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DENUMERABLY MANY POSITIVE SOLUTIONS FOR ITERATIVE SYSTEM OF BOUNDARY VALUE PROBLEMS WITH N-SINGULARITIES ON TIME SCALES

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Abstract. In this paper we consider a iterative system of two-point boundary value problems with integral boundary conditions having n singularities and involve an increasing homeomorphism, positive homomorphism operator. By applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space, we derive sufficient conditions for the existence of denumerably many positive solutions. Finally we provide an example to check validity of our obtained results.

1. Introduction

Theory of time scales was created to unify continuous and discrete analysis. Difference and differential equations can be studied simultaneously by studying dynamic equations on time scales. Since a time scale is any closed and nonempty subset of the real numbers set. So, by this theory, we can extend known results from continuous and discrete analysis to a more general setting. As a matter of fact, this theory allows us to consider time scales which possess hybrid behaviours (both continuous and discrete). These types of time scales play an important role for applications, since most of the phenomena in the environment are neither only discrete nor only continuous, but they possess both behaviours. Moreover, basic results on this issue have been well documented in the articles [1, 2] and monographs of Bohner and Peterson [7, 8].

The study of turbulent flow through porous media is important for a wide range of scientific and engineering applications such as fluidized bed combustion, compact

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heat exchangers, combustion in an inert porous matrix, high temperature gas-cooled reactors, chemical catalytic reactors [9] and drying of different products such as iron ore [15]. To study such type of problems, Leibenson [13] introduced the following p-Laplacian equation

$$\Big(\varphi_p(\varpi'(t))\Big)' = f\Big(t, \varpi(t), \varpi'(t)\Big),$$

where $\varphi_p(\varpi) = |\varpi|^{p-2}\varpi$, $p > 1$, is the p-Laplacian operator its inverse function is denoted by $\varphi_q(\tau)$, with $\varphi_q(\tau) = |\tau|^{q-2}\tau$ and p, q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. It is well known fact that the p-Laplacian operator and fractional calculus arises from many applied fields such as turbulant filtration in porous media, blood flow problems, rheology, modelling of viscoplasticity, material science, it is worth studying the fractional differential equations with p-Laplacian operator.

In this paper, we consider an operator φ called increasing homeomorphism and positive homomorphism operator (IHPHO), which generalizes and improves the p-Laplacian operator for some $p > 1$ and φ is not necessarily odd. Liang and Zhang [14] studied countably many positive solutions for nonlinear singular m–point boundary value problems on time scales with IHPHO,

$$\begin{aligned}
&\Big(\varphi(\varpi^{\Delta}(t))\Big)^{\nabla} + a(t)f\Big(\varpi(t)\Big) = 0, \quad t \in [0, T]_{\mathbb{T}},\\
&\varpi(0) = \sum_{i=1}^{m-2} a_i \varpi(\xi_i), \quad \varpi^{\Delta}(T) = 0,
\end{aligned}$$

by using the fixed-point index theory and a new fixed-point theorem in cones.

In [10], Dogan considered second order p-boundary value problem on time scales,

$$\begin{aligned}
&\Big(\varphi_p(\varpi^{\Delta}(t))\Big)^{\nabla} + \omega(t)f\Big(t, \varpi(t)\Big) = 0, \quad t \in [0, T]_{\mathbb{T}},\\
&\varpi(0) = \sum_{i=1}^{m-2} a_i \varpi(\xi_i), \quad \varphi_p(\varpi^{\Delta}(T)) = \sum_{i=1}^{m-2} b_i \varphi_p(\varpi^{\Delta}(\xi_i)),
\end{aligned}$$

and established existence of multiple positive solutions by applying fixed-point index theory.

Inspired by aforementioned works, in this paper by applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space, we establish the existence of denumerably many positive solutions for dynamical iterative system of two-point boundary value problem with n singularities and involving IHPHO on time scales,

$$\left.\begin{aligned}
&\varphi\Big(\varpi_j^{\Delta\nabla}(t)\Big) + \chi(t)f_j\Big(\varpi_{j+1}(t)\Big) = 0, \quad 1 \le j \le \ell, \quad t \in [0, 1]_{\mathbb{T}},\\
&\varpi_{\ell+1}(t) = \varpi_1(t), \quad t \in [0, 1]_{\mathbb{T}},
\end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned}
&\alpha\varpi_j(0) - \beta\varpi_j^{\Delta}(0) = \int_0^1 \kappa_1(\tau)\varpi_j(\tau)\nabla\tau, \quad 1 \le j \le \ell,\\
&\gamma\varpi_j(1) + \delta\varpi_j^{\Delta}(1) = \int_0^1 \kappa_2(\tau)\varpi_j(\tau)\nabla\tau, \quad 1 \le j \le \ell,
\end{aligned}\right\} \tag{1.2}$$

where $\ell \in \mathbb{N}$, $\chi(t) = \prod_{i=1}^{n} \chi_i(t)$ and each $\chi_i(t) \in L^{p_i}_{\nabla}([0,1]_{\mathbb{T}})$, $p_i \geq 1$, has a singularity in the interval $\left(0, \frac{1}{2}\right)_{\mathbb{T}}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is an IHPHO with $\varphi(0) = 0$.

A projection $\varphi : \mathbb{R} \to \mathbb{R}$ is called a IHPHO, if the following three conditions are fulfilled:

(*a*) $\varphi(\tau_1) \leq \varphi(\tau_2)$ whenever $\tau_1 \leq \tau_2$, for any real numbers τ_1, τ_2;
(*b*) φ is a continuous bijection and its inverse φ^{-1} is continuous;
(*c*) $\varphi(\tau_1\tau_2) = \varphi(\tau_1)\varphi(\tau_2)$ for any real numbers τ_1, τ_2.

We use following notations in the entire paper: $i = 1, 2,\ \mathfrak{z} \in (0, 1/2)_{\mathbb{T}}$,

$$a(t) = \gamma + \delta - \gamma t, \quad b(t) = \beta + \alpha t, d = \alpha\gamma + \alpha\delta + \beta\gamma,$$

$$\aleph_0(t,\tau) = \frac{1}{d}\begin{cases} a(\tau)b(t), & t \leq \tau, \\ a(t)b(\tau), & \tau \leq t, \end{cases} \quad c_i = \int_0^1 \left[\int_0^1 \aleph_0(\tau_1,\tau_2)\kappa_i(\tau_1)\nabla\tau_1\right]\chi(\tau_2)\nabla\tau_2,$$

$$u_a = \frac{1}{d}\int_0^1 \kappa_1(\tau)a(\tau)\nabla\tau, \quad u_b = \frac{1}{d}\int_0^1 \kappa_1(\tau)b(\tau)\nabla\tau, \quad \kappa_i^* = \int_0^1 \kappa_i(\tau)\nabla\tau,$$

$$v_a = \frac{1}{d}\int_0^1 \kappa_2(\tau)a(\tau)\nabla\tau, \quad v_b = \frac{1}{d}\int_0^1 \kappa_2(\tau)b(\tau)\nabla\tau, \quad \kappa_i(\mathfrak{z}) = \int_{\mathfrak{z}}^{1-\mathfrak{z}} \kappa_i(\tau)\nabla\tau,$$

$$\eta(t) = \frac{(1-v_b)a(t) + v_a b(t)}{d[(1-u_a)(1-v_b) - u_b v_a]}, \quad \lambda(t) = \frac{(1-u_a)b(t) + u_b a(t)}{d[(1-u_a)(1-v_b) - u_b v_a]},$$

$$\eta^* = \max_{t\in[0,1]_{\mathbb{T}}} \eta(t), \quad \eta(\mathfrak{z}) = \max_{t\in[\mathfrak{z},1-\mathfrak{z}]_{\mathbb{T}}} \eta(t), \quad \lambda^* = \max_{t\in[0,1]_{\mathbb{T}}} \lambda(t), \quad \lambda(\mathfrak{z}) = \max_{t\in[\mathfrak{z},1-\mathfrak{z}]_{\mathbb{T}}} \lambda(t).$$

We assume the following conditions are true in the entire paper:

(H_1) $f_j : [0,+\infty) \to [0,+\infty)$ and $\kappa_1, \kappa_2 : [0,1]_{\mathbb{T}} \to [0,+\infty)$ are continuous;
(H_2) there exists a sequence $\{t_r\}_{r=1}^{\infty}$ such that $0 < t_{r+1} < t_r < \frac{1}{2}$,

$$\lim_{r\to\infty} t_r = t^* < \frac{1}{2}, \quad \lim_{t\to t_r} \chi_i(t) = +\infty, \quad i = 1, 2, \ldots, n,\ r \in \mathbb{N},$$

and each $\chi_i(t)$ does not vanish identically on any subinterval of $[0,1]_{\mathbb{T}}$. Moreover, there exists $\delta_i > 0$ such that

$$\delta_i < \varphi^{-1}(\chi_i(t)) < \infty \quad \text{a.e. on } [0,1]_{\mathbb{T}}, \quad i = 1, 2, \ldots, n.$$

2. Preliminaries

In this section, we introduce some basic definitions and lemmas which are useful for our later discussions; for details, see [3–5, 7, 12, 17, 18].

