tilde{z}_s+\tilde{y}_s))ds.$$

For any $t\in(t_k,s_k]$, $\tau_1<\tau_2$, $k=1,2,\dots,m$, we have

$$\Big\|(\tilde{\Upsilon}_3\tilde{z})(\tau_2)-(\tilde{\Upsilon}_3\tilde{z})(\tau_1)\Big\|_{\mathcal{X}}=\|\mathfrak{I}_k(u(\tau_2))-(u(\tau_1))\| + \|\mathscr{G}_k(\tau_2,\tilde{z}_{\tau_2}+\tilde{y}_{\tau_2})-\mathscr{G}_k(\tau_1,\tilde{z}_{\tau_1}+\tilde{y}_{\tau_1})\Big\|_{\mathcal{X}}.$$

Similarly, for any $\tau_1,\tau_2\in(s_k,t_{k+1}]$, $\tau_1<\tau_2$, $k=1,2,\dots,m$, we get

$$\begin{aligned}
&\Big\|(\tilde{\Upsilon}_3\tilde{z})(\tau_2)-(\tilde{\Upsilon}_3\tilde{z})(\tau_1)\Big\|_{\mathcal{X}}\\
\le&\Big\|\Big[\mathcal{Q}_r(\tau_2-s_k)-\mathcal{Q}_r(\tau_1-s_k)\Big]\mathfrak{I}_k(u(t_k))+\mathscr{G}_k(s_k,\tilde{z}_{s_k}+\tilde{y}_{s_k})\Big\|_{\mathcal{X}}\\
&+\Bigg\|\Big[\mathcal{Q}_r(\tau_2-s_k)-\mathcal{Q}_r(\tau_1-s_k)\Big]\int_0^{s_k}(s_k-s)^{r-1}\mathcal{P}_r(s_k-s)\\
&\times\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Bigg\|_{\mathcal{X}}\\
&+\Bigg\|\int_0^{\tau_1}(\tau_1-s)^{r-1}[\mathcal{P}_r(\tau_2-s)-(\tau_1-s)]\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Bigg\|_{\mathcal{X}}\\
&+\Bigg\|\int_{\tau_1}^{\tau_2}(\tau_2-s)^{r-1}\mathcal{P}_r(\tau_2-s)\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Bigg\|_{\mathcal{X}}\\
\le&\|\mathcal{Q}_r(\tau_2-s_k)-\mathcal{Q}_r(\tau_1-s_k)\|_{\mathcal{L}(\mathcal{X})}\Big[\mathcal{L}_k+\mathcal{C}_i[\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n\Big]+\bar{\mathcal{C}}_i\\
&+\frac{\mathscr{M}_A r}{\Gamma(r+1)}\|\mathcal{Q}_r(\tau_2-s_k)\mathcal{Q}_r(\tau_1-s_k)\|_{\mathcal{L}(\mathcal{X})}\\
&\times\int_0^{s_k}(s_k-s)^{r-1}m(s)\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))ds\\
&+\int_0^{\tau_1}(\tau_1-s)^{r-1}\|\mathcal{P}_r(\tau_2-s)\mathcal{P}_r(\tau_1-s)\|_{\mathcal{L}(\mathcal{X})}\\
&\times m(s)\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))ds\\
&\times\frac{\mathscr{M}_A r}{\Gamma(r+1)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{r-1}m(s)\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_s+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))ds.
\end{aligned}$$

At the point when $\tau_2\to\tau_1$, the right hand side of the overhead inequality has a tendency to zero, afterwards by H(A6)-H(A8), $\mathcal{P}_r(t),\mathcal{Q}_r(t)$ are uniformly continuous, this demonstrates the equicontinuity.

We end of the step by proving that $\tilde{\Upsilon}_3$ is a $\hbar$-contraction. For any $\tilde{\mathcal{W}}\subset\mathscr{B}_h''$, $\tilde{\mathcal{W}}$ is piecewise equicontinuous since $\mathcal{P}_r(t)$ is equicontinuous. Here $\hbar_{PC}=\sup\{\hbar(\tilde{\mathcal{W}}(t))$, $t\in[s_k,t_{k+1}]\}$, $k=0,1,2,\dots,m$. Then, for each bounded set $\tilde{\mathcal{W}}\in PC$, from the following H(A4)-H(A6), we have for $t\in[0,t_1]$,

$$\begin{aligned}
\hbar(\tilde{\Upsilon}_3\tilde{\mathcal{W}})(t)=&\hbar\Big(\int_0^t(t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\Big(s,\tilde{\mathcal{W}}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big)\\
\le&\frac{\mathscr{M}_A}{\Gamma(r)}\int_0^t(t-s)^{r-1}\hbar\Big(\mathscr{F}\Big(s,\tilde{\mathcal{W}}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau+\tilde{y}_\tau)d\tau\Big)\Big)ds
\end{aligned}$$

$$\le \frac{\mathscr{M}_A}{\Gamma(r)}\int_0^t (t-s)^{r-1}\tilde{\eta}(s)\Big[\sup_{-\infty<\theta\le 0}\hbar(\tilde{\mathcal{W}}(s+\theta)+\tilde{y}(s+\theta))$$
$$+\int_0^s \zeta(s,\tau)\sup_{-\infty<\theta\le 0}\hbar(\tilde{\mathcal{W}}(s+\theta)+\tilde{y}(s+\theta))d\tau\Big]ds$$
$$\le \frac{\mathscr{M}_A}{\Gamma(r)}\int_0^t (t-s)^{r-1}\tilde{\eta}(s)\Big[\sup_{0\le\tau\le s}\hbar(\tilde{\mathcal{W}}(\tau))+\zeta^*\sup_{0\le\tau\le s}\hbar(\tilde{\mathcal{W}}(\tau))\Big]ds$$
$$\le \frac{\mathscr{M}_A(1+\zeta^*)}{\Gamma(r)}\int_0^t (t-s)^{r-1}\tilde{\eta}(s)ds\sup_{0\le s\le T}\hbar(\tilde{\mathcal{W}}(s))$$
$$\le \frac{\mathscr{M}_A(1+\zeta^*)t_1^r}{\Gamma(r+1)}\hbar(\tilde{\mathcal{W}})\int_0^t \tilde{\eta}(s)ds.$$

For any $t\in(t_k,s_k]$, $k=1,2,\dots,m$, we have

$$\hbar(\tilde{\Upsilon}_3\tilde{\mathcal{W}})(t)=\hbar\Big(\mathfrak{I}_k(\tilde{\mathcal{W}}_t+\tilde{y}_t)+\mathscr{G}_k(t,\tilde{\mathcal{W}}_t+\tilde{y}_t)\Big)$$
$$\le \mathcal{L}_k\sup_{-\infty\le\theta\le 0}\hbar\Big(\tilde{\mathcal{W}}(t+\theta)+\tilde{y}(t+\theta)\Big)+\tilde{\nu}_i\sup_{-\infty<\theta\le 0}\hbar\Big(\tilde{\mathcal{W}}(t+\theta)+\tilde{y}(t+\theta)\Big)$$
$$\le (\mathcal{L}_k+\tilde{\nu}_i)\sup_{0\le\tau\le T}\hbar(\tilde{\mathcal{W}}(\tau))$$
$$\le (\mathcal{L}_k+\tilde{\nu}_i)\hbar_{PC}(\tilde{\mathcal{W}}).$$

Similarly, for any $t\in(s_k,t_{k+1}]$, $k=1,2,\dots,m$, we have

$$\hbar(\tilde{\Upsilon}_3\tilde{\mathcal{W}})(t)\le \hbar\Big(\mathcal{Q}_r(t-s_k)\mathfrak{I}_k(\tilde{\mathcal{W}}_t+\tilde{y}_t)\Big)+\mathscr{G}_k(t,\tilde{\mathcal{W}}_t+\tilde{y}_t)$$
$$+\hbar\Big(\mathcal{Q}_r(t-s_k)\int_0^{s_k}(s_k-s)^{r-1}\mathcal{P}_r(s_k-s)$$
$$\mathscr{F}\Big(s,\tilde{\mathcal{W}}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big)$$
$$+\hbar\Big(\int_0^t(t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\Big(s,\tilde{\mathcal{W}}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau+\tilde{y}_\tau)d\tau\Big)ds\Big)$$
$$\le \mathscr{M}_A\hbar\Big(\mathfrak{I}_k(\tilde{\mathcal{W}}_{s_k}+\tilde{y}_{s_k})+\mathscr{G}_k(s_k,\tilde{\mathcal{W}}_{s_k}+\tilde{y}_{s_k})\Big)$$
$$+\frac{\mathscr{M}_A^2}{\Gamma(r)}\int_0^{s_k}(s_k-s)^{r-1}\hbar\Big(\mathscr{F}\Big(s,\tilde{\mathcal{W}}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau+\tilde{y}_\tau)d\tau\Big)\Big)ds$$
$$+\frac{\mathscr{M}_A}{\Gamma(r)}\int_0^t(t-s)^{r-1}\hbar\Big(\mathscr{F}\Big(s,\tilde{\mathcal{W}}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau+\tilde{y}_\tau)d\tau\Big)\Big)ds$$
$$\le \mathscr{M}_A\Big(\mathcal{L}_k\sup_{-\infty<\theta\le 0}\hbar(\tilde{\mathcal{W}}(s_k+\theta)+\tilde{y}(s_k+\theta))$$
$$+\tilde{\nu}_i\sup_{-\infty<\theta\le 0}\hbar(\tilde{\mathcal{W}}(s_k+\theta)+\tilde{y}(s_k+\theta))\Big)$$
$$+\frac{\mathscr{M}_A^2}{\Gamma(r)}\int_0^{s_k}(s_k-s)^{r-1}\tilde{\eta}(s)\Big[\sup_{-\infty<\theta\le 0}\hbar(\tilde{\mathcal{W}}(s+\theta)+\tilde{y}(s+\theta))$$

$$
\begin{aligned}
&+\int_0^s \zeta(s,\tau) \sup_{-\infty<\theta\leq 0} \hbar(\tilde{\mathcal{W}}(s+\theta)+\tilde{y}(s+\theta))d\tau\Big]ds \\
&+\frac{\mathscr{M}_A}{\Gamma(r)}\int_0^t (t-s)^{r-1}\tilde{\eta}(s)\Big[\sup_{-\infty<\theta\leq 0} \hbar(\tilde{\mathcal{W}}(s+\theta)+\tilde{y}(s+\theta)) \\
&+\int_0^s \zeta(s,\tau) \sup_{-\infty<\theta\leq 0} \hbar(\tilde{\mathcal{W}}(s+\theta)+\tilde{y}(s+\theta))d\tau\Big]ds \\
\leq&\mathscr{M}_A(\mathcal{L}_k+\tilde{\nu}_i)\sup_{0\leq\tau\leq T}\hbar(\tilde{\mathcal{W}}(\tau)) \\
&+\frac{\mathscr{M}_A^2}{\Gamma(r)}\int_0^{s_k}(s_k-s)^{r-1}\tilde{\eta}(s)\Big[\sup_{0\leq\tau\leq s}\hbar(\tilde{\mathcal{W}}(\tau))+\zeta^*\sup_{0\leq\tau\leq s}\hbar(\tilde{\mathcal{W}}(\tau))\Big]ds \\
&+\frac{\mathscr{M}_A}{\Gamma(r)}\int_0^{t}(t-s)^{r-1}\tilde{\eta}(s)\Big[\sup_{0\leq\tau\leq s}\hbar(\tilde{\mathcal{W}}(\tau))+\zeta^*\sup_{0\leq\tau\leq s}\hbar(\tilde{\mathcal{W}}(\tau))\Big]ds \\
\leq&\mathscr{M}_A(\mathcal{L}_k+\tilde{\nu}_i)\hbar(\tilde{\mathcal{W}})+\frac{\mathscr{M}_A^2(1+\zeta^*)}{\Gamma(r)}\int_0^{s_k}(s_k-s)^{r-1}\tilde{\eta}(s)ds\sup_{0\leq s\leq T}\hbar(\tilde{\mathcal{W}}(s)) \\
&+\frac{\mathscr{M}_A(1+\zeta^*)}{\Gamma(r)}\int_0^{t}(t-s)^{r-1}\tilde{\eta}(s)ds\sup_{0\leq s\leq T}\hbar(\tilde{\mathcal{W}}(s)) \\
\leq&\mathscr{M}_A(\mathcal{L}_k+\tilde{\nu}_i)\hbar_{PC}(\tilde{\mathcal{W}})+\frac{\mathscr{M}_A^2(1+\zeta^*)(t_{k+1})^r}{\Gamma(r+1)}\hbar_{PC}(\tilde{\mathcal{W}})\int_0^{s_k}\tilde{\eta}(s)ds \\
&+\frac{\mathscr{M}_A(1+\zeta^*)(t_{k+1})^r}{\Gamma(r+1)}\hbar_{PC}(\tilde{\mathcal{W}})\int_0^{t}\tilde{\eta}(s)ds.
\end{aligned}
$$

Therefore, for all $t\in\mathcal{J}$,

$$
\hbar(\tilde{\Upsilon}_3\tilde{\mathcal{W}})(t)\leq(\mathscr{M}_A+1)\Big((\mathcal{L}_k+\tilde{\nu}_i)+\frac{\mathcal{M}_A(1+\zeta^*)T^r}{\Gamma(r+1)}\int_0^T\tilde{\eta}(s)ds\Big)\hbar_{PC}(\tilde{\mathcal{W}})
$$

and

$$
\hbar(\tilde{\Upsilon}_3\tilde{\mathcal{W}})\leq\iota\hbar_{PC}(\tilde{\mathcal{W}})<\hbar_{PC}(\tilde{\mathcal{W}}),
$$

where $\iota=\max_{1\leq k\leq m}(\mathscr{M}_A+1)\Big((\mathcal{L}_k+\tilde{\nu}_i)+\frac{\mathscr{M}_A(1+\zeta^*)T^r}{\Gamma(r+1)}\int_0^T\tilde{\eta}(s)ds\Big)<1$.

