Kragujevac Journal of Mathematics
Volume 47(3) (2023), Pages 459–479.

ON MINIMAXITY AND LIMIT OF RISKS RATIO OF JAMES-STEIN ESTIMATOR UNDER THE BALANCED LOSS FUNCTION

ABDENOUR HAMDAOUI[1], ABDELKADER BENKHALED[2], AND MEKKI TERBECHE[3]

Abstract. The problem of estimating the mean of a multivariate normal distribution by different types of shrinkage estimators is investigated. Under the balanced loss function, we establish the minimaxity of the James-Stein estimator. When the dimension of the parameters space and the sample size tend to infinity, we study the asymptotic behavior of risks ratio of James-Stein estimator to the maximum likelihood estimator. The positive-part of James-Stein estimator is also treated.

1. Introduction

Stein [22] showed that the maximum likelihood estimator (MLE) of the mean $\theta = (\theta_1, \ldots, \theta_p)^\top$ of a multivariate Gaussian distribution $N_p(\theta, \sigma^2 I_p)$ is inadmissible in mean squared sense when the dimension of the parameters space $p \geq 3$. In particular, he proved the existence of a class of estimators which achieve the smaller total mean squared error regardless of the true θ. Perhaps the best known estimator of such kind is James-Stein's estimator introduced by James and Stein [16]. This one is a special case of a larger class of estimators known as shrinkage estimators which is a combination of a model with low bias and high variance, and a model with high bias but low variance. Interestingly, the James-Stein estimator is itself inadmissible, and there exists a wide class of estimators that outperform the MLE, see for example, Lindley [18], Baranchik [1], Bhattacharya [7], Bock [8], Berger [5] and Berger and Wolpert [6]. Some of them, found some particular minimax estimators. Selahattin et al. [19] provided

Key words and phrases. Balanced loss function, James-Stein estimator, minimaxity, multivariate Gaussian random variable, non-central chi-square distribution, risk ratio, shrinkage estimator.

2010 *Mathematics Subject Classification.* Primary: 62F12. Secondary: 62C20.

DOI

Received: March 13, 2020.

Accepted: September 14, 2020.

several alternative methods for derivation of the restricted ridge regression estimator (RRRE). Hansen [15] compared the mean-squared error of ordinary least squares (OLS), James-Stein, and least absolute shrinkage and selection operator (LASSO) shrinkage estimators and showed that neither James-Stein nor LASSO dominates uniformly the other. Xie et al. [24] introduced a class of semi-parametric/parametric shrinkage estimators and established their asymptotic optimality properties.

Casella and Hwang [9] have studied the estimation of the mean θ of the random variable $X \sim N_p(\theta, I_p)$ when the dimension of parameters space p tends to infinity. They showed that if the limit of the ratio $\|\theta\|^2/p$ is a constant $c > 0$, then the risks ratio of the James-Stein estimator δ^{JS} and the positive-part of the James-Stein estimator δ^{JS+}, to the MLE, tends to a constant value $c/(1+c)$. Benmansour and Hamdaoui [3] have taken the model $X \sim N_p(\theta, \sigma^2 I_p)$ where the parameter σ^2 is unknown and estimated by S^2 ($S^2 \sim \sigma^2\chi^2_n$). They established the same results given by Casella and Hwang [9]. Hamdaoui and Benmansour [12] considered the same model given by Benmansour and Hamdaoui [3], but this time, they studied the following class of shrinkage estimators $\delta_\phi = \delta^{JS} + l(S^2\phi(S^2, \|X\|^2)/\|X\|^2)X$ with l is a real parameter. The authors showed that, when the sample size n and the dimension of parameters space p tend to infinity, the estimators δ_ϕ have a lower bound $B_m = c/(1+c)$ and if the shrinkage function ϕ satisfies some conditions, the risks ratio $R(\delta_\phi, \theta)/R(X, \theta)$ attains this lower bound B_m, in particular the risks ratios $R(\delta^{JS}, \theta)/R(X, \theta)$ and $R(\delta^{JS+}, \theta)/R(X, \theta)$. Hamdaoui et al. [14] studied the limit of risks ratio of two forms of shrinkage estimators. The first one has been introduced by Benmansour and Mourid [4], $\delta_\psi = \delta^{JS} + l\left(S^2\psi\left(S^2, \|X\|^2\right)/\|X\|^2\right)X$, where $\psi(\cdot, u)$ is a function with support $[0, b]$, $b \in \mathbb{R}_+$ and satisfies some different conditions from the one given by Hamdaoui and Benmansour [12]. The second is the polynomial form of shrinkage estimator introduced by Li and Kio [17]. Hamdaoui and Mezouar [13] have treated the general class of shrinkage estimators $\delta_\phi = \left(1 - S^2\phi\left(S^2, \|X\|^2\right)/\|X\|^2\right)X$. They showed the same results given by Hamdaoui and Benmansour [12], with different conditions on the shrinkage function ϕ. Benkhaled and Hamdaoui [2] have considered the model $X \sim N_p(\theta, \sigma^2 I_p)$ where σ^2 is unknown. They studied two different forms of shrinkage estimators of θ: estimators of the form $\delta^\psi = (1 - \psi(S^2, \|X\|^2)S^2/\|X\|^2)X$, and estimators of Lindley-Type given by $\delta^\varphi = (1 - \varphi(S^2, T^2)S^2/T^2)(X - \overline{X}) + \overline{X}$ with $\overline{X} = (1/p)\sum_{i=1}^p X_i$ and $T^2 = \sum_{i=1}^p \left(X_i - \overline{X}\right)^2$, that shrink the components of the MLE X to the random variable $\overline{X}$. The authors showed that if the shrinkage function ψ (respectively φ) satisfies the new conditions different from the known results in the literature, then the estimator δ^ψ (respectively δ^φ) is minimax. When the sample size and the dimension of parameters space tend to infinity, they studied the behavior of risks ratio of these estimators to the MLE. Hamdaoui et al. [11] have studied the minimaxity and limits of risks ratios of shrinkage estimators of a multivariate normal mean in the Bayesian case. The authors have considered the model $X \sim N_p(\theta, \sigma^2 I_p)$ where σ^2 is unknown and have taken the prior law $\theta \sim N_p(\upsilon, \tau^2 I_p)$. They constructed a modified Bayes estimator δ_B^* and an empirical modified Bayes estimator δ_{EB}^*. When n and p are finite, they showed that the estimators δ_B^* and δ_{EB}^* are minimax. The authors have also interested in studying the limits of risks ratios of these estimators, to the MLE X, when n and p tend to infinity. The majority of these works has been considered under the quadratic loss function.