Definition 2.1. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined by $\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\}$, $\rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\}$ and $\mu(t) = \rho(t) - t$, respectively.

- The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively.
- If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$. Otherwise, $\mathbb{T}_k = \mathbb{T}$.

- If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$.
- A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.
- A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of all ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{ld} = C_{ld}(\mathbb{T}) = C_{ld}(\mathbb{T}, \mathbb{R})$.
- By an interval time scale, we mean the intersection of a real interval with a given time scale, i.e., $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$ other intervals can be defined similarly.

Definition 2.2. Let μ_Δ and μ_∇ be the Lebesgue Δ-measure and the Lebesgue ∇-measure on $\mathbb{T}$, respectively. If $A \subset \mathbb{T}$ satisfies $\mu_\Delta(A) = \mu_\nabla(A)$, then we call A is measurable on $\mathbb{T}$, denoted $\mu(A)$ and this value is called the Lebesgue measure of A. Let P denote a proposition with respect to $t \in \mathbb{T}$.

(i) If there exists $\Gamma_1 \subset A$ with $\mu_\Delta(\Gamma_1) = 0$ such that P holds on $A\backslash\Gamma_1$, then P is said to hold Δ-a.e. on A.

(ii) If there exists $\Gamma_2 \subset A$ with $\mu_\nabla(\Gamma_2) = 0$ such that P holds on $A\backslash\Gamma_2$, then P is said to hold ∇-a.e. on A.

Definition 2.3. Let $E \subset \mathbb{T}$ be a ∇-measurable set and $p \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, +\infty\}$ be such that $p \geq 1$ and let $f : E \to \bar{\mathbb{R}}$ be ∇-measurable function. We say that f belongs to $L^p_\nabla(E)$ provided that either

$$\int_E |f|^p(s)\nabla s < \infty \quad \text{if } p \in \mathbb{R},$$

or there exists a constant $M \in \mathbb{R}$ such that

$$|f| \leq M \quad \nabla\text{-a.e. on } E, \text{ if } p = +\infty.$$

Lemma 2.1. *Let $E \subset \mathbb{T}$ be a ∇-measurable set. If $f : \mathbb{T} \to \mathbb{R}$ is a ∇-integrable on E, then*

$$\int_E f(s)\nabla s = \int_E f(s)ds + \sum_{i\in I_E} \Big(t_i - \rho(t_i)\Big) f(t_i),$$

where $I_E := \{i \in I : t_i \in E\}$ and $\{t_i\}_{i\in I}$, $I \subset \mathbb{N}$, is the set of all left-scattered points of $\mathbb{T}$.

Lemma 2.2. *For any $\varrho(t) \in C([0, 1]_{\mathbb{T}})$, the boundary value problem,*

$$-\varphi(\varpi_1^{\Delta\nabla}(t)) = \varrho(t), \quad t \in [0, 1]_{\mathbb{T}}, \tag{2.1}$$

$$\left.\begin{aligned} \alpha\varpi_1(0) - \beta\varpi_1^\Delta(0) &= \int_0^1 \kappa_1(\tau)\varpi_1(\tau)\nabla, \\ \gamma\varpi_1(1) + \delta\varpi_1^\Delta(1) &= \int_0^1 \kappa_2(\tau)\varpi_1(\tau)\nabla, \end{aligned}\right\} \tag{2.2}$$

has a unique solution

$$\varpi_1(t) = \int_0^1 \aleph(t, \tau)\varphi^{-1}(\varrho(\tau))\nabla\tau,$$

where

$$\aleph(t,\tau)=\aleph_0(t,\tau)+\eta(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_1(\tau_1)\nabla\tau_1+\lambda(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_2(\tau_1)\nabla\tau_1.$$

Proof. Suppose ϖ_1 is a solution of (2.1), then

$$\begin{aligned}\varpi_1(t)&=-\int_0^t\int_0^\tau \varphi^{-1}(\varrho(\tau_1))\nabla\tau_1\Delta\tau+At+B\\ &=-\int_0^t(t-\tau)\varphi^{-1}(\varrho(\tau))\nabla\tau+A_1t+A_2,\end{aligned}$$

where $A_1=\varpi_1^\Delta(0)$ and $A_2=\varpi_1(0)$. By the conditions (2.2), we get

$$A_1=\frac{1}{d}\int_0^1[\alpha\kappa_2(\tau)-\gamma\kappa_1(\tau)]\vartheta_1(\tau)\nabla\tau+\frac{1}{d}\int_0^1\alpha[\gamma(1-\tau)+\delta]\varphi^{-1}(\varrho(\tau))\nabla\tau$$

and

$$A_2=\frac{1}{d}\int_0^1[(\gamma+\delta)\kappa_1(\tau)+\beta\kappa_2(\tau)]\vartheta_1(\tau)\nabla\tau+\frac{1}{d}\int_0^1\beta[\gamma(1-\tau)+\delta]\varphi^{-1}(\varrho(\tau))\nabla\tau.$$

So, we have

(2.3)

$$\varpi_1(t)=\int_0^1\aleph_0(t,\tau)\varphi^{-1}(\varrho(\tau))\nabla\tau+\frac{a(t)}{d}\int_0^1\kappa_1(\tau)\vartheta_1(\tau)\nabla\tau+\frac{b(t)}{d}\int_0^1\kappa_2(\tau)\vartheta_1(\tau)\nabla\tau.$$

By simple computations, we find that

$$\int_0^1\kappa_1(\tau)\vartheta_1(\tau)\nabla\tau=\frac{c_1(1-v_b)+c_2u_b}{(1-u_a)(1-v_b)-u_bv_a},\tag{2.4}$$

$$\int_0^1\kappa_2(\tau)\vartheta_1(\tau)\nabla\tau=\frac{c_2(1-u_a)+c_1v_a}{(1-u_a)(1-v_b)-u_bv_a}.\tag{2.5}$$

Plugging (2.4) and (2.5) into (2.3), we received

$$\begin{aligned}\varpi_1(t)&=\int_0^1\aleph_0(t,\tau)\varphi^{-1}(\varrho(\tau))\nabla\tau+c_1\eta(t)+c_2\lambda(t)\\ &=\int_0^1\left[\aleph_0(t,\tau)+\eta(t)\int_0^1\aleph_0(\tau_1,\tau)\kappa_1(\tau_1)\nabla\tau_1+\lambda(t)\int_0^1\aleph_0(\tau_1,\tau)\kappa_2(\tau_1)\nabla\tau_1\right]\\ &\quad\times\varphi^{-1}(\varrho(\tau))\nabla\tau\\ &=\int_0^1\aleph(t,\tau)\varphi^{-1}(\varrho(\tau))\nabla\tau.\end{aligned}$$

This completes the proof. □

Lemma 2.3. *Suppose* (H_1)-(H_2) *hold. For* $\mathfrak{z}\in(0,\frac{1}{2})_{\mathbb{T}}$, *let*

$$\mathcal{L}(\mathfrak{z})=\min\left\{\frac{\alpha\mathfrak{z}+\beta}{\alpha+\beta},\frac{\gamma\mathfrak{z}+\delta}{\gamma+\delta}\right\}<1.$$

Then $\aleph_0(t,\tau)$ *have the following properties:*

(i) $0\le\aleph_0(t,\tau)\le\aleph_0(\tau,\tau)$ *for all* $t,\tau\in[0,1]_{\mathbb{T}}$*;*

(ii) $\mathcal{L}(\mathfrak{z})\aleph_0(\tau,\tau) \le \aleph_0(t,\tau)$ *for all* $t \in [\mathfrak{z}, 1-\mathfrak{z}]_{\mathbb{T}}$ *and* $\tau \in [0,1]_{\mathbb{T}}$.