Therefore, $\tilde{\Upsilon}$ is $\hbar$-contraction. By Lemma 2.4, we conclude that $\tilde{\Upsilon}$ has at least one fixed point in $\tilde{y}\in\tilde{\mathcal{W}}\subset\mathscr{B}_h''$. Let $u(t)=\tilde{y}(t)+\tilde{z}(t)$ on $t\in(-\infty,\mathcal{T}]$. Then, u is a fixed point of the operator $\tilde{\Upsilon}$ which is the mild solution of the system (1.1)–(1.3) and the proof of theorem is complete. □

Theorem 3.2. *Assume that the hypotheses* H(A1)-H(A9) *are satisfied, then the system* (1.1)–(1.3) *has atleast one mild solution on* $\mathcal{J}$, *for some*

$$
\mathcal{Z}^*=\Big[\mathscr{M}_A\Big(\mathcal{L}_k+\tilde{\nu}_i\Big)+\frac{2\mathscr{M}_A^2\mathscr{M}_6 r\mathcal{T}^r}{\Gamma(1+r)}\int_0^t\eta(s)ds+\frac{2\mathscr{M}_A\mathscr{M}_6 r\mathcal{T}^r}{\Gamma(1+r)}\int_0^t\eta(s)ds\Big]<1.
$$

Proof. Define the operator $\Upsilon : \mathscr{B}_h^{'} \to \mathscr{B}_h^{'}$ by

$$(\Upsilon u)(t) = \begin{cases} \Phi(t), \quad t \in (-\infty, 0], \\ \mathcal{Q}_r(t)\Big[\Phi(0) + \mathcal{G}(0, \Phi(0))\Big] - \mathcal{G}(t, u_t) + \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, u_s)ds \\ + \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\Big(s, u_s, \int_0^s \mathcal{H}(s,\tau,u_\tau)d\tau\Big)ds, \quad t \in [0, t_k], \\ k = 0, 1, 2, \dots, m, \\ \mathfrak{I}_k(u(t_k)) + \mathscr{G}(t, u_t), \quad t \in (t_k, s_k], \quad k = 1, 2, 3, \dots, m, \\ \mathcal{Q}_r(t - s_k)\mathscr{D}_k - \mathcal{G}(t, u_t) + \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, u_s)ds \\ + \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\mathscr{F}\Big(s, u_s, \int_0^s \mathcal{H}(s,\tau,u_\tau)d\tau\Big)ds, \quad t \in (s_k, t_{k+1}], \\ k = 1, 2, \dots, m. \end{cases}$$

Here Υ is well defined and shows that the operator Υ satisfied the hypotheses of Lemma 2.4. By applying same techniques as in Theorem 3.1. The proof consists of following steps.

Step 1. We show that there exists some $q > 0$ such that $\tilde{\Upsilon}(\mathscr{B}_q) \subseteq \mathscr{B}_q$. If it is not true, then for each positive number q, there exist a function $\tilde{z}^q(.) \in \mathscr{B}_q$ and some $t \in \mathfrak{J}$ such that $\|(\tilde{\Upsilon}\tilde{z}^q)(t)\| > q$.

Step 2. $\tilde{\Upsilon} : \mathscr{B}_h^{''} \to \mathscr{B}_h^{''}$ is continuous.

Step 3. $(\tilde{\Upsilon}_3 u)$ maps bounded sets into equicontinuous sets of $\mathscr{B}_h^{''}$.

Step 4. Mönch's condition holds.

Suppose that $\tilde{\mathcal{W}} \subseteq \mathscr{B}_h^{''}$ is countable and $\tilde{\mathcal{W}} \subseteq \text{conv}(\{0\} \cup \tilde{\Upsilon}_3(\tilde{\mathcal{W}}))$. We show that $\hbar(\tilde{\mathcal{W}}) = 0$, where $\hbar$ is the Hausdorff measure of noncompactness. Without loss of generality, we may suppose that $\tilde{\mathcal{W}} = \{u_n\}_{n=1}^{+\infty}$. We can easily verify that $\tilde{\mathcal{W}}$ is bounded and equicontinuous.

Now we need to show that $\tilde{\Upsilon}_3(\tilde{\mathcal{W}}(t))$ is relatively compact in $\mathcal{X}$ for each $t \in \mathfrak{J}$.

Case 1. For each $t \in [0, t_1]$, we get

$$\begin{aligned} &\hbar\Big(\{\tilde{\Upsilon}_3\tilde{\mathcal{W}}(t)\}_{n=1}^{+\infty}\Big) \\ \leq &\hbar\Big(\Big\{\int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s) \\ &\times \mathscr{F}\Big(s, \tilde{\mathcal{W}}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau + \tilde{y}_\tau)d\tau\Big)ds\Big\}_{n=1}^{+\infty}\Big) \\ \leq &\frac{2\mathscr{M}_A r}{\Gamma(1+r)}\int_0^t (t-s)^{r-1}\hbar\Big(\Big\{\mathscr{F}\Big(s, \tilde{\mathcal{W}}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau + \tilde{y}_\tau)d\tau\Big)\Big\}_{n=1}^{+\infty}\Big)ds \\ \leq &\frac{2\mathscr{M}_A r}{\Gamma(1+r)}\int_0^t (t-s)^{r-1}\eta(s)\Big[\sup_{-\infty<\theta\leq 0}\hbar\Big(\Big\{\tilde{\mathcal{W}}(s+\theta) + \tilde{y}(s+\theta)\Big\}_{n=1}^{+\infty}\Big) \\ &+ \hbar\Big(\Big\{\int_0^s \mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau + \tilde{y}_\tau)d\tau\Big\}_{n=1}^{+\infty}\Big)\Big]\,ds \\ \leq &\Big[\frac{2\mathscr{M}_A r}{\Gamma(1+r)}(1 + 2\zeta^*)\mathscr{M}_6\Big]\sup_{-\infty<\theta\leq 0}\hbar(\tilde{\mathcal{W}}(\tau))ds \end{aligned}$$

$$\leq \frac{\mathscr{M}_A t_1^r}{\Gamma(1+r)}(1+2\zeta^*)\mathscr{M}_{6\ \hbar_{PC}}(\tilde{\mathcal{W}}).$$

Case 2. For any $t \in (t_k, s_k],\ k = 1, 2, \ldots, m$, we have

$$\begin{aligned}
\hbar\Big(\{\tilde{\Upsilon}_3\tilde{\mathcal{W}}(t)\}_{n=1}^{+\infty}\Big) =&\hbar\Big(\mathfrak{I}_k(\tilde{\mathcal{W}}_t + \tilde{y}_t) + \mathscr{G}_k(t, \tilde{\mathcal{W}}_t + \tilde{y}_t)\Big)\\
\leq&\mathcal{L}_k \sup_{-\infty\leq\theta\leq 0} \hbar(\tilde{\mathcal{W}}(t+\theta) + \tilde{y}(t+\theta))\\
&+ \tilde{\nu}_i \sup_{-\infty<\theta\leq 0} \hbar(\tilde{\mathcal{W}}(t+\theta) + \tilde{y}(t+\theta))\\
\leq&(\mathcal{L}_k + \tilde{\nu}_i) \sup_{0\leq\tau\leq T} \hbar(\tilde{\mathcal{W}}(\tau)) \leq (\mathcal{L}_k + \tilde{\nu}_i)\hbar_{PC}(\tilde{\mathcal{W}}).
\end{aligned}$$

Case 3. Now, for every $t \in (s_k, t_{k+1}],\ k = 1, 2, \ldots, m$, we have

$$\begin{aligned}
\hbar\Big(\{\tilde{\Upsilon}_3\tilde{\mathcal{W}}(t)\}_{n=1}^{+\infty}\Big) \leq&\hbar(\mathcal{Q}_r(t-s_k)\mathfrak{I}_k(u_{s_k}) + \mathscr{G}_k(s_k, \tilde{\mathcal{W}}_{s_k} + \tilde{y}_{s_k}))\\
&+ \hbar\Big(\mathcal{Q}_r(t-s_k)\int_0^{s_k}(s_k-s)^{r-1}\mathcal{P}_r(s_k-s)\\
&\times \mathscr{F}\Big(s, \tilde{\mathcal{W}}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau + \tilde{y}_\tau)d\tau\Big)ds\Big)\\
&+ \hbar\Big(\int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\\
&\quad \mathscr{F}(s, \tilde{\mathcal{W}}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau + \tilde{y}_\tau)d\tau\Big)ds\Big)\\
\leq&\mathscr{M}_A\hbar(\mathfrak{I}_k(u_{s_k}) + \mathscr{G}_k(s_k, \tilde{\mathcal{W}}_{s_k} + \tilde{y}_{s_k})) + \frac{2\mathscr{M}_A^2 r}{\Gamma(1+r)}\int_0^{s_k}(s_k-s)^{r-1}\\
&\times \hbar\Big(\mathscr{F}\Big(s, \tilde{\mathcal{W}}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau + \tilde{y}_\tau)d\tau\Big)\Big)ds\\
&+ \frac{2\mathscr{M}_A r}{\Gamma(1+r)}\int_0^t (t-s)^{r-1}\\
&\times \hbar\Big(\mathscr{F}\Big(s, \tilde{\mathcal{W}}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{\mathcal{W}}_\tau + \tilde{y}_\tau)d\tau\Big)\Big)ds\\
\leq&\mathscr{M}_A\Big(\mathcal{L}_k + \tilde{\nu}_i \sup_{-\infty<\theta\leq 0}\hbar(\tilde{\mathcal{W}}(s_k+\theta) + \tilde{y}(s_k+\theta))\Big)\\
&+ \frac{2\mathscr{M}_A^2 r}{\Gamma(1+r)}\int_0^{s_k}(s_k-s)^{r-1}\eta(s)\Big[\sup_{-\infty<\theta\leq 0}\hbar(\tilde{\mathcal{W}}(s+\theta) + \tilde{y}(s+\theta))\\
&+ \int_0^s \zeta(s,\tau)\sup_{-\infty<\theta\leq 0}\hbar(\tilde{\mathcal{W}}(s+\theta) + \tilde{y}(s+\theta))d\tau\Big]ds\\
&+ \frac{2\mathscr{M}_A r}{\Gamma(1+r)}\int_0^t (t-s)^{r-1}\eta(s)\Big[\sup_{-\infty<\theta\leq 0}\hbar(\tilde{\mathcal{W}}(s+\theta) + \tilde{y}(s+\theta))\\
&+ \int_0^s \zeta(s,\tau)\sup_{-\infty<\theta\leq 0}\hbar(\tilde{\mathcal{W}}(s+\theta) + \tilde{y}(s+\theta))d\tau\Big]ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathscr{M}_A\Big(\mathcal{L}_k + \tilde{\nu}_i \sup_{-\infty\leq t\leq \mathfrak{I}} \hbar(\tilde{\mathcal{W}}(\tau))\Big) + \frac{2\mathscr{M}_A^2 r}{\Gamma(1+r)}\int_0^{s_k}(s_k-s)^{r-1}\\
&\quad\times \eta(s)\Big[\sup_{0\leq\tau\leq s}\hbar(\tilde{\mathcal{W}}(\tau)) + \zeta^*\sup_{0\leq\tau\leq s}\hbar(\tilde{\mathcal{W}}(\tau))\Big]ds\\
&\quad+\frac{2\mathscr{M}_A r}{\Gamma(1+r)}\int_0^t (t-s)^{r-1}\\
&\quad\times \eta(s)\Big[\sup_{0\leq\tau\leq s}\hbar(\tilde{\mathcal{W}}(\tau)) + \zeta^*\sup_{0\leq\tau\leq s}\hbar(\tilde{\mathcal{W}}(\tau))\Big]ds\\
&\leq \mathscr{M}_A\Big(\mathcal{L}_k + \tilde{\nu}_i\hbar_{PC}(\tilde{\mathcal{W}})\Big) + \frac{2\mathscr{M}_A^2 r}{\Gamma(1+r)}(1+2\zeta^*)\mathscr{M}_6\\
&\quad\times\int_0^{s_k}(s_k-s)^{r-1}\eta(s)ds\sup_{0\leq s\leq\mathfrak{I}}\hbar(\tilde{\mathcal{W}}(s))\\
&\quad+\frac{2\mathscr{M}_A r}{\Gamma(1+r)}(1+2\zeta^*)\mathscr{M}_6\int_0^t(t-s)^{r-1}\eta(s)ds\sup_{0\leq s\leq\mathfrak{I}}\hbar(\tilde{\mathcal{W}}(s))\\
&\leq \mathscr{M}_A\Big(\mathcal{L}_k + \tilde{\nu}_i\hbar_{PC}(\tilde{\mathcal{W}})\Big)\\
&\quad+\frac{2\mathscr{M}_A^2 r(t_{k+1})^r}{\Gamma(1+r)}(1+2\zeta^*)\mathscr{M}_6\hbar_{PC}(\tilde{\mathcal{W}})\int_0^{s_k}\eta(s)ds\\
&\quad+\frac{2\mathscr{M}_A r(t_{k+1})^r}{\Gamma(1+r)}(1+2\zeta^*)\mathscr{M}_6\hbar_{PC}(\tilde{\mathcal{W}})\int_0^{t}\eta(s)ds.\\
&\leq \mathscr{M}_A\Big[\Big(\mathcal{L}_k+\tilde{\nu}_i\Big) + \Big(\frac{2\mathscr{M}_A r\mathfrak{I}^r}{\Gamma(1+r)}(1+2\zeta^*)\mathscr{M}_6\\
&\quad+\frac{2r\mathfrak{I}^r}{\Gamma(1+r)}(1+2\zeta^*)\mathscr{M}_6\Big)\int_0^{\mathfrak{I}}\eta(s)ds\Big]\hbar_{PC}(\tilde{\mathcal{W}}).
\end{aligned}
$$