Proof. (i) is evident. We establish (ii), for this, let $t \in [\mathfrak{z}, 1-\mathfrak{z}]_{\mathbb{T}}$ and $t \le \tau$. Then

$$\frac{\aleph_0(t,\tau)}{\aleph_0(\tau,\tau)} = \frac{b(t)}{b(\tau)} = \frac{\alpha t + \beta}{\alpha\tau + \beta} \ge \frac{\alpha\mathfrak{z} + \beta}{\alpha + \beta} \ge \mathcal{L}(\mathfrak{z}).$$

For $\tau \le t$,

$$\frac{\aleph_0(t,\tau)}{\aleph_0(\tau,\tau)} = \frac{a(t)}{a(\tau)} = \frac{\gamma + \delta - \gamma t}{\gamma + \delta - \gamma\mathfrak{z}} \ge \frac{\gamma\mathfrak{z} + \delta}{\gamma + \delta} \ge \mathcal{L}(\mathfrak{z}).$$

This completes the proof. □

Lemma 2.4. *Suppose* (H_1)*-*(H_2) *hold. Then* $\aleph(t,\tau)$ *satisfies properties:*

(i) $0 \le \aleph(t,\tau) \le \Xi\,\aleph_0(\tau,\tau)$ *for all* $t, \tau \in\in [0,1]_{\mathbb{T}}$*;*

(ii) $0 \le \Xi_{\mathfrak{z}}\aleph_0(\tau,\tau) \le \aleph(t,\tau)$ *for all* $t \in [\mathfrak{z}, 1-\mathfrak{z}]_{\mathbb{T}}$ *and* $\tau \in [0,1]_{\mathbb{T}}$, *where*

$$\Xi = 1 + \eta^*\kappa_1^* + \lambda^*\kappa_2^*$$

and

$$\Xi_{\mathfrak{z}} = \mathcal{L}(\mathfrak{z})\Big[1 + \eta(\mathfrak{z})\kappa_1(\mathfrak{z}) + \lambda(\mathfrak{z})\kappa_2(\mathfrak{z})\Big].$$

Proof. From Lemma 2.3, we get

$$\begin{aligned}
\aleph(t,\tau) &= \aleph_0(t,\tau) + \eta(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_2(\tau_1)\nabla\tau_1\\
&\le \aleph_0(\tau,\tau) + \eta(t)\int_0^1 \aleph_0(\tau,\tau)\kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_0^1 \aleph_0(\tau,\tau)\kappa_2(\tau_1)\nabla\tau_1\\
&\le \Big[1 + \eta(t)\int_0^1 \kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_0^1 \kappa_2(\tau_1)\nabla\tau_1\Big]\aleph_0(\tau,\tau)\\
&\le \Big[1 + \eta^*\kappa_1^* + \lambda^*\kappa_2^*\Big]\aleph_0(\tau,\tau).
\end{aligned}$$

On the other hand, for $t \in [\mathfrak{z}, 1-\mathfrak{z}]_{\mathbb{T}}$ and $\tau \in [0,1]_{\mathbb{T}}$, we have

$$\begin{aligned}
\aleph(t,\tau) &= \aleph_0(t,\tau) + \eta(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_0^1 \aleph_0(\tau_1,\tau)\kappa_2(\tau_1)\nabla\tau_1\\
&\ge \aleph_0(t,\tau) + \eta(t)\int_{\mathfrak{z}}^{1-\mathfrak{z}} \aleph_0(\tau_1,\tau)\kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_{\mathfrak{z}}^{1-\mathfrak{z}} \aleph_0(\tau_1,\tau)\kappa_2(\tau_1)\nabla\tau_1\\
&\ge \mathcal{L}(\mathfrak{z})\left[1 + \eta(t)\int_{\mathfrak{z}}^{1-\mathfrak{z}} \kappa_1(\tau_1)\nabla\tau_1 + \lambda(t)\int_{\mathfrak{z}}^{1-\mathfrak{z}} \kappa_2(\tau_1)\nabla\tau_1\right]\aleph_0(\tau,\tau)\\
&\ge \mathcal{L}(\mathfrak{z})\Big[1 + \eta^{**}\kappa_1^{**} + \lambda^{**}\kappa_2^{**}\Big]\aleph_0(\tau,\tau).
\end{aligned}$$

This completes the proof. □

Notice that an ℓ-tuple $(\varpi_1(t), \varpi_2(t), \varpi_3(t), \dots, \varpi_\ell(t))$ is a solution of the iterative boundary value problem (1.1)–(1.2) if and only if

$$\begin{aligned}
&\varpi_j(t) = \int_0^1 \aleph(t,\tau)\varphi^{-1}\Big[\chi(\tau)f_j(\varpi_{j+1}(\tau))\Big]\nabla\tau, \quad t \in [0,1]_{\mathbb{T}},\ 1 \le j \le \ell,\\
&\varpi_{\ell+1}(t) = \varpi_1(t), \quad t \in [0,1]_{\mathbb{T}},
\end{aligned}$$

i.e.,

$$\begin{aligned}\varpi_1(t) = &\int_0^1 \aleph(t,\tau_1)\varphi^{-1}\Bigg[\chi(\tau_1)f_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\chi(\tau_2)f_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\\ &\times \varphi^{-1}\Bigg[\chi(\tau_3)f_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\ &\times f_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\Bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1.\end{aligned}$$

Let $\mathtt{B}$ be the Banach space $C_{ld}([0,1]_{\mathbb{T}},\mathbb{R})$ with the norm $\|\varpi\| = \max_{t\in[0,1]_{\mathbb{T}}}|\varpi(t)|$. For $\mathfrak{z} \in \left(0,\frac{1}{2}\right)$, we define the cone $\mathtt{K}_{\mathfrak{z}} \subset \mathtt{B}$ as

$$\mathtt{K}_{\mathfrak{z}} = \left\{\varpi \in \mathtt{B} : \varpi(t) \text{ is nonnegative and } \min_{t\in[\mathfrak{z},\,1-\mathfrak{z}]_{\mathbb{T}}} \varpi(t) \geq \frac{\Xi_{\mathfrak{z}}}{\Xi}\|\varpi(t)\|\right\}.$$

For any $\varpi_1 \in \mathtt{K}_{\mathfrak{z}}$, define an operator $\Omega : \mathtt{K}_{\mathfrak{z}} \to \mathtt{B}$ by

$$\begin{aligned}(\Omega\varpi_1)(t) = &\int_0^1 \aleph(t,\tau_1)\varphi^{-1}\Bigg[\chi(\tau_1)f_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\chi(\tau_2)f_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\\ &\times \varphi^{-1}\Bigg[\chi(\tau_3)f_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\ &\times f_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\Bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1.\end{aligned}$$

Lemma 2.5. *Assume that* (H_1)*-*(H_2) *hold. Then for each* $\mathfrak{z} \in \left(0,\frac{1}{2}\right)$, $\Omega(\mathtt{K}_{\mathfrak{z}}) \subset \mathtt{K}_{\mathfrak{z}}$ *and* $\Omega : \mathtt{K}_{\mathfrak{z}} \to \mathtt{K}_{\mathfrak{z}}$ *is completely continuous.*

Proof. From Lemma 2.3, $\aleph(t,\tau) \geq 0$ for all $t,\tau \in [0,1]_{\mathbb{T}}$. So, $(\Omega\varpi_1)(t) \geq 0$. Also, for $\varpi_1 \in \mathtt{K}$, we have

$$\begin{aligned}(\Omega\varpi_1)(t) \leq &\,\Xi\int_0^1 \aleph_0(\tau_1,\tau_1)\varphi^{-1}\Bigg[\chi(\tau_1)f_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\chi(\tau_2)\\ &\times f_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\varphi^{-1}\Bigg[\chi(\tau_3)f_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\ &\times f_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\Bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1.\end{aligned}$$

So,

$$\begin{aligned}\|\Omega\varpi_1\| \leq &\,\Xi\int_0^1 \aleph_0(\tau_1,\tau_1)\varphi^{-1}\Bigg[\chi(\tau_1)f_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\chi(\tau_2)\\ &\times f_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\varphi^{-1}\Bigg[\chi(\tau_3)f_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\end{aligned}$$

$$\times f_{\ell-1}\bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1.$$

Again from Lemma 2.3, we get

$$\begin{aligned}&\min_{t\in[\mathfrak{z},1-\mathfrak{z}]_{\mathbb{T}}}\{(\Omega\varpi_1)(t)\}\\ \geq&\Xi_{\mathfrak{z}}\int_0^1 \aleph_0(\tau_1,\tau_1)\varphi^{-1}\Bigg[\chi(\tau_1)f_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\chi(\tau_2)\\ &\times f_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\varphi^{-1}\Bigg[\chi(\tau_3)f_3\bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\ &\times f_{\ell-1}\bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1.\end{aligned}$$

It follows from the above two inequalities that

$$\min_{t\in[\mathfrak{z},1-\mathfrak{z}]_{\mathbb{T}}}\{(\Omega\varpi_1)(t)\}\geq\frac{\Xi_{\mathfrak{z}}}{\Xi}\|\Omega\varpi_1\|.$$

So, $\Omega\varpi_1\in\mathtt{K}_{\mathfrak{z}}$ and thus $\Omega(\mathtt{K}_{\mathfrak{z}})\subset\mathtt{K}_{\mathfrak{z}}$. Next, by standard methods and Arzela-Ascoli theorem, it can be proved easily that the operator Ω is completely continuous. The proof is complete. □

3. Denumerably Many Positive Solutions

For the existence of denumerably many positive solutions for iterative system of boundary value problem (1.1), we apply following theorems.

Theorem 3.1 ([11])**.** *Let $\mathcal{E}$ be a cone in a Banach space $\mathcal{X}$ and $\mathtt{M}_1$, $\mathtt{M}_2$ are open sets with $0\in\mathtt{M}_1,\overline{\mathtt{M}}_1\subset\mathtt{M}_2$. Let $\mathcal{A}:\mathcal{E}\cap(\overline{\mathtt{M}}_2\backslash\mathtt{M}_1)\to\mathcal{E}$ be a completely continuous operator such that*

(a) $\|\mathcal{A}z\|\leq\|z\|$, $z\in\mathcal{E}\cap\partial\mathtt{M}_1$, *and* $\|\mathcal{A}z\|\geq\|z\|$, $z\in\mathcal{E}\cap\partial\mathtt{M}_2$, *or*
(b) $\|\mathcal{A}z\|\geq\|z\|$, $z\in\mathcal{E}\cap\partial\mathtt{M}_1$, *and* $\|\mathcal{A}z\|\leq\|z\|$, $z\in\mathcal{E}\cap\partial\mathtt{M}_2$.

Then $\mathcal{A}$ has a fixed point in $\mathcal{E}\cap(\overline{\mathtt{M}}_2\backslash\mathtt{M}_1)$.

Theorem 3.2 ([8, 16])**.** *Let $f\in L^p_\nabla(J)$, with $p>1$, $g\in L^q_\nabla(J)$, with $q>1$, and $\frac{1}{p}+\frac{1}{q}=1$. Then $fg\in L^1_\nabla(J)$ and $\|fg\|_{L^1_\nabla}\leq\|f\|_{L^p_\nabla}\|g\|_{L^q_\nabla}$, where*

$$\|f\|_{L^p_\nabla}:=\begin{cases}\left[\int_J|f|^p(s)\nabla s\right]^{\frac{1}{p}}, & p\in\mathbb{R},\\ \inf\Big\{M\in\mathbb{R}\,/\,|f|\leq M\ \nabla\text{-a.e. on } J\Big\}, & p=\infty,\end{cases}$$

and $J=(a,b]_{\mathbb{T}}$.

Theorem 3.3 (Hölder's)**.** *Let* $f \in L_{\nabla}^{p_i}(J)$*, with* $p_i > 1$*, for* $i = 1, 2, \dots, n$ *and* $\sum_{i=1}^{n} \frac{1}{p_i} = 1$*. Then* $\prod_{i=1}^{n} f_i \in L_{\nabla}^{1}(J)$ *and*

$$\left\| \prod_{i=1}^{n} f_i \right\|_1 \leq \prod_{i=1}^{n} \|f_i\|_{p_i}.$$

Further, if $f \in L_{\nabla}^{1}(J)$ *and* $g \in L_{\nabla}^{\infty}(J)$*, then* $fg \in L_{\nabla}^{1}(J)$ *and* $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$*.*

Consider the following three possible cases for $\chi_i \in L_{\nabla}^{p_i}([0,1]_{\mathbb{T}})$:

(i) $\sum_{i=1}^{n} \frac{1}{p_i} < 1$;
(ii) $\sum_{i=1}^{n} \frac{1}{p_i} = 1$;
(iii) $\sum_{i=1}^{n} \frac{1}{p_i} > 1$.

Firstly, we seek denumerably many positive solutions for the case $\sum_{i=1}^{n} \frac{1}{p_i} < 1$.

Theorem 3.4. *Suppose* (H_1)*-*(H_3) *hold, let* $\{\mathfrak{z}_r\}_{r=1}^{\infty}$ *be a sequence with* $\mathfrak{z}_r \in (t_{r+1}, t_r)$*. Let* $\{\Gamma_r\}_{r=1}^{\infty}$ *and* $\{\Theta_r\}_{r=1}^{\infty}$ *be such that*

$$\Gamma_{r+1} < \frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r < \Theta_r < \mathfrak{Z}\Theta_r < \Gamma_r, \quad r \in \mathbb{N},$$

where

$$\mathfrak{Z} = \max\left\{ \left[\Xi_{\mathfrak{z}_1} \prod_{i=1}^{n} \delta_i \int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1} \aleph_0(\tau, \tau) \nabla \tau \right]^{-1}, \ 1 \right\}.$$

Assume that f *satisfies*

(C_1) $f_j(\varpi) \leq \varphi(\mathfrak{N}_1 \Gamma_r)$ *for all* $t \in [0,1]_{\mathbb{T}}$*,* $0 \leq \varpi \leq \Gamma_r$*, where*

$$\mathfrak{N}_1 < \left[\Xi \, \|\aleph_0\|_{L_{\nabla}^{q}} \prod_{i=1}^{n} \left\| \varphi^{-1}(\chi_i) \right\|_{L_{\nabla}^{p_i}} \right]^{-1};$$

(C_2) $f_j(\varpi) \geq \varphi(\mathfrak{Z}\Theta_r)$ *for all* $t \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}}$*,* $\frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r \leq \varpi \leq \Theta_r$*.*

Then the iterative boundary value problem (1.1)*–*(1.2) *has denumerably many solutions* $\{(\varpi_1^{[r]}, \varpi_2^{[r]}, \dots, \varpi_\ell^{[r]})\}_{r=1}^{\infty}$ *such that* $\varpi_j^{[r]}(t) \geq 0$ *on* $[0,1]_{\mathbb{T}}$*,* $j = 1, 2, \dots, \ell$ *and* $r \in \mathbb{N}$*.*

Proof. Let

$$\mathtt{M}_{1,r} = \{\varpi \in \mathtt{B} : \|\varpi\| < \Gamma_r\},$$
$$\mathtt{M}_{2,r} = \{\varpi \in \mathtt{B} : \|\varpi\| < \Theta_r\},$$

be open subsets of $\mathtt{B}$. Let $\{\mathfrak{z}_r\}_{r=1}^{\infty}$ be given in the hypothesis and we note that

$$t^* < t_{r+1} < \mathfrak{z}_r < t_r < \frac{1}{2},$$

for all $r \in \mathbb{N}$. For each $r \in \mathbb{N}$, we define the cone $\mathtt{K}_{\mathfrak{z}_r}$ by

$$\mathtt{K}_{\mathfrak{z}_r} = \left\{ \varpi \in \mathtt{B} : \varpi(t) \geq 0, \min_{t \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}}} \varpi(t) \geq \frac{\Xi_{\mathfrak{z}_r}}{\Xi} \|\varpi(t)\| \right\}.$$