Along these lines, for all $t\in\mathfrak{I}$, we get

$$
\begin{aligned}
\hbar_{PC}(\tilde{\Upsilon}\tilde{\mathcal{W}})(t) \leq \Big[&\mathscr{M}_A\Big(\mathcal{L}_k+\tilde{\nu}_i\Big) + \frac{2\mathscr{M}_A^2\mathscr{M}_6 r\mathfrak{I}^r}{\Gamma(1+r)}\int_0^t\eta(s)ds\\
&+\frac{2\mathscr{M}_A\mathscr{M}_6 r\mathfrak{I}^r}{\Gamma(1+r)}\int_0^t\eta(s)ds\Big]\hbar_{PC}(\tilde{\mathcal{W}}(\tau)),
\end{aligned}
$$

which implies, by Lemma 2.2, $\hbar_{PC}(\tilde{\Upsilon}(\tilde{\mathcal{W}}))\leq \mathcal{Z}^*\hbar(\tilde{\mathcal{W}})$, where $\mathcal{Z}^*$ is defined in condition (3.2). Thus, from Mönch's condition, we get

$$
\hbar_{PC}(\tilde{\mathcal{W}}) \leq \hbar_{PC}\Big(\mathrm{conv}(\{0\}\cup(\tilde{\Upsilon}(\tilde{\mathcal{W}})))\Big) = \hbar_{PC}(\tilde{\Upsilon}(\mathcal{W})) \leq \mathcal{Z}^*\hbar_{PC}(\tilde{\mathcal{W}}),
$$

which implies that $\hbar_{PC}(\tilde{\mathcal{W}})=0$.

Hence, using Lemma 2.4, $\tilde{\Upsilon}$ has a fixed point $\tilde{y}$ in $\mathscr{B}_q$. Then, $u=\tilde{y}+\tilde{z}$ is a mild solution of system (1.1)–(1.3). This completes the proof. □

4. Controllability Results

We consider the Controllability of Fractional Neutral Integro-Differential Equation and Non-Instantaneous impulses with infinite delay of the form

$$
\begin{aligned}
{}^{C}D_t^r[u(t) - \mathcal{G}(t,u_t)] =& \mathfrak{A}u(t) + \mathscr{F}\Big(t, u_t, \int_0^t \mathcal{H}(t,s,u_s)ds\Big) + \mathfrak{B}u(t), \\
& t \in (s_k, t_{k+1}], \quad k = 0,1,2,\ldots,m, && (4.1)\\
u(t) =& \mathfrak{I}(u(t_k)) + \mathscr{G}_k(t,u_t), \quad t \in (s_k, t_k], \quad k = 1,2,\ldots,m && (4.2)\\
u(t) =& \Phi(t), \quad t \in (-\infty, 0], && (4.3)
\end{aligned}
$$

where ${}^{C}D_t^r$ denotes the Caputo derivative with $r \in (0,1)$. The control function $u(.)$ is given by $L^2(\mathcal{J}, U)$, a Banach space of admissible control function, with U as a Banach space. $\mathfrak{B}$ is a bounded linear operator from U into $\mathcal{X}$. $u_t : (-\infty, 0] \to \mathcal{X}$, defined by $u_t(s) = u(t+s)$, belongs to some abstract phase space $\mathscr{B}_h$. $\mathcal{G}, \mathscr{F}$, $k = 0,1,2,\ldots,m$, $\mathscr{G}_k$, $k = 1,2,\ldots,m$ are appropriate function $0 = s_0 < t_1 < t_2 < \cdots < t_m < b$ are fixed number and $\Delta u(t_k) = \mathfrak{I}(u(t_k)) = u(t_k^+) - u(t_k^-)$. Let $u(t_k^+)$ and $u(t_k^-)$ denote the right and left limits of u at $t = t_k$.

Definition 4.1. A function $u : (-\infty, b] \to \mathcal{X}$ is called a mild solution of the control system (4.1)–(4.3) if $u_0 = \Phi \in \mathscr{B}_h$ on $(-\infty, 0]$ and the integral equation

$$
u(t) = \begin{cases}
\mathcal{Q}_r(t)[\Phi(0) + \mathcal{G}(0,\Phi(0))] - \mathcal{G}(t,u_t) + \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s,u_s)ds \\
+ \int_0^t (t-s)^{r-1}\mathcal{P}_r(t-s)\Big[\mathscr{F}\Big(s,u_s,\int_0^s \mathcal{H}(s,\tau,u_\tau)d\tau\Big) + \mathfrak{B}u(s)\Big]ds, \\
t \in [0,t_1], \quad k = 0,1,\ldots,m, \\
\mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(t,u_t), \quad t \in (t_k, s_k], \quad k = 1,2,\ldots,m, \\
\mathcal{Q}_r(t-s_k)\mathscr{D}_k - \mathcal{G}(t,u_t) + \int_0^t (t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s,u_s)ds \\
+ \int_0^t (t-s)^{r-1}\mathcal{P}_t(t-s)\Big[\mathscr{F}\Big(s,u_s,\int_0^s \mathcal{H}(s,\tau,u_\tau)d\tau\Big) + \mathfrak{B}u(s)\Big]ds, \\
t \in (s_k, t_{k+1}], k = 1,2,\ldots,m,
\end{cases}
$$

where

$$
\begin{aligned}
\mathscr{D}_k =& (\mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(s_k, u_{s_k})) - \int_0^{s_k} (s_k - s)^{r-1}\mathfrak{A}\mathcal{P}_r(s_k - s)\mathcal{G}(s, u(s_k))ds \\
& - \int_0^{s_k} (s_k - s)^{r-1}\mathcal{P}_r(s_k - s)\Big[\mathscr{F}\Big(s, u_s, \int_0^s \mathcal{H}(s,\tau,u_\tau)d\tau\Big) + \mathfrak{B}u(s)\Big]ds, \\
& k = 1,2,\ldots,m. && (4.4)
\end{aligned}
$$

Definition 4.2. System (4.1)–(4.3) is said to be controllable on $\mathcal{J}$ if for every continuous inital function $\Phi \in \mathscr{B}_h$, $u_1 \in \mathcal{X}$, there exists a control $u \in L^2(\mathcal{J}, U)$ such that the mild solution $u(t)$ of satisfies $u(b) = u_1$.

For the study of the system (4.1)–(4.3), we introduce the following assumption.

H(A10): (i) The linear operator $\mathfrak{B}: L^2(\mathfrak{I}, U) \to L^1(\mathfrak{I}, U)$ is bounded, $\Xi : L^2(\mathfrak{I}, U) \to \mathfrak{X}$ is defined by

$$\Xi\, u = \int_0^b (b-s)^{r-1}\mathcal{P}_r(t-s)\mathfrak{B}u(s)ds,$$

has an inverse operator Ξ^{-1} which takes values in $L^2(\mathfrak{I}, U)/Ker\ \Xi$ and there exist two constants $\mathscr{M}_2, \mathscr{M}_3 > 0$ such that $\|\mathfrak{B}\| \le \mathscr{M}_2$ and $\|\Xi^{-1}\| \le \mathscr{M}_3$.

(ii) There exist a constant $q_0 \in (0, q)$ and $\mathcal{K}_\Xi \in L^{\frac{1}{q_0}}(\mathfrak{I}, \mathbb{R}^+)$ such that for any bounded subset $\mathfrak{Q} \subset \mathfrak{X}$, $\hbar((\Xi^{-1}\mathfrak{Q})(t)) \le \mathcal{K}_\Xi(t)\hbar(\mathfrak{Q})$.

The result is based on Mönch's fixed point theorem.

Theorem 4.1. *Assume that hypotheses* H(A1)-H(A8), H(A10) *are satisfied. Then, the system* (4.1)–(4.3) *is controllable on* $\mathfrak{I}$, *provided by*

$$\left(\left[1+\frac{2\mathscr{M}_A\mathscr{M}_2\mathscr{M}_5 r}{\Gamma(1+r)}\right](1+\zeta^*)\frac{2\mathscr{M}_A\mathscr{M}_6 r}{\Gamma(1+r)}+(\mathcal{L}_k+\tilde{\nu})\right)<1.$$

Proof. Using hypotheses H(A10), for an arbitrary function $u(\cdot) \in C$, we define the control $\widehat{X}_u(t)$ by

$$u_x(t)=\begin{cases}\Xi^{-1}\Bigg[\Big[u_b-\mathcal{Q}_r(b)\left(\Phi(0)+\mathcal{G}(0,\Phi(0))\right)-\mathcal{G}(b,u_b)\\ +\int_0^b(b-s)^{r-1}\mathfrak{A}\mathcal{P}_r(b-s)\mathcal{G}(s,u_s)ds\\ -\int_0^b(b-s)^{r-1}\mathcal{P}_r(b-s)\Big[\mathscr{F}\Big(s,u_s,\int_0^s\mathcal{H}(s,\tau,u_\tau)d\tau\Big)ds\Big]\Bigg](t),\\ t\in[0,t_1],\quad k=0,1,\dots,m,\\ \mathfrak{I}_k(u(t_k))+\mathscr{G}_k(t,u_t),\quad t\in(t_k,s_k],\quad k=1,2,\dots,m,\\ u_b-\mathcal{Q}_r(b-s_k)\mathscr{D}_k-\mathcal{G}(b,u_b)+\int_0^b(b-s)^{r-1}\mathfrak{A}\mathcal{P}_r(b-s)\mathcal{G}(s,u_s)ds\\ +\int_{s_k}^b(b-s)^{r-1}\mathcal{P}_r(b-s)\Big[\mathscr{F}\Big(s,x_s,\int_0^s\mathcal{H}(s,\tau,u_\tau)d\tau\Big)+\mathfrak{B}u_x(s)\Big]ds\Big],\\ t\in(s_k,t_{k+1}],\quad k=1,2,\dots,m.\end{cases}$$

We show that, using this control, the operator $\Upsilon : \mathscr{B}_h' \to \mathscr{B}_h'$ defined by

$$\Upsilon u(t)=\begin{cases}\Phi(t),\quad t\in-(\infty,0],\\ \mathcal{Q}_r(t)[\Phi(0)+\mathcal{G}(0,\Phi(0))]-\mathcal{G}(t,u_t)+\int_0^t(t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s,u_s)ds\\ +\int_0^t(t-s)^{r-1}\mathcal{P}_r(t-s)\Big[\mathscr{F}\Big(s,u_s,\int_0^s\mathcal{H}(s,\tau,u_\tau)d\tau\Big)+\mathfrak{B}u_x(s)\Big]ds,\\ t\in[0,t_1],\quad k=0,1,\dots,m,\\ \mathfrak{I}_k(u(t_k))+\mathscr{G}(t,u_t),\quad t\in(t_k,s_k],\quad k=1,2,\dots,m\\ \mathcal{Q}_r(t-s_k)\mathscr{D}_k-\mathcal{G}(t,u_t)+\int_0^t(t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s,u_s)ds\\ +\int_0^t(t-s)^{r-1}\mathcal{P}_r(t-s)\Big[\mathscr{F}\Big(s,u_s,\int_0^s\mathcal{H}(s,\tau,u_\tau)d\tau\Big)+\mathfrak{B}u_x(s)\Big]ds,\\ t\in(s_k,t_{k+1}],\quad k=1,2,\dots,m,\end{cases}$$

has a fixed points. The fixed point is then a solution of the given system. Clearly, $\Phi u(b) = u_1$, which implies the fractional system is controllable on $\mathcal{J}$.

Let $\Phi \in \mathscr{B}_h$, we define $\tilde{y}$ by

$$\tilde{y}(t) = \begin{cases} \Phi(t), & t \in (-\infty, 0], \\ \mathcal{Q}_r(t)\Phi(0), & t \in \mathcal{J}, \end{cases}$$

then $u(t) = \tilde{y}(t) + \tilde{z}(t)$, $t \in \mathcal{J}$. It is easy to see that u satisfies (4.4) if and only if $\tilde{z}$ satisfies $\tilde{z}_0 = 0$ and

$$\tilde{z}(t) = \begin{cases} 0, \quad t \leq 0, \\ \mathcal{Q}_r(t)\mathcal{G}(0, \Phi) - \mathcal{G}(t, \tilde{y}_t + \tilde{z}_t) - \int_0^t (t-s)^{r-1} \mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, \tilde{y}_s + \tilde{z}_s) ds \\ + \int_0^t (t-s)^{r-1} \mathcal{P}_r(t-s) \Big[\mathscr{F}\Big(s, \tilde{y}_s + \tilde{z}_s, \mathcal{H}(s, \tau, \tilde{y}_\tau + \tilde{z}_\tau) d\tau\Big) + \mathfrak{B} u_y(s) \Big] ds, \\ t \in [0, t_1], \quad k = 0, 1, \ldots, m, \\ \mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(t, \tilde{y}_t + \tilde{z}_t), t \in (t_k, s_k], \quad k = 1, 2, \ldots, m, \\ \mathcal{Q}_r(t - s_k)\mathscr{D}_k - \mathcal{G}(t, \tilde{y}_t + \tilde{z}_t) + \int_0^t (t-s)^{r-1} \mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s, \tilde{y}_s + \tilde{z}_s) ds \\ + \int_{s_k}^t (t-s)^{r-1} \mathcal{P}_r(t-s) \Big[\mathscr{F}\Big(s, x_s, \int_0^s \mathcal{H}(s, \tau, u_\tau) d\tau\Big) + \mathfrak{B} u_y(s) \Big] ds, \\ t \in (s_k, t_{k+1}], \quad k = 1, 2, \ldots, m, \end{cases}$$

where

$$\begin{aligned} u_y(s) =& \Xi^{-1} \Big[u_1 - \mathcal{Q}_r(b)[\Phi(0) - \mathcal{G}(0, \Phi(0))] - \mathcal{G}(s, \tilde{y}_s + \tilde{z}_s) + \mathfrak{I}(u(t_k)) + \mathscr{G}_k(s, \tilde{y}_s + \tilde{z}_s) \\ &+ \int_0^b (b-s)^{r-1} \mathfrak{A}\mathcal{P}_r(b-s)\mathcal{G}(s, \tilde{y}_s + \tilde{z}_s) ds \\ &+ \int_0^b (b-s)^{r-1} \mathcal{P}_r(b-s) \mathscr{F}\Big(s, \tilde{y}_s + \tilde{z}_s, \int_0^s \mathcal{H}(s, \tau, \tilde{y}_\tau + \tilde{z}_\tau) d\tau\Big) ds \Big](t). \end{aligned}$$

Let $\mathscr{B}_h^{''} = \{\tilde{z} \in \mathscr{B}_h^{'} : \tilde{z}_0 = 0 \in \mathscr{B}_h\}$. For any $\tilde{z} \in \mathscr{B}_h^{''}$,

$$\|\tilde{z}\|_b = \|\tilde{z}_0\|_{\mathscr{B}_h} + \sup\{\|\tilde{z}(s)\| : 0 \leq s \leq b\} = \sup\{\|\tilde{z}(s)\| : 0 \leq s \leq b\},$$

thus $(\mathscr{B}_h^{''}, \|\cdot\|_b)$ is a Banach space. Set $\mathscr{B}_q = \{\tilde{z} \in \mathscr{B}_h^{''} : \|\tilde{z}\|_b \leq q\}$ for some $q > 0$, $\mathscr{B}_q \subseteq \mathscr{B}_h^{''}$ is uniformly bounded, and for every $\tilde{z} \in \mathscr{B}_q$.

Define the operator $\tilde{\Upsilon} : \mathscr{B}_h^{''} \to \mathscr{B}_h^{''}$ by

$$\tilde{\Upsilon}\tilde{z}(t) = \begin{cases} \Phi(t), \quad t \in -(\infty, 0], \\ \mathcal{Q}_r(t)[\Phi(0) + \mathcal{G}(0, \Phi(0))] - \mathcal{G}(t, u_t) + \int_0^t (t-s)^{r-1} \mathfrak{A} \mathcal{P}_r(t-s) \mathcal{G}(s, u_s) ds \\ + \int_0^t (t-s)^{r-1} \mathcal{P}_r(t-s) \Big[\mathscr{F}\Big(s, u_s, \int_0^s \mathcal{H}(s, \tau, u_\tau) d\tau\Big) + \mathfrak{B} u_y(s) \Big] ds, \\ t \in [0, t_1], \quad k = 0, 1, \dots, m, \\ \mathfrak{I}_k(u(t_k)) + \mathscr{G}(t, u_t), \quad t \in (t_k, s_k], \quad k = 1, 2, \dots, m \\ \mathcal{Q}_r(t - s_k) \mathscr{D}_k - \mathcal{G}(t, u_t) + \int_0^t (t-s)^{r-1} \mathfrak{A} \mathcal{P}_r(t-s) \mathcal{G}(s, u_s) ds \\ + \int_0^t (t-s)^{r-1} \mathcal{P}_r(t-s) \Big[\mathscr{F}\Big(s, u_s, \int_0^s \mathcal{H}(s, \tau, u_\tau) d\tau\Big) + \mathfrak{B} u_y(s) \Big] ds, \\ t \in (s_k, t_{k+1}], \quad k = 1, 2, \dots, m. \end{cases}$$

Thus, the operator Υ has a fixed point is equivalent to $\tilde{\Upsilon}$ has one. So our goal is to show that $\tilde{\Upsilon}$ has a fixed point and the proof is given in the following steps.

Step 1. There exists $q > 0$ such that $\tilde{\Upsilon}(\mathscr{B}_q) \subseteq \mathscr{B}_q$. If it is not true, then for each positive number q, there exists a function $\tilde{z}^q(\cdot) \in \mathscr{B}_q$ and some $t \in \mathcal{J}$ such that $\|(\tilde{\Upsilon}\tilde{z}^q)(t)\| > q$ for some $t \in \mathcal{J}$.

Then by hypotheses H(A4) (iii), H(A5) (iii), H(A6) (ii), H(A10) (ii) and Lemma 2.1 (1), we have

$$\begin{aligned} q <& \|(\tilde{\Upsilon}\tilde{z}^q)(t)\| \leq \|\mathcal{Q}_r(t)\|_{\mathcal{L}(\mathcal{X})} \|\mathcal{G}(0, \Phi)\|_{\mathcal{X}} + \|\mathcal{G}(t, \tilde{z}_t + \tilde{y}_t)\|_{\mathcal{X}} \\ &+ \Big\| \int_0^t (t-s)^{r-1} \mathfrak{A} \mathcal{P}_r(t-s) \mathcal{G}(s, \tilde{z}_s + \tilde{y}_s) ds \Big\|_{\mathcal{X}} \\ &+ \Big\| \int_0^t (t-s)^{r-1} \mathcal{P}_r(t-s) \Big[\mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s, \tau, \tilde{z}_\tau + \tilde{y}_\tau) d\tau\Big) + \mathfrak{B} u_{y^q}(s) \Big] ds \Big\| \\ \leq& \mathscr{M}_A \mathscr{M}_0 [\mathscr{C}_1 \|\Phi\|_{\mathscr{B}_h} + \mathscr{C}_2] + \mathscr{M}_0 \mathscr{C}_1 (\mathcal{D}_1 \|\tilde{z}\|_t + \tilde{c}_n) + \mathscr{M}_0 \mathscr{C}_2 \\ &+ \frac{\mathscr{M}_{1-\beta} \Gamma(\beta+1)}{\Gamma(r\beta+1)} (\mathscr{C}_1 (\mathcal{D}_1 \|\tilde{z}\|_t + \tilde{c}_n) + \mathscr{C}_2) \frac{t_1^{r\beta}}{\beta} \\ &+ \frac{\mathscr{M}_A t_1^r}{\Gamma(r+1)} \Omega(\mathcal{D}_1 \|\tilde{z}\|_s + \tilde{c}_n + bv(\tau)(\mathcal{D}_1 \|\tilde{z}\|_\tau + \tilde{c}_n)) \sup_{t \in \mathcal{J}} m(s) \\ &+ \frac{\mathscr{M}_A \mathscr{M}_2 r}{\Gamma(r+1)} \sqrt{\frac{b^{2r-1}}{2r-1}} \|u_{y^q}\|_{L^2} \\ =& \mathcal{J}_1, \end{aligned}$$

where

$$\begin{aligned} \|u_{y^q}\|_{L^2} = \mathscr{M}_3 \Big[& \|u_1\| + \mathscr{M}_A \mathscr{M}_0 [\mathscr{C}_1 \|\Phi\|_{\mathscr{B}_h} + \mathscr{C}_2] + \mathscr{M}_0 \mathscr{C}_1 (\mathcal{D}_1 \|\tilde{z}\|_t + \tilde{c}_n) + \mathscr{M}_0 \mathscr{C}_2 \\ &+ \mathcal{L}_k + \mathcal{C}_i [\mathcal{D}_1 \|\tilde{z}\|_t + \tilde{c}_n] + \bar{\mathcal{C}}_i \\ &+ \frac{\mathscr{M}_{1-\beta} \Gamma(\beta+1)}{\Gamma(r\beta+1)} (\mathscr{C}_1 (\mathcal{D}_1 \|\tilde{z}\|_t + \tilde{c}_n) + \mathscr{C}_2) \frac{t_1^{r\beta}}{\beta} \end{aligned}$$

$$+\frac{\mathscr{M}_A\mathscr{M}_4t_1^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathcal{J}}m(s)\Big].$$

For any $t\in(t_k,s_k]$, $k=1,2,\dots,m$, we have

$$\begin{aligned}q<\|(\tilde{\Upsilon}\tilde{z}^q)(t)\|&\leq\|\mathfrak{I}_k(u(t_k))\|_{\mathcal{X}}+\|\mathscr{G}_k(t,\tilde{z}_t+\tilde{y}_t)\|_{\mathcal{X}}\\&\leq\mathcal{L}_k+\mathcal{C}_i[\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n]+\bar{\mathcal{C}}_i\\&=\mathcal{J}_2,\end{aligned}$$

and for $t\in(s_k,t_{k+1}]$, $k=1,2,\dots,m$,

$$\begin{aligned}q<\|(\tilde{\Upsilon}\tilde{z}^q)(t)\|\leq&\|\mathcal{Q}_r(t-s_k)\|_{\mathcal{L}(\mathcal{X})}\Big[\|\mathfrak{I}_k(u(t_k))\|+\|\mathscr{G}_k(s_k,\tilde{z}_{s_k}+\tilde{y}_{s_k})\|_{\mathcal{X}}\\&+\|\mathcal{G}(s_k,\tilde{z}_{s_k}+\tilde{y}_{s_k})\|_{\mathcal{X}}\\&+\Big\|\int_0^{s_k}(s_k-s)^{r-1}\mathfrak{A}\mathcal{P}_r(s_k-s)\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)ds\Big\|\\&+\Big\|\int_0^{s_k}(s_k-s)^{r-1}\mathcal{P}_r(s_k-s)\\&\times\Big[\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)+\mathfrak{B}_{u_y^q}(s)\Big]ds\Big\|_{\mathcal{X}}\Big]\\&+\|\mathscr{G}(t,\tilde{z}_t+\tilde{y}_t)\|_{\mathcal{X}}+\Big\|\int_0^t(t-s)^{r-1}\mathfrak{A}\mathcal{P}_r(t-s)\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)ds\Big\|_{\mathcal{X}}\\&+\Big\|\int_0^t(t-s)^{r-1}\mathcal{P}_r(t-s)\Big[\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)\\&+\mathfrak{B}_{u_y^q}(s)\Big]ds\Big\|_{\mathcal{X}}\\\leq&\mathscr{M}_A\Bigg[\mathcal{L}_k+\mathcal{C}_i[\mathcal{D}_1\|\tilde{z}\|_{s_k}+\tilde{c}_n]+\bar{\mathcal{C}}_i+\mathscr{M}_0\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_{s_k}+\tilde{c}_n)+\mathscr{M}_0\mathscr{C}_2\\&+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_{s_k}+\tilde{c}_n)+\mathscr{C}_2)\frac{(s_k)^{r\beta}}{\beta}\\&+\frac{\mathscr{M}_A\mathscr{M}_4(s_k)^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathcal{J}}m(s)\\&+\frac{\mathscr{M}_A\mathscr{M}_2(s_k^r)}{\Gamma(r+1)}\sqrt{\frac{b^{2r-1}}{2r-1}}\mathscr{M}_3\Big[\|u_1\|+\mathscr{M}_A\mathscr{M}_0[\mathscr{C}_1\|\Phi\|_{\mathscr{B}_h}+\mathscr{C}_2]\\&+\mathscr{M}_0\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{M}_0\mathscr{C}_2+\mathcal{L}_k+\mathcal{C}_i[\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n]+\bar{\mathcal{C}}_i\\&+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}(\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{C}_2)\frac{t_1^{r\beta}}{\beta}\\&+\frac{\mathscr{M}_A\mathscr{M}_4t_1^r}{\Gamma(r+1)}\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathcal{J}}m(s)\Big]\Bigg]\\&+\mathscr{M}_0\mathscr{C}_1(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{M}_0\mathscr{C}_2\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}(\mathscr{C}_1(\mathfrak{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{C}_2)\frac{(t_{k+1})^{r\beta}}{\beta}\\
&+ \frac{\mathscr{M}_A\mathscr{M}_4(t_{k+1})^r}{\Gamma(r+1)}\Omega(\mathfrak{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathfrak{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathfrak{I}} m(s)\\
&+ \frac{\mathscr{M}_A\mathscr{M}_2(s_k^r)}{\Gamma(r+1)}\sqrt{\frac{b^{2r-1}}{2r-1}}\mathscr{M}_3\Big[\|u_1\|+\mathscr{M}_A\mathscr{M}_0[\mathscr{C}_1\|\Phi\|_{\mathscr{B}_h}+\mathscr{C}_2]\\
&+ \mathscr{M}_0\mathscr{C}_1(\mathfrak{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{M}_0\mathscr{C}_2+\mathcal{L}_k+\mathfrak{C}_i[\mathfrak{D}_1\|\tilde{z}\|_t+\tilde{c}_n]+\bar{\mathfrak{C}}_i\\
&+ \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}(\mathscr{C}_1(\mathfrak{D}_1\|\tilde{z}\|_t+\tilde{c}_n)+\mathscr{C}_2)\frac{t_1^{r\beta}}{\beta}\\
&+ \frac{\mathscr{M}_A\mathscr{M}_4 t_1^r}{\Gamma(r+1)}\Omega(\mathfrak{D}_1\|\tilde{z}\|_s+\tilde{c}_n+bv(\tau)(\mathfrak{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))\sup_{t\in\mathfrak{I}} m(s)\Big].
\end{aligned}
$$