Let $\varpi_1 \in \mathtt{K}_{\mathfrak{z}_r} \cap \partial \mathtt{M}_{1,r}$. Then $\varpi_1(\tau) \leq \Gamma_r = \|\varpi_1\|$ for all $\tau \in [0,1]_{\mathbb{T}}$. By (C_1) and for $\tau_{\ell-1} \in [0,1]_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell &\leq \Xi\int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\\ &\leq \Xi\,\mathfrak{N}_1\Gamma_r\int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)\Big]\nabla\tau_\ell\\ &\leq \Xi\,\mathfrak{N}_1\Gamma_r\int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\Bigg[\prod_{i=1}^n\chi_i(\tau_\ell)\Bigg]\nabla\tau_\ell\\ &\leq \Xi\,\mathfrak{N}_1\Gamma_r\int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\prod_{i=1}^n\varphi^{-1}(\chi_i(\tau_\ell))\nabla\tau_\ell.\end{aligned}$$

There exists a $q>1$ such that $\frac{1}{q}+\sum_{i=1}^n\frac{1}{p_i}=1$. So,

$$\begin{aligned}\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell &\leq \Xi\,\mathfrak{N}_1\Gamma_r\Big\|\aleph_0\Big\|_{L^q_\nabla}\Bigg\|\prod_{i=1}^n\varphi^{-1}(\chi_i)\Bigg\|_{L^{p_i}_\nabla}\\ &\leq \Xi\,\mathfrak{N}_1\Gamma_r\|\aleph_0\|_{L^q_\nabla}\prod_{i=1}^n\Big\|\varphi^{-1}(\chi_i)\Big\|_{L^{p_i}_\nabla}\\ &\leq \Gamma_r.\end{aligned}$$

It follows in similar manner for $\tau_{\ell-2} \in [0,1]_{\mathbb{T}}$ that

$$\begin{aligned}&\int_0^1 \aleph(\tau_{\ell-2},\tau_{\ell-1})\varphi^{-1}\Bigg[\chi(\tau_{\ell-1})f_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\Bigg)\Bigg]\nabla\tau_{\ell-1}\\ \leq&\int_0^1 \aleph(\tau_{\ell-2},\tau_{\ell-1})\varphi^{-1}\Big[\chi(\tau_{\ell-1})f_{\ell-1}(\Gamma_r)\Big]\nabla\tau_{\ell-1}\\ \leq&\,\Xi\int_0^1 \aleph_0(\tau_{\ell-1},\tau_{\ell-1})\varphi^{-1}\Big[\chi(\tau_{\ell-1})f_{\ell-1}(\Gamma_r)\Big]\nabla\tau_{\ell-1}\\ \leq&\,\Xi\,\mathfrak{N}_1\Gamma_r\int_0^1 \aleph_0(\tau_{\ell-1},\tau_{\ell-1})\varphi^{-1}\Big[\chi(\tau_{\ell-1})\Big]\nabla\tau_{\ell-1}\\ \leq&\,\Xi\,\mathfrak{N}_1\Gamma_r\int_0^1 \aleph_0(\tau_{\ell-1},\tau_{\ell-1})\varphi^{-1}\Bigg[\prod_{i=1}^n\chi_i(\tau_{\ell-1})\Bigg]\nabla\tau_{\ell-1}\\ \leq&\,\Xi\,\mathfrak{N}_1\Gamma_r\int_0^1 \aleph_0(\tau_{\ell-1},\tau_{\ell-1})\prod_{i=1}^n\varphi^{-1}(\chi_i(\tau_{\ell-1}))\nabla\tau_{\ell-1}\\ \leq&\,\Xi\,\mathfrak{N}_1\Gamma_r\|\aleph_0\|_{L^q_\nabla}\prod_{i=1}^n\Big\|\varphi^{-1}(\chi_i)\Big\|_{L^{p_i}_\nabla}\\ \leq&\,\Gamma_r.\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$(\Omega\varpi_1)(t)=\int_0^1 \aleph(t,\tau_1)\varphi^{-1}\Bigg[\chi(\tau_1)f_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\varphi^{-1}\Bigg[\chi(\tau_2)f_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)$$

$$\times\varphi^{-1}\Big[\chi(\tau_3)f_3\Big(\int_0^1\aleph(\tau_3,\tau_4)\cdots$$
$$\times f_{\ell-1}\Big(\int_0^1\aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\Big)\cdots\nabla\tau_3\Big]\nabla\tau_2\Big]\nabla\tau_1$$
$$\leq\Gamma_r.$$

Since $\Gamma_r=\|\varpi_1\|$ for $\varpi_1\in\mathtt{K}_{\mathfrak{z}_r}\cap\partial\mathtt{M}_{1,r}$ we get

$$\|\Omega\varpi_1\|\leq\|\varpi_1\|. \tag{3.1}$$

Let $t\in[\mathfrak{z}_r,1-\mathfrak{z}_r]_{\mathbb{T}}$. Then

$$\Theta_r=\|\varpi_1\|\geq\varpi_1(t)\geq\min_{t\in[\mathfrak{z}_r,1-\mathfrak{z}_r]_{\mathbb{T}}}\varpi_1(t)\geq\frac{\Xi_{\mathfrak{z}_r}}{\Xi}\|\varpi_1\|\geq\frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r.$$

By (C_2) and for $\tau_{\ell-1}\in[\mathfrak{z}_r,1-\mathfrak{z}_r]_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^1\aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell&\geq\Xi_{\mathfrak{z}_r}\int_{\mathfrak{z}_r}^{1-\mathfrak{z}_r}\aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\\&\geq\Xi_{\mathfrak{z}_r}\mathfrak{Z}\Theta_r\int_{\mathfrak{z}_r}^{1-\mathfrak{z}_r}\aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}(\chi(\tau_\ell))\nabla\tau_\ell\\&\geq\Xi_{\mathfrak{z}_r}\mathfrak{Z}\Theta_r\int_{\mathfrak{z}_r}^{1-\mathfrak{z}_r}\aleph_0(\tau_\ell,\tau_\ell)\prod_{i=1}^n\varphi^{-1}(\chi_i(\tau_\ell))\nabla\tau_\ell\\&\geq\Xi_{\mathfrak{z}_1}\mathfrak{Z}\Theta_r\prod_{i=1}^n\delta_i\int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1}\aleph_0(\tau_\ell,\tau_\ell)\nabla\tau_\ell\\&\geq\Theta_r.\end{aligned}$$

Continuing with bootstrapping argument we get

$$(\Omega\varpi_1)(t)=\int_0^1\aleph(t,\tau_1)\varphi^{-1}\Big[\chi(\tau_1)f_1\Big(\int_0^1\aleph(\tau_1,\tau_2)\varphi^{-1}\Big[\chi(\tau_2)f_2\Big(\int_0^1\aleph(\tau_2,\tau_3)$$
$$\times\varphi^{-1}\Big[\chi(\tau_3)f_3\Big(\int_0^1\aleph(\tau_3,\tau_4)\cdots$$
$$\times f_{\ell-1}\Big(\int_0^1\aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\Big)\cdots\nabla\tau_3\Big]\nabla\tau_2\Big]\nabla\tau_1$$
$$\geq\Theta_r.$$

Thus, if $\varpi_1\in\mathtt{K}_{\mathfrak{z}_r}\cap\partial\mathtt{K}_{2,r}$, then

$$\|\Omega\varpi_1\|\geq\|\varpi_1\|. \tag{3.2}$$

It is evident that $0\in\mathtt{M}_{2,k}\subset\overline{\mathtt{M}}_{2,k}\subset\mathtt{M}_{1,k}$. From (3.1) and (3.2), it follows from Theorem 3.1 that the operator Ω has a fixed point $\varpi_1^{[r]}\in\mathtt{K}_{\mathfrak{z}_r}\cap\Big(\overline{\mathtt{M}}_{1,r}\backslash\mathtt{M}_{2,r}\Big)$ such that $\varpi_1^{[r]}(t)\geq0$

on $[0,1]_{\mathbb{T}}$, and $r \in \mathbb{N}$. Next setting $\varpi_{\ell+1} = \varpi_1$, we obtain denumerably many positive solutions $\{(\varpi_1^{[r]}, \varpi_2^{[r]}, \dots, \varpi_\ell^{[r]})\}_{r=1}^{\infty}$ of (1.1)–(1.2) given iteratively by

$$\varpi_j(t) = \int_0^1 \aleph(t,\tau)\varphi^{-1}\Big[\chi(\tau) f_j(\varpi_{j+1}(\tau))\Big]\nabla\tau, \quad t \in [0,1]_{\mathbb{T}},\ j = \ell, \ell-1, \dots, 1.$$

The proof is completed. □

For $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, we have the following theorem.