Then, for all $t \in \mathfrak{I}$, we find that

$$
\begin{aligned}
\|(\tilde{\Upsilon}\tilde{z}^q)(t)\| \leq& \mathscr{E}^* + \left(1+\frac{\mathscr{M}_A\mathscr{M}_2\mathscr{M}_3 r}{\Gamma(r+1)}\sqrt{\frac{b^{2r-1}}{2r-1}}\right)\\
&\times\left[\Big[(\mathcal{L}_k+\mathfrak{C}_i+\mathscr{M}_0\mathscr{C}_1)+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\mathscr{C}_1\frac{T^{r\beta}}{\beta}\Big](\mathscr{D}_1\|\tilde{z}\|_t+\tilde{c}_n)\right.\\
&\left.+\frac{\mathscr{M}_A\mathscr{M}_4(\mathscr{M}_A+1)T^r}{\Gamma(r+1)}\Omega\Big(\mathfrak{D}_1\|\tilde{z}\|_t+\tilde{c}_n+bv(\tau)(\mathfrak{D}_1\|\tilde{z}\|_t+\tilde{c}_n)\Big)\right]\\
&\times\sup_{t\in\mathfrak{I}} m(s)\frac{\mathscr{M}_A\mathscr{M}_2\mathscr{M}_3(\mathscr{M}_A+1)T^r}{\Gamma(r+1)}\sqrt{\frac{b^{2r-1}}{2r-1}},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathscr{E}^* =& \max_{1\leq k\leq m}\Big\{\mathscr{M}_A\mathscr{M}_0[\mathscr{C}_1\|\Phi\|_{\mathscr{B}_h}+\mathscr{C}_2]+\Big(\frac{\mathscr{M}_A\mathscr{M}_2\mathscr{M}_3 r}{\Gamma(r+1)}\sqrt{\frac{2r-1}{2r-1}}+1\Big)\\
&\times\Big(\mathscr{M}_0\mathscr{C}_2+\tilde{C}_i+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\mathscr{C}_2\frac{T^{r\beta}}{\beta}\Big)\Big\}\\
&+\frac{\mathscr{M}_A T^r}{\Gamma(r+1)}\Omega\Big(\mathfrak{D}_1\|\tilde{z}\|_t+\tilde{c}_n+bv(\tau)(\mathfrak{D}_1\|\tilde{z}\|_t+\tilde{c}_n)\Big)\Big]\sup_{t\in\mathfrak{I}} m(s).
\end{aligned}
$$

Combining the above equation

$$
\begin{aligned}
q <& \|(\tilde{\Upsilon}\tilde{z}^q)(t)\|\\
\leq& \mathscr{E}^* + \left(1+\frac{\mathscr{M}_A\mathscr{M}_2\mathscr{M}_3 r}{\Gamma(r+1)}\sqrt{\frac{b^{2r-1}}{2r-1}}\right)\Big[\Big[(\mathcal{L}_k+\mathfrak{C}_i+\mathscr{M}_0\mathscr{C}_1)+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\mathscr{C}_1\frac{T^{r\beta}}{\beta}\Big]\\
&\times(\mathscr{D}_1\|\tilde{z}\|_t+\tilde{c}_n)
\end{aligned}
$$

$$+\frac{\mathscr{M}_A\mathscr{M}_4(\mathscr{M}_A+1)T^r}{\Gamma(r+1)}\Omega\Big(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)\Big)\Big]$$
$$\times\sup_{t\in\mathfrak{I}}m(s)\frac{\mathscr{M}_A\mathscr{M}_2\mathscr{M}_3(\mathscr{M}_A+1)T^r}{\Gamma(r+1)}\sqrt{\frac{b^{2r-1}}{2r-1}}.$$

Now dividing on both sides by q and taking the limit as $q\to\infty$, we get

$$\begin{aligned}1\leq&\|(\tilde{\Upsilon}\tilde{z}^q)(t)\|\\ \leq&\mathscr{E}^*+\left(1+\frac{\mathscr{M}_A\mathscr{M}_2\mathscr{M}_3 r}{\Gamma(r+1)}\sqrt{\frac{b^{2r-1}}{2r-1}}\right)\\ &\times\left[\Big[(\mathcal{L}_k+\mathfrak{C}_i+\mathscr{M}_0\mathscr{C}_1)+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\mathscr{C}_1\frac{T^{r\beta}}{\beta}\Big](\mathscr{D}_1\|\tilde{z}\|_t+\tilde{c}_n)\right.\\ &\left.+\frac{\mathscr{M}_A\mathscr{M}_4(\mathscr{M}_A+1)T^r}{\Gamma(r+1)}\Omega\Big(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_t+\tilde{c}_n)\Big)\right]\sup_{t\in\mathfrak{I}}m(s)\\ &\times\frac{\mathscr{M}_A\mathscr{M}_2\mathscr{M}_3(\mathscr{M}_A+1)T^r}{\Gamma(r+1)}\sqrt{\frac{b^{2r-1}}{2r-1}}.\end{aligned}$$

We get $1\leq 0$. This is contradiction. Hence, for some integer $\tilde{\Upsilon}(\mathscr{B}_q)\subseteq\mathscr{B}_q$.

Step 2. $\tilde{\Upsilon}:\mathscr{B}_h^{''}\to\mathscr{B}_h^{''}$ is continuous,

$$F_n(s)=\mathscr{F}\Big(s,\tilde{z}_s^{(n)}+\tilde{c}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau^{(n)}+\tilde{y}_\tau)d\tau\Big),$$
$$F(s)=\mathscr{F}\Big(s,\tilde{z}_s+\tilde{c}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big).$$

For this purpose let $\{\tilde{z}^{(n)}\}_{n=0}^{+\infty}\subseteq\mathscr{B}_h^{''}$ with $\tilde{z}^{(n)}\to\tilde{z}$ in $\mathscr{B}_h^{''}$. Then there is a number $c'>0$ such that $\|\tilde{z}^{(n)}(t)\|\leq c'$ for all n and a.e. $t\in\mathfrak{I}$, so $\tilde{z}^{(n)}\in\mathscr{B}_{c'}=\{\tilde{z}\in\mathscr{B}_h^{''}:\|\tilde{z}\|_{\mathscr{B}_h^{''}}\leq c'\}\subseteq\mathscr{B}_h^{''}$ and $\tilde{z}\in B_{c'}$. From remark, we have $\|\tilde{z}_t+\tilde{y}_t\|_{\mathscr{B}_h}\leq c^{''}$, $t\in\mathfrak{I}$.

By H(A4), H(A5), Remark $P_{12},P_{13},P_{14},P_{15}$, and Lebesgue's dominated convergence theroem, we obtain, for $t\in[0,t_1]$,

$$\begin{aligned}&\|(\tilde{\Upsilon}\tilde{z}^n)(t)-(\tilde{\Upsilon}\tilde{z})(t)\|_{\mathfrak{X}}\\ \leq&\mathscr{M}_0\Big\|(\mathfrak{A})^\beta\Big[\mathscr{G}(t,\tilde{z}_t^n+\tilde{y}_t)-\mathscr{G}(t,\tilde{z}_t+\tilde{y}_t)\Big]\Big\|_{\mathfrak{X}}+\frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)}\cdot\frac{(t_1)^{r\beta}}{\beta}\\ &\times\int_0^t(t-s)^{r-1}\Big\|(\mathfrak{A})^\beta\Big[\mathcal{G}(s,\tilde{z}_s^n+\tilde{y}_s)-\mathcal{G}(s,\tilde{z}_s+\tilde{y}_s)\Big]\Big\|_{\mathfrak{X}}ds\\ &+\frac{\mathscr{M}_A(t_1)^r}{\Gamma(r+1)}\int_0^t(t-s)^{r-1}\left[\Big\|\mathscr{F}\Big(s,\tilde{z}_s^n+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau^n+\tilde{y}_\tau)d\tau\Big)+\mathfrak{B}u_y^n\right.\\ &\left.-\mathscr{F}\Big(s,\tilde{z}_s+\tilde{y}_s,\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau+\tilde{y}_\tau)d\tau\Big)+\mathfrak{B}u_y\Big\|_{\mathfrak{X}}\right]ds\to 0\quad\text{as } n\to+\infty.\end{aligned}$$

For all $t \in (t_k, s_k]$, $k = 1, 2, \ldots, m$, we obtain

$$\|(\tilde{\Upsilon}\tilde{z}^n)(t) - (\tilde{\Upsilon}\tilde{z})(t)\|_{\mathcal{X}} = 0.$$

In the same way, for all $t \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, we have

$$\begin{aligned}
&\|(\tilde{\Upsilon}\tilde{z}^n)(t) - (\tilde{\Upsilon}\tilde{z})(t)\|_{\mathcal{X}} \\
\leq &\|\mathcal{Q}_r(t - s_k)\|_{\mathcal{L}(\mathcal{X})}\Big[\|\mathfrak{I}_k(u(t_k))\|_{\mathcal{X}} + \|\mathscr{G}_k(s_k, \tilde{z}^n_{s_k} + \tilde{y}_{s_k}) - \mathscr{G}_k(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k})\|_{\mathcal{X}} \\
&+ \Big\|(\mathfrak{A})^{-\beta}\Big\|\Big\|(\mathfrak{A})^{\beta}\mathcal{G}(s_k, \tilde{z}^n_{s_k} + \tilde{y}_{s_k}) - (\mathfrak{A})^{\beta}\mathcal{G}(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k})\Big\|_{\mathcal{X}} \\
&+ \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)} \cdot \frac{(s_k)^{r\beta}}{\beta}\int_0^{s_k}(s_k - s)^{r-1} \\
&\times \Big\|(\mathfrak{A})^{\beta}\Big[\mathcal{G}(s_k, \tilde{z}^n_{s_k} + \tilde{y}_{s_k}) - \mathcal{G}(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k})\Big]\Big\|_{\mathcal{X}} ds \\
&+ \frac{\mathscr{M}_A(s_k)^r}{\Gamma(r+1)}\int_0^{s_k}(s_k - s)^{r-1}\Big\|\mathscr{F}\Big(s, \tilde{z}^n_{s_k} + \tilde{y}_{s_k}, \int_0^s \mathcal{H}(s, \tau, \tilde{z}^n_\tau + \tilde{y}_\tau)d\tau\Big) + \mathfrak{B}u^n_y \\
&- \mathscr{F}\Big(s, \tilde{z}_{s_k} + \tilde{y}_{s_k}, \int_0^s \mathcal{H}(s, \tau, \tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big) + \mathfrak{B}u_y\Big\|_{\mathcal{X}} ds\Big] \\
&+ \Big\|(\mathfrak{A})^{-\beta}\Big\|\Big\|(\mathfrak{A})^{\beta}\mathcal{G}(t, \tilde{z}^n_t + \tilde{y}_t) - (\mathfrak{A})^{\beta}\mathcal{G}(t, \tilde{z}_t + \tilde{c}_t)\Big\|_{\mathcal{X}} \\
&+ \frac{\mathscr{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(r\beta+1)} \cdot \frac{(t_{k+1})^{r\beta}}{\beta}\int_0^t \Big[\Big\|(\mathfrak{A})^{\beta}\Big[\mathcal{G}(s, \tilde{z}^n_s + \tilde{y}_s) - (\mathfrak{A})^{\beta}\mathcal{G}(s, \tilde{z}_s + \tilde{y}_s)\Big]\Big\|_{\mathcal{X}} \\
&+ \frac{\mathscr{M}_A(t_{k+1})^r}{\Gamma(r+1)}\int_0^t(s_k - s)^{r-1}\Big\|\mathscr{F}\Big(s, \tilde{z}^n_s + \tilde{y}_s, \int_0^s \mathcal{H}(s, \tau, \tilde{z}^n_\tau + \tilde{y}_\tau)d\tau\Big) + \mathfrak{B}u^n_y \\
&- \mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s, \tau, \tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big) + \mathfrak{B}u_y\Big\|_{\mathcal{X}}\Big]ds \to 0 \quad \text{as } n \to +\infty.
\end{aligned}$$

It is simple to see that

$$\lim_{n\to+\infty}\Big\|(\tilde{\Upsilon}\tilde{z}^n) - (\tilde{\Upsilon}\tilde{z})\Big\|_{\mathscr{B}''_h} = 0.$$

Thus, $\tilde{\Upsilon}$ is continuous.

Step 3. $(\tilde{\Upsilon}_2\tilde{z})$ maps bounded into equicontinuous set of $\mathscr{B}''_h$.

Let $0 < \tau_1 < \tau_2 \leq t_1$. For each $\tilde{z} \in \mathscr{B}''_h$, we have

$$\begin{aligned}
&\Big\|(\tilde{\Upsilon}_2\tilde{z})(\tau_2) - (\tilde{\Upsilon}_2\tilde{z})(\tau_1)\Big\|_{\mathscr{B}''_h} \\
\leq &\Big\|\int_0^{\tau_2}(\tau_2 - s)^{r-1}\mathcal{P}_r(\tau_2 - s)\Big[\mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s, \tau, \tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big) + \mathfrak{B}u_y\Big]ds\Big\| \\
&- \Big\|\int_0^{\tau_1}(\tau_1 - s)^{r-1}\mathcal{P}_r(\tau_1 - s)\Big[\mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s, \tau, \tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big) + \mathfrak{B}u_y\Big]ds\Big\| \\
\leq &\Big\|\int_0^{\tau_1}(\tau_2 - s)^{r-1}\Big[\mathcal{P}_r(\tau_2 - s) - \mathcal{P}_r(\tau_1 - s)\Big]
\end{aligned}$$

$$\times \Big[\mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big) + \mathfrak{B}u_y\Big]ds$$
$$+ \int_{\tau_1}^{\tau_2} \Big\|(\tau_2 - s)^{r-1}\mathcal{P}_r(\tau_2 - s) - (\tau_1 - s)^{r-1}\mathcal{P}_r(\tau_1 - s)\Big\|$$
$$\times \Big[\mathscr{F}\Big(s, \tilde{z}_s + \tilde{c}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big) + \mathfrak{B}u_y\Big]ds.$$

For all $\tau_1, \tau_2 \in (t_k, s_k]$, $\tau_1 < \tau_2$, $k = 1, 2, \ldots, m$, we have

$$\Big\|(\tilde{\Upsilon}_2\tilde{z})(\tau_2) - (\tilde{\Upsilon}_2\tilde{z})(\tau_1)\Big\|_{\mathscr{B}_h''} = \Big\|\mathfrak{I}_k(u(\tau_2) - u(\tau_1))\Big\|$$
$$+ \Big\|\mathscr{G}_k(\tau_2, \tilde{z}_{\tau_2} + \tilde{y}_{\tau_2}) - \mathscr{G}_k(\tau_1, \tilde{z}_{\tau_1} + \tilde{y}_{\tau_1})\Big\| = 0,$$

and for $\tau_2, \tau_1 \in (s_k, t_{k+1}]$, $\tau_1 < \tau_2$, $k = 1, 2, \ldots, m$, we get

$$\Big\|(\tilde{\Upsilon}_2\tilde{z})(\tau_2) - (\tilde{\Upsilon}_2\tilde{z})(\tau_1)\Big\|_{\mathscr{B}_h''}$$
$$\leq \Big\|\Big[\mathcal{Q}_r(\tau_2 - s_k) - \mathcal{Q}_r(\tau_1 - s_k)\Big]\mathfrak{I}_k(u(t_k)) + \mathscr{G}_k(s_k, \tilde{z}_{s_k} + \tilde{y}_{s_k})\Big\|$$
$$+ \Big\|\Big[\mathcal{Q}_r(\tau_2 - s_k) - \mathcal{Q}_r(\tau_1 - s_k)\Big]\int_0^{s_k}(s_k - s)^{r-1}\mathcal{P}_r(s_k - s)$$
$$\times \Big[\mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big)ds + \mathfrak{B}u_y\Big]\Big\|ds$$
$$+ \Big\|\int_0^{\tau_1}(\tau_1 - s)^{r-1}[\mathcal{P}_r(\tau_2 - s) - (\tau_1 - s)]$$
$$\times \Big[\mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big)ds + \mathfrak{B}u_y\Big]\Big\|ds$$
$$+ \Big\|\int_{\tau_1}^{\tau_2}(\tau_2 - s)^{r-1}\mathcal{P}_r(\tau_2 - s)$$
$$\times \Big[\mathscr{F}\Big(s, \tilde{z}_s + \tilde{y}_s, \int_0^s \mathcal{H}(s,\tau,\tilde{z}_\tau + \tilde{y}_\tau)d\tau\Big)ds + \mathfrak{B}u_y\Big]\Big\|ds$$
$$\leq \Big\|\mathcal{Q}_r(\tau_2 - s_k) - \mathcal{Q}_r(\tau_1 - s_k)\Big\|_{\mathcal{L}(\mathcal{X})}\Big[\mathcal{L}_k + \mathcal{C}_i[\mathcal{D}_1\|\tilde{z}\|_t + \tilde{c}_n]\Big] + \bar{\mathcal{C}}_i$$
$$+ \frac{\mathscr{M}_A r}{\Gamma(r+1)}\|\mathcal{Q}_r(\tau_2 - s_k)\mathcal{Q}_r(\tau_1 - s_k)\|_{\mathcal{L}(\mathcal{X})}$$
$$\times \int_0^{s_k}(s_k - s)^{r-1}m(s)\Omega(\mathcal{D}_1\|\tilde{z}\|_s + \tilde{y}_s + bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau + \tilde{y}_n))ds$$
$$+ \frac{\mathscr{M}_A\mathscr{M}_2 r}{\Gamma(1+r)}\sqrt{\frac{b^{2q-1}}{2q-1}}\|u_{y^q}\|_{L^2} + \int_0^{\tau_1}(\tau_1 - s)^{r-1}\|\mathcal{P}_r(\tau_2 - s)\mathcal{P}_r(\tau_1 - s)\|_{\mathcal{L}(\mathcal{X})}$$
$$\times m(s)\Omega(\mathcal{D}_1\|\tilde{z}\|_s + \tilde{c}_s + bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau + \tilde{c}_n))ds + \frac{\mathscr{M}_A\mathscr{M}_2 r}{\Gamma(1+r)}\sqrt{\frac{b^{2q-1}}{2q-1}}\|u_{y^q}\|_{L^2}$$
$$+ \int_0^{\tau_1}(\tau_1 - s)^{r-1}\|\mathcal{P}_r(\tau_2 - s)\mathcal{P}_r(\tau_1 - s)\|_{\mathcal{L}(\mathcal{X})}$$

$$\times m(s)\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_s+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))ds+\frac{\mathscr{M}_A\mathscr{M}_2r}{\Gamma(1+r)}\sqrt{\frac{b^{2q-1}}{2q-1}}\|u_{y^q}\|_{L^2}$$
$$+\frac{\mathscr{M}_Ar}{\Gamma(r+1)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{r-1}m(s)\Omega(\mathcal{D}_1\|\tilde{z}\|_s+\tilde{c}_s+bv(\tau)(\mathcal{D}_1\|\tilde{z}\|_\tau+\tilde{c}_n))ds$$
$$+\frac{\mathscr{M}_A\mathscr{M}_2r}{\Gamma(1+r)}\sqrt{\frac{b^{2q-1}}{2q-1}}\|u_{y^q}\|_{L^2}.$$

At the point when $\tau_2\to\tau_1$, the right hand side of the above inequality has a tendency to zero. Therefore, $(\tilde{\Upsilon}_2\tilde{z})$ is equicontinuous on $\mathcal{J}$.

Step 4. Mönch's condition holds.

Suppose that $\Xi\subseteq\mathscr{B}_h$ is countable and $\Xi\subseteq\operatorname{conv}(\{0\}\cup\tilde{\Upsilon}_2(\Xi))$. We show that $\hbar(\Xi)=0$, where $\hbar$ is the Hausdorff MNC. Without loss of generality, we may suppose that $\Xi=\{\tilde{z}_n\}_{n=1}^{\infty}$ we can easily verify that Ξ is bounded and equicontinuous. Now we need to show that $\tilde{\Upsilon}_2(\Xi(t))$ is relatively compact in $\mathcal{X}$ for each $t\in\mathcal{J}$.

Case 1. For each $t\in[0,t_1]$, by Theorem 2.1 and we get

$$\begin{aligned}\hbar\Big(\{\tilde{\Upsilon}_2\tilde{z}_{y_n}(t)\}_{n=1}^{+\infty}\Big)\le&\hbar\Big(\int_0^b(b-s)^{r-1}\mathcal{P}_r(b-s)F_n(s)ds\Big)\\ \le&\mathcal{K}_\Xi(s)\frac{2\mathscr{M}_Ar}{\Gamma(r+1)}\int_0^b(b-s)^{r-1}\\ &\times\eta(s)\Big[\sup_{\infty<\theta\le0}\hbar\Big(\Big\{\tilde{z}^n(s+\theta)+\tilde{y}(s+\theta)\Big\}_{n=1}^{+\infty}\Big)\\ &+\hbar\Big(\Big\{\int_0^s\mathcal{H}(s,\tau,\tilde{z}_\tau^n+\tilde{y}_\tau)d\tau\Big\}_{n=1}^{+\infty}\Big)\Big]ds\\ \le&\mathcal{K}_\Xi(s)\frac{2\mathscr{M}_Ar}{\Gamma(r+1)}\int_0^b(b-s)^{r-1}\eta(s)(1+2\zeta^*)\sup_{0\le\tau\le s}\hbar(\Xi(\tau))ds.\end{aligned}$$

This implies that

$$\begin{aligned}\hbar\Big(\{\Upsilon\tilde{z}_{y_n}(t)\}_{n=1}^{+\infty}\Big)\le&\hbar\Big(\Big\{\int_0^b(b-s)^{r-1}\mathcal{P}_r(b-s)F_n(s)ds\Big\}_{n=1}^{+\infty}\Big)\\ &+\hbar\Big(\Big\{\int_0^b(b-s)^{r-1}\mathcal{P}_r(b-s)\mathfrak{B}u_{y^n}(s)ds\Big\}_{n=1}^{+\infty}\Big)\\ \le&\frac{2\mathscr{M}_Ar}{\Gamma(r+1)}\int_0^b(b-s)^{r-1}\eta(s)ds(1+2\zeta^*)\sup_{0\le\tau\le s}\hbar(\Xi(\tau))\\ &+(1+2\zeta^*)\frac{2\mathscr{M}_A\mathscr{M}_2r}{\Gamma(r+1)}\Big(\int_0^b(b-s)^{r-1}\mathcal{K}_\Xi(s)ds\Big)\\ &\times\Big[\frac{2\mathscr{M}_Ar}{\Gamma(r+1)}\int_0^b(b-s)^{r-1}\eta(s)ds\Big]\sup_{0\le\tau\le s}\hbar(\Xi(\tau))ds\\ \le&\Big[\frac{2\mathscr{M}_At_1^r}{\Gamma(r+1)}(1+2\zeta^*)\mathscr{M}_6\end{aligned}$$

$$+\frac{2\mathscr{M}_A\mathscr{M}_2t_1^r}{\Gamma(1+r)}\mathscr{M}_5\frac{2\mathscr{M}_At_1^r}{\Gamma(1+r)}(1+2\zeta^*)\mathscr{M}_6\Big]\sup_{0\le\tau\le s}\hbar(\Xi(\tau))ds.$$

Case 2. For each $t\in(t_k,s_k]$, $k=1,2,\dots,m$, we have

$$\begin{aligned}\hbar\Big(\{\tilde\Upsilon_2\tilde z_{y_n}(t)\}_{n=1}^{\infty}\Big)=&\hbar\Big(\mathfrak{I}(u(t_k))+\mathscr{G}_k(t,\Xi_t+\tilde y_t)\Big)\\ \le&\mathcal{L}_k\sup_{-\infty<\theta\le0}\hbar\Big(\Big\{\Xi(t+\theta)+\tilde y(t+\theta)\Big\}_{n=1}^{+\infty}\\ &+\tilde\nu_i\sup_{-\infty<\theta\le0}\hbar\Big(\Big\{\Xi(t+\theta)+\tilde y(t+\theta)\Big\}_{n=1}^{+\infty}\Big)\\ \le&\mathcal{L}_k\sup_{0\le\tau\le\mathfrak{T}}\hbar(\Xi(\tau))+\tilde\nu_i\sup_{0\le\tau\le\mathfrak{T}}\hbar(\Xi(\tau))\le(\mathcal{L}_k+\tilde\nu_i)\hbar_{PC}(\Xi).\end{aligned}$$