Theorem 3.5. *Suppose* (H_1)-(H_3) *hold, let* $\{\mathfrak{z}_r\}_{r=1}^{\infty}$ *be a sequence with* $\mathfrak{z}_r \in (t_{r+1}, t_r)$. *Let* $\{\Gamma_r\}_{r=1}^{\infty}$ *and* $\{\Theta_r\}_{r=1}^{\infty}$ *be such that*

$$\Gamma_{r+1} < \frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r < \Theta_r < \mathfrak{Z}\Theta_r < \Gamma_r, \quad r \in \mathbb{N},$$

where

$$\mathfrak{Z} = \max\left\{ \left[\Xi_{\mathfrak{z}_1} \prod_{i=1}^{n} \delta_i \int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1} \aleph_0(\tau,\tau)\nabla\tau\right]^{-1},\ 1\right\}.$$

Assume that f satisfies

(C_3) $f_j(\varpi) \le \varphi(\mathfrak{N}_2\Gamma_r)$ *for all* $t \in [0,1]_{\mathbb{T}}$, $0 \le \varpi \le \Gamma_r$, *where*

$$\mathfrak{N}_2 < \min\left\{ \left[\Xi\,\|\aleph_0\|_{L^\infty_\nabla} \prod_{i=1}^{n} \left\|\varphi^{-1}(\chi_i)\right\|_{L^{p_i}_\nabla}\right]^{-1}, \mathfrak{Z}\right\};$$

(C_4) $f_j(\varpi) \ge \varphi(\mathfrak{Z}\Theta_r)$ *for all* $t \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}}$, $\frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r \le \varpi \le \Theta_r$.

Then the iterative boundary value problem (1.1)–(1.2) *has denumerably many solutions* $\{(\varpi_1^{[r]}, \varpi_2^{[r]}, \dots, \varpi_\ell^{[r]})\}_{r=1}^{\infty}$ *such that* $\varpi_j^{[r]}(t) \ge 0$ *on* $[0,1]_{\mathbb{T}}$, $j = 1, 2, \dots, \ell$, *and* $r \in \mathbb{N}$.

Proof. For a fixed r, let $\mathtt{M}_{1,r}$ be as in the proof of Theorem 3.4 and let $\varpi_1 \in \mathtt{K}_{\mathfrak{z}_r} \cap \partial\mathtt{M}_{2,r}$. Again $\varpi_1(\tau) \le \Gamma_r = \|\varpi_1\|$ for all $\tau \in [0,1]_{\mathbb{T}}$. By (C_3) and for $\tau_{\ell-1} \in [0,1]_{\mathbb{T}}$, we have

$$\begin{aligned}
\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell) f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell &\le \Xi \int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell) f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell \\
&\le \Xi\,\mathfrak{N}_2\Gamma_r \int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell)\Big]\nabla\tau_\ell \\
&\le \Xi\,\mathfrak{N}_2\Gamma_r \int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\varphi^{-1}\left[\prod_{i=1}^{n}\chi_i(\tau_\ell)\right]\nabla\tau_\ell \\
&\le \Xi\,\mathfrak{N}_2\Gamma_r \int_0^1 \aleph_0(\tau_\ell,\tau_\ell)\prod_{i=1}^{n}\varphi^{-1}(\chi_i(\tau_\ell))\nabla\tau_\ell \\
&\le \Xi\,\mathfrak{N}_2\Gamma_r \|\aleph_0\|_{L^\infty_\nabla} \prod_{i=1}^{n}\left\|\varphi^{-1}(\chi_i)\right\|_{L^{p_i}_\nabla} \\
&\le \Gamma_r.
\end{aligned}$$

It follows in similar manner for $\tau_{\ell-2} \in [0,1]_{\mathbb{T}}$ that

$$\int_0^1 \aleph(\tau_{\ell-2},\tau_{\ell-1})\varphi^{-1}\left[\chi(\tau_{\ell-1}) f_{\ell-1}\left(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell) f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\right)\right]\nabla\tau_{\ell-1}$$

$$
\begin{aligned}
&\le \int_0^1 \aleph(\tau_{\ell-2}, \tau_{\ell-1})\varphi^{-1}\Big[\chi(\tau_{\ell-1}) f_{\ell-1}(\Gamma_r)\Big]\nabla\tau_{\ell-1} \\
&\le \Xi \int_0^1 \aleph_0(\tau_{\ell-1}, \tau_{\ell-1})\varphi^{-1}\Big[\chi(\tau_{\ell-1}) f_{\ell-1}(\Gamma_r)\Big]\nabla\tau_{\ell-1} \\
&\le \Xi\, \mathfrak{N}_2 \Gamma_r \int_0^1 \aleph_0(\tau_{\ell-1}, \tau_{\ell-1})\varphi^{-1}\Big[\chi(\tau_{\ell-1})\Big]\nabla\tau_{\ell-1} \\
&\le \Xi\, \mathfrak{N}_2 \Gamma_r \int_0^1 \aleph_0(\tau_{\ell-1}, \tau_{\ell-1})\varphi^{-1}\left[\prod_{i=1}^{n} \chi_i(\tau_{\ell-1})\right]\nabla\tau_{\ell-1} \\
&\le \Xi\, \mathfrak{N}_2 \Gamma_r \int_0^1 \aleph_0(\tau_{\ell-1}, \tau_{\ell-1}) \prod_{i=1}^{n} \varphi^{-1}(\chi_i(\tau_{\ell-1}))\nabla\tau_{\ell-1} \\
&\le \Xi\, \mathfrak{N}_2 \Gamma_r \|\aleph_0\|_{L^\infty_\nabla} \prod_{i=1}^{n} \left\|\varphi^{-1}(\chi_i)\right\|_{L^{p_i}_\nabla} \\
&\le \Gamma_r.
\end{aligned}
$$

Continuing with this bootstrapping argument, we get

$$
\begin{aligned}
(\Omega\varpi_1)(t) = \int_0^1 &\aleph(t, \tau_1)\varphi^{-1}\left[\chi(\tau_1) f_1\left(\int_0^1 \aleph(\tau_1, \tau_2)\varphi^{-1}\left[\chi(\tau_2) f_2\left(\int_0^1 \aleph(\tau_2, \tau_3)\right.\right.\right.\right. \\
&\times \varphi^{-1}\left[\chi(\tau_3) f_3\left(\int_0^1 \aleph(\tau_3, \tau_4)\cdots\right.\right. \\
&\times f_{\ell-1}\left(\int_0^1 \aleph(\tau_{\ell-1}, \tau_\ell)\varphi^{-1}\Big[\chi(\tau_\ell) f_\ell(\varpi_1(\tau_\ell))\Big]\nabla\tau_\ell\right)\cdots\nabla\tau_3\Big]\nabla\tau_2\Big]\nabla\tau_1 \\
\le\, &\Gamma_r.
\end{aligned}
$$

Since $\Gamma_r = \|\varpi_1\|$ for $\varpi_1 \in \mathtt{K}_{\mathfrak{z}_r} \cap \partial\mathtt{M}_{1,r}$, we get $\|\Omega\varpi_1\| \le \|\varpi_1\|$. Now define $\mathtt{M}_{2,r} = \{\varpi_1 \in \mathtt{B} : \|\varpi_1\| < \Theta_r\}$. Let $\varpi_1 \in \mathtt{K}_{\mathfrak{z}_r} \cap \partial\mathtt{M}_{2,r}$ and let $\tau \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}}$. Then the argument leading to (3.2) can be done to the present case. Hence, the theorem is proved. □

Lastly, the case $\sum_{i=1}^{n} \frac{1}{p_i} > 1$.