Case 3. Now, for any $t\in(s_k,t_{k+1}]$, $k=1,2,\dots,m$, we have

$$\begin{aligned}\hbar\Big(\{\tilde\Upsilon_2\tilde z_{y_n}(t)\}_{n=1}^{\infty}\Big)\le&\hbar\Big(\mathfrak{Q}_r(t-s_k)\mathfrak{I}(u(t_k))+\mathscr{G}_k(s_k,u_t)\Big)\\ &+\hbar\Big[\Big(\mathfrak{Q}_r(t-s_k)\int_0^{s_k}(s_k-s)^{r-1}\mathfrak{P}_r(s_k-s)\\ &\times\mathscr{F}\Big(s,\Xi_s+\tilde y_s,\int_0^s\mathcal{H}(s,\tau,\Xi_\tau+\tilde y_\tau)d\tau+\mathfrak{B}u_y(s)\Big)ds\Big)\\ &+\int_0^t(t-s)^{r-1}\mathfrak{P}_r(t-s)\\ &\times\mathscr{F}\Big(s,\Xi_s+\tilde y_s,\int_0^s\mathcal{H}(s,\tau,\Xi_\tau+\tilde y_\tau)d\tau+\mathfrak{B}u_y(s)\Big)ds\Big]\\ \le&\mathcal{L}_k+\tilde\nu_i\sup_{0\le\tau\le\mathfrak{T}}\hbar(\Xi(\tau))+\Big[\frac{2\mathscr{M}_As_k^r}{\Gamma(r+1)}(1+2\zeta^*)\mathscr{M}_6+\frac{2\mathscr{M}_A\mathscr{M}_2s_k^r}{\Gamma(r+1)}\mathscr{M}_5\\ &\times\frac{2\mathscr{M}_As_k^r}{\Gamma(r+1)}(1+2\zeta^*)\mathscr{M}_6\Big]\sup_{0\le\tau\le s}\hbar(\Xi(\tau))\\ \le&\Big[\mathcal{L}_k+\tilde\nu_i\hbar_{PC}(\Xi)+\frac{2\mathscr{M}_As_k^r}{\Gamma(1+r)}(1+2\zeta^*)\mathscr{M}_6+\frac{2\mathscr{M}_A\mathscr{M}_2s_k^r}{\Gamma(r+1)}\mathscr{M}_5\\ &\times\frac{2\mathscr{M}_As_k^r}{\Gamma(r+1)}(1+2\zeta^*)\mathscr{M}_6\Big]\sup_{0\le\tau\le s}\hbar(\Xi(\tau)).\end{aligned}$$

Therefore,

$$\hbar\Big(\tilde\Upsilon(\Xi)\Big)(t)\le\Big(\Big[1+\frac{2\mathscr{M}_A\mathscr{M}_2\mathscr{M}_5r}{\Gamma(r+1)}\Big](1+2\zeta^*)\frac{2\mathscr{M}_A\mathscr{M}_6r}{\Gamma(r+1)}+\mathcal{L}_k+\tilde\nu_i\Big)\sup_{0\le\tau\le s}\hbar(\Xi(\tau)),$$

which implies that Lemma 2.4, $\hbar(\tilde\Upsilon(\Xi))\le\check{\mathfrak{S}}^*\hbar(\Xi)$. Thus, from Mönch's condition, we get

$$\hbar(\Xi)\le\hbar\Big(\operatorname{conv}\{0\}\cup(\tilde\Upsilon(\Xi))\Big)=\hbar(\tilde\Upsilon(\Xi))\le\check{\mathfrak{S}}^*\hbar(\Xi),$$

which implies that $\hbar(\Xi)=0$.

Hence, using Lemma 2.4, $\tilde{\Upsilon}$ has a fixed point $\tilde{y}$ in $\mathscr{B}_h''$. Then $u = \tilde{y} + \tilde{z}$ is a mild solutions of system (4.1)-(4.3) satisfying $u(b) = u_1$. Therefore, system (4.1)–(4.3) is controllable on $\mathfrak{J}$. This completes the proof. □

5. Examples

Example 5.1. Now, we consider the space $\mathfrak{X} = \mathcal{L}^2([0,\pi],\mathbb{R})$ and the following fractional neutral partial diffferential equation with infinite delay:

$$(5.1)\quad {}^C D_t^r\Big[u(t,x) + \int_{-\infty}^{t} a(\theta,x)u(t,\theta)d\theta\Big]$$
$$= \frac{\partial^2}{\partial x^2}u(t,x) + \mu\Big(t, x_t(\cdot,u), \int_0^t \mu_i(t,s,x_t(\cdot,u))ds\Big), \quad t\in[s_k,t_{k+1}], \quad u\in[0,\pi],$$

$$(5.2)\quad u(t,x) = \mathfrak{I}(u(t,x)) + \mathscr{G}_k(t,u(t,x)), \quad t\in(t_k,s_k], \quad u\in[0,\pi],$$

$$(5.3)\quad u(t,0) = u(t,\pi) = 0, \quad t\in[0,\mathfrak{T}],$$

$$(5.4)\quad u(t,x) = \Phi(t,x), \quad -\infty \le t \le 0, \quad 0\le x\le\pi,$$

where $s_k \in (t_k, t_{k+1}]$, $k = 1,2,\dots,m$, in the partition $0 = t_0 < t_1 < \cdots < t_{m+1} = \mathfrak{T}$ of the interval $[0,\mathfrak{T}]$ with $s_0 = 0$ and u_t indicates the portion of the solution $u(\cdot,\cdot) : (-\infty,\mathfrak{T}]\times[0,\pi]\to\mathfrak{X}$, that is for any $t\ge 0$, $u_t(\cdot,\cdot) : (-\infty,0]\times[0,\pi]\to\mathfrak{X}$ is given by

$$u_t(\theta,x) = u(t+\theta,x), \quad \text{for } \theta\in(-\infty,0].$$

Let $\mathfrak{X} = L^2[0,\pi]$ and define $\mathfrak{A} : \mathfrak{D}(\mathfrak{A})\subset\mathfrak{X}\to\mathfrak{X}$ by $\mathfrak{A}u = u''$, on

$$\mathfrak{D}(\mathfrak{A}) = \Big\{u\in\mathfrak{X} : \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \in \mathfrak{X} \text{ and } u(0) = u(\pi) = 0\Big\}.$$

Then $\mathfrak{A}$ generates a infinitesimal generator of a analytic semigroup $\mathfrak{Q}(t)_{t\ge 0}$ on $\mathfrak{X}$ and $\mathfrak{Q}(t)$ is not a compact semigroup on $\mathfrak{X}$, with $\hbar(\mathfrak{Q}(t)\mathfrak{D}) \le \hbar(\mathfrak{D})$, where $\hbar$ is the Hausdorff measure of noncompactness and there exists a constant an $\mathscr{M}_A \ge 1$ such that $\sup_{t\in\mathscr{I}}\|\mathfrak{Q}(t)\| \le \mathscr{M}_A$. Define $f, g : [0,\pi]\times\mathfrak{X}\to\mathfrak{X}$ by

$$u(t)x = u(t,x),$$
$$g(u)x = \int_0^{\pi} a(\theta,x)u(\theta)d\theta,$$
$$u(t,x) = \mathfrak{I}(u(t,x)) + \mathscr{G}_k(t,u(t,x)), \quad x\in[0,\pi],$$
$$f\Big(t,\phi,\int_0^t \mathcal{H}(t,s,\phi)x = \mu\left(t,\phi(\theta,x),\int_0^t \mu_1(t,s,\phi(\theta,x)ds\right), \quad \theta\in(-\infty,0],$$
$$\Phi(\theta)(x) = \Phi(\theta,x), \quad \theta\in(-\infty,0], \quad x\in[0,\pi],$$

with the following assumptions.

(i) For each $k = 0,1,2,\dots,m$, the function $\mathscr{F}$ is defined above by is continuous and we impose a suitable condition on F to satisfy the hypotheses H(A4)-H(A5).

(ii) For each $k = 1,2,\dots,m$, the function $\mathscr{G}_k$ is defined above by is continuous and we impose a suitable condition on G to satisfy the hypothesis H(A6).

With the above setting the system of equations (5.1)–(5.4) reduces to the system of equations (1.1)–(1.3) satisfying the hypotheses of Theorem 3.1 and hence, ensuring a mild solution on $(-\infty, \mathcal{T}]$.

Example 5.2. We consider the following fractional control impulsive system:

$$ {}^C D_{0,t}^{\frac{3}{4}}[u(t,y) + \int_{-\infty}^{t} \mu_1(t,y,s)\Phi(s)(y)ds] \tag{5.5}$$

$$= \frac{\partial^2}{\partial u^2} u(t,y) + \Xi \mu_2(t,y)$$

$$+ \mu_2\Big(t, \int_{-\infty}^{t} \mu_3(s-t)u(s,y)ds, \int_0^t \int_{-\infty}^{0} \mu_4(s,y,\tau_1 - s)u(\tau_1,y)d\tau_1 ds\Big), \tag{5.6}$$

$$u(t,y) = \Phi(t,y), \quad t \in (-\infty, 0], \quad y \in [0,1], \tag{5.7}$$

$$u(t,y) = \mathfrak{I}(u(t_{\frac{1}{2}}, y)) + g(t, u(t,y)), \quad t \in \left(\frac{1}{2}, \frac{2}{3}\right], \tag{5.8}$$

where ${}^C D_{0,t}^{\frac{3}{4}}$ is a Caputo fractional derivative of order $\Phi \in \mathscr{B}_h$, $\mu_2 : \mathcal{J} \times [0,1] \times [0,1]$ is continuous in t and Φ is continuous and satisfies certain smoothness conditions.

Let $U = Y = L^2(0,1)$ be endowed with the usual norm $\|\cdot\|_{L^2}$, and Let $\mathfrak{A} : \mathfrak{D}(\mathfrak{A}) \subset \mathcal{X} \to \mathcal{X}$ be defined by $\mathfrak{A}\mathfrak{W} = \mathfrak{W}''; \mathfrak{W} \in \mathfrak{D}(\mathfrak{A})$, where $\mathfrak{D}(\mathfrak{A}) = \{\mathfrak{W} \in \mathcal{X} : \mathcal{W}'' \in \mathcal{X}, \mathfrak{W}(0) = \mathfrak{W}(1) = 0\}$. It is well know that $\mathfrak{A}$ is an infinitesimal generator of a semigroup that $\{\mathcal{Q}_r(t) : t \geq 0\}$ in $\mathcal{X}$ and is given by $\mathcal{Q}_r(t)\mathfrak{W}(s) = \mathfrak{W}(t+s)$ for $\mathfrak{W} \in \mathcal{X}$. $\mathcal{Q}_r(t)$ is not a compact semigroup on $\mathcal{X}$ with $\hbar(\mathcal{Q}_r(t)\mathfrak{D}) \leq \hbar(\mathfrak{D})$, where $\hbar$ is the Hausdorff measure of noncompactness, and there exists $\mathscr{M}_A \leq 1$ such that $\sup_{t\in\mathcal{J}} \|\mathcal{Q}_r(t)\| \leq \mathscr{M}_A$. For the phase space, we choose $h = e^{2s}$, $s < 0$, then $l = \int_{\infty}^{0} h(s)ds = \frac{1}{2} < +\infty$ for $t \leq 0$, and we determine

$$\|\phi\|_{\mathscr{B}_h} = \int_{-\infty}^{0} h(s) \sup_{\theta\in[s,0]} \|\phi(\theta)\| ds.$$

Hence, for $(t,\phi) \in [0,\mathcal{T}] \times \mathscr{B}_h$, where $\phi(\theta)(x) = \phi(\theta,x)$, $(\theta,x) \in (-\infty,0] \times [0,\phi]$. Moreover, $t \to \mathfrak{W}(t^{\frac{3}{4}}\theta + s)x$ is equicontuinuous for $t \geq 0$ and $\theta \in (0,+\infty)$.