Theorem 3.6. *Suppose* (H_1)-(H_2) *hold, let* $\{\mathfrak{z}_r\}_{r=1}^{\infty}$ *be a sequence with* $\mathfrak{z}_r \in (t_{r+1}, t_r)$. *Let* $\{\Gamma_r\}_{r=1}^{\infty}$ *and* $\{\Theta_r\}_{r=1}^{\infty}$ *be such that*

$$\Gamma_{r+1} < \frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r < \Theta_r < \mathfrak{Z}\Theta_r < \Gamma_r, \quad r \in \mathbb{N},$$

where

$$\mathfrak{Z} = \max\left\{\left[\Xi_{\mathfrak{z}_1} \prod_{i=1}^{n} \delta_i \int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1} \aleph_0(\tau, \tau)\nabla\tau\right]^{-1},\ 1\right\}.$$

Assume that f *satisfies*

(C_5) $f_j(\varpi) \le \varphi(\mathfrak{N}_3\Gamma_r)$ *for all* $t \in [0,1]_{\mathbb{T}}$, $0 \le \varpi \le \Gamma_r$, *where*

$$\mathfrak{N}_3 < \min\left\{\left[\Xi\, \|\aleph_0\|_{L^\infty_\nabla} \prod_{i=1}^{n} \left\|\varphi^{-1}(\chi_i)\right\|_{L^1_\nabla}\right]^{-1}, \mathfrak{Z}\right\};$$

(C_6) $f_j(\varpi) \geq \varphi(\mathfrak{Z}\Theta_r)$ *for all* $t \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}}$, $\frac{\Xi_{\mathfrak{z}_r}}{\Xi}\Theta_r \leq \varpi \leq \Theta_r$.

Then the iterative boundary value problem (1.1)–(1.2) *has denumerably many solutions* $\{(\varpi_1^{[r]}, \varpi_2^{[r]}, \dots, \varpi_\ell^{[r]})\}_{r=1}^{\infty}$ *such that* $\varpi_j^{[r]}(t) \geq 0$ *on* $[0,1]_{\mathbb{T}}$, $j = 1, 2, \dots, \ell$, *and* $r \in \mathbb{N}$.

Proof. The proof is similar to the proof of Theorem 3.1. So, we omit the details here. □

4. Examples

In this section, we present an example to check validity of our main results.

Example 4.1. Consider the following boundary value problem on $\mathbb{T} = [0,1]$

$$\left.\begin{aligned} &\varphi(\varpi_j''(t)) + \chi(t) f_j(\varpi_{j+1}(t)) = 0, \quad j = 1, 2,\ t \in [0,1],\\ &\varpi_3(t) = \varpi_1(t), \end{aligned}\right\} \tag{4.1}$$

$$\left.\begin{aligned} \varpi_j(0) - \varpi_j'(0) &= \int_0^1 \frac{1}{2}\varpi_j(\tau)d\tau,\\ \varpi_j(1) + \varpi_j'(1) &= \int_0^1 \frac{1}{2}\varpi_j(\tau)d\tau, \end{aligned}\right\} \tag{4.2}$$

where

$$\varphi(\varpi) = \begin{cases} \dfrac{\varpi^3}{1+\varpi^2}, & \varpi \leq 0,\\ \varpi^2, & \varpi > 0, \end{cases}$$

$$\chi(t) = \chi_1(t) \cdot \chi_2(t),$$

in which

$$\chi_1(t) = \frac{1}{|t - \frac{1}{4}|^{\frac{1}{2}}} \quad \text{and} \quad \chi_2(t) = \frac{1}{|t - \frac{1}{3}|^{\frac{1}{2}}},$$

$$f_1(\varpi) = f_2(\varpi)$$
$$= \begin{cases} 0.05 \times 10^{-8}, & \varpi \in (10^{-4}, +\infty),\\ \frac{5604\times 10^{-(8r+6)} - 0.05\times 10^{-8r}}{10^{-(4r+3)} - 10^{-4r}}(\varpi - 10^{-4r}) & \\ +0.05 \times 10^{-8r}, & \varpi \in \left[10^{-(4r+3)}, 10^{-4r}\right],\\ 5604 \times 10^{-(8r+6)}, & \varpi \in \left(0.98 \times 10^{-(4r+3)}, 10^{-(4r+3)}\right),\\ \frac{5604\times 10^{-(8r+6)} - 0.05\times 10^{-8r}}{0.98\times 10^{-(4r+3)} - 10^{-(4r+4)}}(\varpi - 10^{-(4r+4)}) & \\ +0.05 \times 10^{-8r}, & \varpi \in \left(10^{-(4r+4)}, 0.98 \times 10^{-(4r+3)}\right]. \end{cases}$$

Let

$$t_r = \frac{31}{64} - \sum_{k=1}^{r} \frac{1}{4(k+1)^4}, \quad \mathfrak{z}_r = \frac{1}{2}(t_r + t_{r+1}), \quad \text{for } r = 1, 2, 3, \dots,$$

then

$$\mathfrak{z}_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32} \quad \text{and} \quad t_{r+1} < \mathfrak{z}_r < t_r, \quad \mathfrak{z}_r > \frac{1}{5}, \quad \text{for } r = 1, 2, 3, \dots$$

Therefore, $\frac{\mathfrak{z}_r}{1} = \mathfrak{z}_r > \frac{1}{5}$, $j = 1, 2, 3, \ldots$ It is clear that

$$t_1 = \frac{15}{32} < \frac{1}{2}, \quad t_r - t_{r+1} = \frac{1}{4(r+2)^4}, \quad r = 1, 2, 3, \ldots$$

Since $\sum_{x=1}^{\infty} \frac{1}{x^4} = \frac{\pi^4}{90}$ and $\sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6}$, it follows that

$$t^* = \lim_{r\to\infty} t_r = \frac{31}{64} - \sum_{k=1}^{\infty} \frac{1}{4(r+1)^4} = \frac{47}{64} - \frac{\pi^4}{360} = 0.4637941914,$$

$\chi_1, \chi_2 \in L^p[0,1]$ for $0 < p < 2$, so $\delta_1 = \delta_2 = \frac{1}{\sqrt{3}}$,

$$a(t) = 2 - t, \quad b(t) = 1 + t, \quad d = 3, \quad \aleph_0(t, \tau) = \frac{1}{3}\begin{cases} (2-\tau)(1+t), & t \le \tau, \\ (2-t)(1+\tau), & \tau \le t, \end{cases}$$

$$c_i = \int_0^1 \left[\int_0^1 \aleph_0(\tau_1, \tau_2)\kappa_i(\tau_1)\nabla\tau_1\right]\chi(\tau_2)\nabla\tau_2 = 2.774076198,$$

$$u_a = u_b = v_a = v_b = \frac{1}{4}, \quad \kappa_1^* = \kappa_2^* = \frac{1}{2}, \quad \kappa_1(\mathfrak{z}_1) = \kappa_2(\mathfrak{z}_1) = 0.06558641976,$$

$$\mathcal{L}(\mathfrak{z}_1) = \min\left\{\frac{\alpha\mathfrak{z}_1 + \beta}{\alpha + \beta}, \frac{\gamma\mathfrak{z}_1 + \delta}{\gamma + \delta}\right\} = \frac{1 + \mathfrak{z}_1}{2} = 0.7336033950,$$

$$\eta(t) = \frac{(1 - v_b)a(t) + v_a b(t)}{d[(1 - u_a)(1 - v_b) - u_b v_a]} = \frac{7 - 2t}{6}, \quad \eta^* = \frac{7}{6}, \quad \eta(\mathfrak{z}_1) = 1.010931070,$$

$$\lambda(t) = \frac{(1 - u_a)b(t) + u_b a(t)}{d[(1 - u_a)(1 - v_b) - u_b v_a]} = \frac{5 - 2t}{6}, \quad \lambda^* = \frac{5}{6}, \quad \lambda(\mathfrak{z}_1) = 0.6775977366,$$

$$\Xi = 1 + \eta^*\kappa_1^* + \lambda^*\kappa_2^* = 2,$$

$$\Xi_{\mathfrak{z}_1} = \mathcal{L}(\mathfrak{z}_1)\Big[1 + \eta(\mathfrak{z}_1)\kappa_1(\mathfrak{z}_1) + \lambda(\mathfrak{z}_1)\kappa_2(\mathfrak{z}_1)\Big] = 0.8148459802.$$