Define

$$\begin{aligned}
{}^C D_{0,t}^{\frac{3}{4}} u(t)(y) &= \frac{\partial^{\frac{3}{4}}}{\partial t^{\frac{3}{4}}} u(t,y),\\
u(t)(y) &= u(t,y),\\
\mathcal{G}(t,\Phi)(y) &= \int_{-\infty}^{t} \mu_4(t,y,s)\Phi(s)(y)ds,\\
\mathscr{F}\Big(t,\Phi,\int_0^t \mathcal{H}(s,\Phi)ds\Big)(y) &= \mu_2\Big(t, \int_{-\infty}^{0} \mu_3(s)\Phi(s)(y)ds, \int_0^t \mathcal{H}(s,\Phi)(y)ds\Big).
\end{aligned}$$

Let $\mathfrak{B} : \mathcal{X} \to \mathcal{X}$ be defined by

$$(\mathfrak{B}u)(t)(y) = \Xi\mu_2(t,y), \quad 0 < y < 1,$$

with this choice of $\mathfrak{A}, \mathfrak{B}$ and $\mathscr{F}$, system (5.6) can be rewritten as

$$\begin{aligned} {}^{C}D_t^r[u(t)+\mathcal{G}(t,u_t)] =& \mathfrak{A}u(t)+\mathscr{F}\Big(t,u_t,\int_0^t \mathcal{H}(t,s,u_s)ds\Big)+\mathfrak{B}u(t), \quad t\in(s_k,t_{k+1}],\\ k =& 0,1,2,\dots,m,\\ u(t) =& \mathfrak{I}(u(t_k))+\mathscr{G}(t,u_t), \quad t\in(t_k,s_k],\\ u_0 =& \Phi\in\mathscr{B}_h, \quad (-\infty,0], \quad k=1,2,\dots,m. \end{aligned}$$

For $y\in(0,1)$, the linear operator Ξ is given by

$$(\Xi u)(y)=\int_0^1 (1-s)^{\frac{-1}{4}}\mathcal{P}_r(1-s)\mathfrak{W}\mu_2(s,y)ds,$$

where

$$\begin{aligned} \mathcal{P}_r(t)\mathfrak{W}(s) &= \frac{3}{4}\int_0^\theta \theta\eta_{\frac{3}{4}}(\theta)\mathfrak{W}(t^{\frac{3}{4}}\theta+s)d\theta,\\ \wp_{\frac{3}{4}}(\theta) &= \frac{4}{3}\theta^{\frac{-7}{3}}\bar{\mathfrak{W}}_{\frac{3}{4}}(\theta^{\frac{-4}{3}}),\\ \bar{\mathfrak{W}}_{\frac{3}{4}}(\theta) &= \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{\frac{-3n+4}{4}}\frac{\Gamma(\frac{3n+4}{4})}{n!}\sin\Big(\frac{3n\pi}{4}\Big), \quad \theta\in(0,+\infty). \end{aligned}$$

Thus, under appropriate conditions on the functions $\mathscr{F}, \mathcal{G}, \mathscr{G}_k$ and $\mathfrak{I}_k$ as those in H(A1)-H(A9). We assume that Ξ satisfies H(A10), then all the conditions of Theorem 4.1 are satisfied. Hence, the system (5.5)–(5.8) is controllable on $\mathcal{J}$.

6. Conclusion

In this paper, we have studied the existence, uniqueness and controllability results for fractional neutral integro-differential equation and non-instantaneous impulses with delay involving the Caputo derivatives in a Banach space. More precisely, some appropriate assumption, by utilizing the ideas and techniques of sectorial operator, the theory of fractional calculus, Darbo-sadovskii and Mönch's fixed point theorem via Hausdorff measure of noncompactness. Finally, an example is presented in the end to show the applications of the obtained abstract results.

Acknowledgements. We thank the referees for their careful review and constructive comments on the manuscript. The first author would like to acknowledgemnt the financial aid from Erode Arts and Science College, Erode under the Seed Money Research Project, with Reference No. EASC/RPC/SG/2021-2022/009.

References

[1] R. P. Agarwal, S. Hristova and D. O'Regan, *Non-instantaneous Impulses in Differential Equations,* Springer Cham, 2017. https://doi.org/10.1007/978-3-319-66384-5.
[2] R. P. Agarwal, M. Meehan and D. O'Regan, *Fixed Point Theory and Applications,* Cambridge University Press, Cambridge, **141** (2001). https://doi.org/10.1017/CBO9780511543005

[3] A. Anguraj and S. Kanjanadevi, *Non-Instantaneous impulsive fractional neutral differential equations with state-dependent delay,* Progr. Fract. Differ. Appl. **3**(3) (2017), 207–218. https://doi.org/10.18576/pfda/030304

[4] D. Baleanu, K. Diethelm, E. Scalas and JJ. Trujillo, *Fractional Calculus Models and Numerical Methods,* World Scientific Publishing, New York, 2012. https://doi.org/10.1142/9789814355216

[5] M. Benchohra, J. Henderson and S. Ntouyas, *Impulsive Differential Equations and Inclusion,* Hindawi Publishing Corporation, New York, (2006). https://doi.org/10.1155/9789775945501

[6] K. Balachandran and J. Y. Park, *Controllability of fractional integro-differential systems in Banach spaces,* Nonlinear Anal. Hybrid Syst. **3**(4) (2009), 363–367. https://doi.org/10.1016/j.nahs.2009.01.014

[7] K. Balachandran, V. Govindaraj, L. Rodriguez-Germa and J. J. Trujillo, *Controllability results for nonlinear fractional-order dynamical systems,* J. Optim. Theory Appl. **156**(1) (2013), 33–44. https://doi.org/10.1007/s10957-012-0212-5

[8] J. Banas and K. Goebel, *Measures of Noncompactness in Banach Spaces,* Marcel Dekker, New York, 1980.

[9] M. Bohner, O. Tunç and C. Tunç, *Qualitative analysis of Caputo fractional integro-differential equation with constant delay,* Comp. Appl. Math. **40**(6) (2021), Paper ID 214. https://doi.org/10.1007/s40314-021-01595-3

[10] Y. K. Chang, *Controllability of impulsive functional differential systems with infinite delay in Banach spaces,* Chaos Solitons Fractals **33**(5) (2007), 1601–1609. https://doi.org/1016/j.chaos.2006.03.006

[11] D. N. Chalishajar, K. Malar and K. Karthikeyan, *Approximate controllability of abstract impulsive fractional neutral evolution equations with infinite delay in Banach spaces*, Electron. J. Differential Equations **2013**(275) (2013), 1–21. https://ejde.math.txstate.edu

[12] D. N. Chalishajar, K. Malar and R. Ilavarasi, *Existence and controllability results of impulsive fractional neutral integro-diffetential equation with sectorial operator and infinite delay,* Sixth Inter. Confer. on New Trends in the Applications of Diff. Equ. in Sciences, AIP Conference Proceedings **2159**(1) (2019), Paper ID 030006. http://doi.org/10.1063/1.5127471

[13] J. Dabas and A. Chauhan, *Existence and uniqueness of mild solution for an impulsive fractional integro-differential equation with infinite delay,* Math. Comput. Model. **57**(3-4) (2013), 754–763. https://doi.org/10.1016/j.mcm.2012.09.001

[14] A. Debboucha and D. Baleanu, *Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems,* Comput. Math. Appl. **62**(3) (2011), 1442–1450. https://doi.org/10.1016/j.camwa.2011.03.075

[15] J. R. Graef, J. Henderson and A. Ouahab, *Impulsive Differential Inclusions: A Fixed Point Approach,* De Gruyter, Berlin/-Boston, 2013. https://doi.org/10.1515/9783110295313

[16] M. Gabeleh, E. Malkowsky, M. Mursaleen, V. Rakočević, *A new survey of measures of noncompactness and their applications,* Axioms **11** (2022), Paper ID 299. https://doi.org/10.3390/axioms11060299

[17] J. K. Hale and J. Kato, *Phase space for retarded equations with infinite delay,* Funkcialaj, Ekvacioj **21**(1) (1978), 11–41.

[18] E. Hernández and D. O'Regan, *On a new class of abstract impulsive differential equations,* Proceeding of American Mathematical Society **141**(5) (2013), 1641–1649. https://doi.org/10.1090/S0002-9939-2012-11613-2

[19] R. Hilfer, *Applications of Fractional Calculus in Physics,* Singapore, World Scientific, 2000. https://doi.org/10.1142/3779

[20] Y. Hino, S. Murakami and T. Naito, *Functional Differential Equations with Infinite Delay,* Lecture Notes in Mathematics **1473**, Springer-v=Verlag, Berlin, 1991. https://doi.org/10.1007/BFb0084432

[21] E. Hernández, D. O'Regan and K. Balachandran, *On recent developments in the theory of abstract differential equations with fractional derivatives,* Nonlinear Anal. **73**(10) (2010), 3462–3471. `https://doi.org/10.1016/j.na.2010.07.035`
[22] S. Ji, G. Li and M. Wang, *Controllability of impulsive differential systems with nonlocal conditions,* Appl. Math. Comput. **217**(16) (2011), 6981–6989. `https:doi.org/10.1016/j.amc.2011.01.107`
[23] A. Kilbas, H. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations,* Elsevier, Amesterdam, 2006.
[24] J. Klamka, *Controllability of Dynamical Systems,* Kluwer Academic, Dordrecht, 1993.
[25] V. Lakshmikantham, S. Leela and J. V. Devi, *Theory of Fractional Dynamics Systems,* Scientific Publishers, Cambridge, 2009.
[26] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations,* World Scientific Singapore, 1989. `https://doi.org/10.1142/0906`
[27] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations,* Wiley, New York, 1993.
[28] H. Mönch, *Boundary value problems for nonlinear ordinary differential equations of second order in Banach Spaces,* Nonlinear Anal. **4**(5) (1980), 985–999. `https:doi.org/10.1016/0362-546X(80)90010-3`
[29] K. Malar and A. Anguraj, *Existence results of abstract impulsive integrodifferential systems with measure of non-compactness,* Journal of Statistical Science and Application **4**(3–4) (2016), 108–117. `https:doi.org/10.17265/2328-224X/2015.0304-004`
[30] A. Meraj and D. N. Pandey, *Existence of mild solutions for fractional non-instantaneous impulsive integro-differential equations with nonlocal conditions*, Arab J. Math. Sci. **26**(1/2) (2018), 3–13. `https://doi.org/10.1016/j.ajmsc.2018.11.002`
[31] A. Meraj and D. N. Pandey, *Approximate controllability of fractional integro-differential evolution equations with nonlocal and non-instantaneous impulsive conditions*, J. Fract. Calc. Appl. **10**(2) (2019), 3–17. `https://doi.org/10.21608/jfca.2019.308531`
[32] D. O'Regan and R. Precup, *Existence criteria for integral equations in Banach spaces,* Journal Inequal. Appl. **6** (2001), 77–97. `https://doi.org/10.1155/S1025583401000066`
[33] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-verlag, New York, 1983. `https://doi.org/10.1007/978-1-4612-5561-1`
[34] I. Podlubny, *Fractional Differential Equations,* Academic Press, San Diego, 1999.
[35] M. Pierri, D. O'Regan and V. Rolnik, *Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses,* Appl. Math. Comput. **219**(12) (2013), 6743–6749. `https://doi.org/10.1016/j.amc.2012.12.084`
[36] C. Ravichandran, N. Valliammal and J. J. Nieto, *New results on exact controllability of a class of fractional neutral integro-differential systems with state-dependent delay in Banach Spaces,* J. Franklin Inst. **356**(3) (2019), 1535–1565. `https://doi.org/10.1016/j.jfranklin.2018.12.001`
[37] X. B. Shu and F. Xu, *The existence of solutions for impulsive fractional partial neutral differential equation,* J. Math. (2013), Article ID 147193, 9 pages. `https://doi.org/10.1155/2013/147193`
[38] O. Tunç and C. Tunç, *Solution estimates to Caputo proportional fractional derivative delay integro-differential equations,* Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **117**(12) (2023). `https://doi.org/10.1007/s13398-022-01345-y`
[39] C. Tunç, O. Tunç and J. C. Yao, *On the new qualitative results in integro-differential equations with caputo fractional derivative and multiple kernels and delays,* J. Nonlinear Convex Anal. **23**(11) (2022), 2577–2591.
[40] V. Slyn'ko and C. Tunc, *Stability of abstract linear switched impulsive differential equations*, Automatica **107** (2019), 433–441. `https://doi.org/10.1016/j.automatica.2019.06.001`

[41] J. R. Wang, Z. Fan and Y. Zhou, *Nonlocal controllability of semilinear dynamic systems with fractional derivative in Banach spaces,* J. Optim. Theory Appl. **154**(1) (2012), 292–302. https://doi.org/10.1007/s10957-012-9999-3

[42] J. R. Wang, A. G. Ibrahim, D. O'Regan and Y. Zhou, *A general class of noninstantaneous impulsive fractional differential inclusions in Banach spaces,* Adv. Difference Equ. **287** (2017), 1–28. https://doi.org/10.1186/s13662-017-1342-8

[43] B. Yan, *Boundary value problems on the half-line with impulses and infinite delay,* J. Math. Anal. Appl. **259**(1) (2001), 94–114. https://doi.org/10.1006/jmaa.2000.7392

[44] Z. Yan, *Controllability of fractional-order partial neutral functional integro differential inclusions with infinite delay,* J. Franklin Inst. **348**(8) (2011), 2156–2173. https://doi.org/j.jfranklin.2011.06.009

[45] K. Yosida, *Functional Analysis,* 6th Edn, Springer, Berlin, 1980. https://doi.org/10.1007/978-3-642-61859-8

[46] Y. Zhou and F. Jiao, *Existence of mild solutions for fractional neutral evolution equations,* Comput. Math. Appl. **59**(3) (2010), 1063–1077. https://doi.org/10.1016/j.camwa.2009.06.026

[47] Y. Zhou, *Fractional Evolution Equations and Inclusions, Analysis and Control*, Academic Press, Elsevier, 2016. https://doi.org/10.1016/C2015-0-00813-9

[1]Department Of Mathematics,
Erode Arts and Science College,
Erode-638009, Tamil Nadu, India
Email address: malarganesaneac@gmail.com
ORCID iD: https://orcid.org/0000-0002-5143-7048
Email address: ilavarasi99ravi@gmail.com
ORCID iD: https://orcid.org/0000-0001-8756-7162