Note that $\Xi_{\mathfrak{z}}$ is increasing, it follows that $1.969391539 = \Xi_{\mathfrak{z}\infty} < \Xi_{\mathfrak{z}r} < \Xi_{\mathfrak{z}1} = 2$, $0.9846957695 \le \frac{\Xi_{\mathfrak{z}r}}{\Xi} \le 2$ and

$$\int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1} \aleph_0(\tau, \tau)\nabla\tau = \int_{\frac{15}{32} - \frac{1}{648}}^{1 - \frac{15}{32} + \frac{1}{648}} \frac{(2-\tau)(1+\tau)}{3} d\tau = 0.04918197800.$$

Thus, we get

$$\mathfrak{Z} = \max\left\{\left[\Xi_{\mathfrak{z}_1}\prod_{i=1}^{n}\delta_i \int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1} \aleph_0(\tau, \tau)\nabla\tau\right]^{-1}, 1\right\} = \max\left\{74.85826138, 1\right\}$$
$$= 74.85826138$$

and

$$\|\aleph_0\|_{L^q_\nabla} = \left[\int_0^1 |\aleph_0(\tau, \tau)|^q d\tau\right]^{\frac{1}{q}} < 1, \quad \text{for } 0 < q < 2.$$

Next, let $0 < \mathfrak{a} < 1$ be fixed. Then $\chi_1, \chi_2 \in L^{1+\mathfrak{a}}[0,1]$. It follows that

$$\|\varphi^{-1}(\chi_1)\|_{1+\mathfrak{a}} = \left[\frac{1}{3 - \mathfrak{a}}\left(3^{\frac{3-\mathfrak{a}}{4}} + 1\right)2^{\frac{1+\mathfrak{a}}{2}}\right]^{\frac{1}{1+\mathfrak{a}}}$$

and

$$\|\varphi^{-1}(\chi_2)\|_{1+\mathfrak{a}} = \left[\frac{4}{3-\mathfrak{a}}\left(2^{\frac{3-\mathfrak{a}}{4}}+1\right)(1/3)^{\frac{3-\mathfrak{a}}{4}}\right]^{\frac{1}{1+\mathfrak{a}}}.$$

So, for $0 < \mathfrak{a} < 1$, we have

$$0.2509961333 \leq \left[\Xi \,\|\aleph_0\|_{L^q_\nabla} \prod_{i=1}^{n} \left\|\varphi^{-1}(\chi_i)\right\|_{L^{p_i}_\nabla}\right]^{-1} \leq 0.2856331500.$$

Taking $\mathfrak{N}_1 = 0.2$. In addition, if we take

$$\Gamma_r = 10^{-4r}, \quad \Theta_r = 10^{-(4r+3)},$$

then

$$\begin{aligned}
\Gamma_{r+1} = 10^{-(4r+4)} &< 0.9846957695 \times 10^{-(4r+3)} < \frac{\Xi_{\mathfrak{z}r}}{\Xi}\Theta_r < \Theta_r = 10^{-(4r+3)} \\
&< \Gamma_r = 10^{-4r}, \\
\mathfrak{Z}\Theta_r &= 74.85826138 \times 10^{-(4r+3)} < 0.2 \times 10^{-4r} = \mathfrak{N}_1\Gamma_r, \quad r = 1, 2, 3, \dots,
\end{aligned}$$

and f_1, f_2 satisfies the following growth conditions:

$$\begin{aligned}
f_1(\varpi) = f_2(\varpi) &\leq \varphi(\mathfrak{N}_1\Gamma_r) = \mathfrak{N}_1^2\Gamma_r^2 = 0.04 \times 10^{-8r}, \quad \varpi \in \left[0, 10^{-4r}\right] \\
f_1(\varpi) = f_2(\varpi) &\geq \varphi(\mathfrak{Z}\Theta_r) = \mathfrak{Z}^2\Theta_r^2 \\
&= 5603.759297 \times 10^{-(8r+6)}, \quad \varpi \in \left[0.98 \times 10^{-(4r+3)}, 10^{-(4r+3)}\right].
\end{aligned}$$

Then all the conditions of Theorem 3.4 are satisfied. Therefore, by Theorem 3.4, the iterative boundary value problem (4.1)–(4.2) has denumerably many solutions $\left\{\left(\varpi_1^{[r]}, \varpi_2^{[r]}\right)\right\}_{r=1}^{\infty}$ such that $\varpi_j^{[r]}(t) \geq 0$ on $[0, 1]$, $j = 1, 2$ and $r \in \mathbb{N}$.

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References

[1] R. P. Agarwal and M. Bohner, *Basic calculus on time scales and some of its applications*, Results Math. **35** (1999), 3–22. https://doi.org/10.1007/BF03322019

[2] R. P. Agarwal, M. Bohner, D. O'Regan and A. Peterson, *Dynamic equations on time scales: a survey*, J. Comput. Appl. Math. **141** (2002), 1–26. https://doi.org/10.1016/S0377-0427(01)00432-0

[3] R. P. Agarwal, V. Otero-Espinar, K. Perera and D. R. Vivero, *Basic properties of Sobolev's spaces on time scales*, Adv. Differ. Equ. **2006**(1) (2006), 1–14. https://doi.org/10.1155/ADE/2006/38121

[4] G. A. Anastassiou, *Intelligent Mathematics: Computational Analysis*, Springer, Heidelberg, 2011. https://doi.org/10.1007/978-3-642-17098-0

[5] M. Bohner and H. Luo, *Singular second-order multi-point dynamic boundary value problems with mixed derivatives*, Adv. Differ. Equ. **2006**(1) (2006), 1–15. https://doi.org/10.1155/ADE/2006/54989

[6] Z. He, *Double positive solutions of three-point boundary value problems for p-Laplacian dynamic equations on time scales*, J. Comput. Appl. Math. **182** (2005), 304–315. https://doi.org/10.1016/j.cam.2004.12.012

[7] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001. https://doi.org/10.1007/978-1-4612-0201-1

[8] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003. https://doi.org/10.1007/978-0-8176-8230-9

[9] M. J. De Lemos, *Turbulence in Porous Media: Modeling and Applications*, Elsevier, London, 2012. https://doi.org/10.1016/C2011-0-06981-8

[10] A. Dogan, *Positive solutions of the p-Laplacian dynamic equations on time scales with sign changing nonlinearity*, Electron. J. Differential Equations **39** (2018), 1–17.

[11] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988. https://doi.org/10.1016/C2013-0-10750-7

[12] G. S. Guseinov, *Integration on time scales*, J. Math. Anal. App. **285**(1) (2003), 107–127. https://doi.org/10.1016/S0022-247X(03)00361-5

[13] L. S. Leibenson, *General problem of the movement of a compressible fluid in a porous medium*, Izvestiia Akademii Nauk Kirgizskoi SSR **9** (1983), 7–10.

[14] S. Liang and J. Zhang, *The existence of countably many positive solutions for nonlinear singular m-point boundary value problems on time scales*, J. Comput. Appl. Math. **223**(1) (2009), 291–303. https://doi.org/10.1016/j.cam.2008.01.021

[15] A. L. Ljung, V. Frishfelds, T. S. Lundstrom and B. D. Marjavaara, *Discrete and continuous modeling of heat and mass transport in drying of a bed of iron ore pellets*, Dry. Technol. **30**(7) (2012), 760–773. https://doi.org/10.1080/07373937.2012.662567

[16] U. M. Ozkan, M. Z. Sarikaya and H. Yildirim, *Extensions of certain integral inequalities on time scales*, Appl. Math. Lett. **21**(10) (2008), 993–1000. https://doi.org/10.1016/j.aml.2007.06.008

[17] B. P. Rynne, *L^2 spaces and boundary value problems on time scales*, J. Math. Anal. App. **328**(2) (2007), 1217–1236. https://doi.org/10.1016/j.jmaa.2006.06.008

[18] P. A. Williams, *Unifying fractional calculus with time scales*, PhD thesis, University of Melbourne, 2012.

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