In the field of the estimation of a multivariate normal mean under the balanced loss function we cite for example, Farsipour and Asgharzadeh [10] have considered the model: $X_1, \ldots, X_n$ to be a random sample from a $N_p(\theta, \sigma^2)$ with σ^2 known and the aim is to estimate the parameter θ. They studied the admissibility of the estimator of the form $a\overline{X} + b$ under the balanced loss function. Selahattin and Issam [20] introduced and derived the optimal extended balanced loss function (EBLF) estimators and predictors and discussed their performances.

In this work, we deal with the model $X \sim N_p(\theta, \sigma^2 I_p)$, where the parameter σ^2 is unknown and estimated by S^2 ($S^2 \sim \sigma^2\chi_n^2$). Our aim is to estimate the unknown parameter θ by shrinkage estimators deduced by the MLE. The criterion adopted for comparing two estimators is the risk associated to the balanced loss function. The paper is organized as follows. In Section 2, we recall some preliminaries that are useful for our main results. In the first part of the Section 3, we study the minimaxity of the James-Stein estimator and the positive-part of James-Stein estimator. In the second part of this Section, we show that the positive-part of James-Stein estimator is not only minimax but also dominates the James-Stein estimator. In Section 4, we treat the asymptotic behavior of risks ratios of James-Stein estimator and the positive-part of the James-Stein estimator to the MLE, when the dimension p tends to infinity and the sample size n is fixed on one hand, and on the other hand when p and n tend simultaneously to infinity. We compute lower and upper bounds of each risks ratio, that allow us to calculate the limit of risks ratio. In Section 5, we graphically illustrate some obtained results. We end the manuscript by giving an Appendix which contains technical lemmas that are used in the proofs of our main results.

2. Preliminaries

We recall that if X is a multivariate Gaussian random $N_p(\theta, \sigma^2 I_p)$ in $\mathbb{R}^p$, then $\frac{\|X\|^2}{\sigma^2} \sim \chi_p^2\,(\lambda)$ where $\chi_p^2(\lambda)$ denotes the non-central chi-square distribution with p degrees of freedom and non-centrality parameter $\lambda = \frac{\|\theta\|^2}{2\sigma^2}$.

In the next we also recall the following results that are useful in our proofs.

Definition 2.1. Let $U \sim \chi_p^2(\lambda)$. For any measurable function $f : \mathbb{R}_+ \to \mathbb{R}$, $\chi_p^2(\lambda)$ integrable, we have

$$\begin{aligned} E\left[f(U)\right] &= E_{\chi_p^2(\lambda)}\left[f(U)\right] \\ &= \int_{\mathbb{R}_+} f(u)\chi_p^2(\lambda)\, du \end{aligned}$$

$$= \sum_{k=0}^{+\infty} \left[\int_{\mathbb{R}_+} f(u) \chi^2_{p+2k}(0)\, du \right] e^{-\frac{\lambda}{2}} \frac{\left(\frac{\lambda}{2}\right)^k}{k!}$$
$$= \sum_{k=0}^{+\infty} \left[\int_{\mathbb{R}_+} f(u) \chi^2_{p+2k} du \right] P\left(\frac{\lambda}{2}; dk\right),$$

where $P\left(\frac{\lambda}{2}\right)$ is a Poisson random variable with parameter $\frac{\lambda}{2}$ and χ^2_{p+2k} is the central chi-square distribution with $p+2k$ degrees of freedom.

From the Definition 2.1, we deduce that if $X \sim N_p(\theta, \sigma^2 I_p)$, then for $p \geq 3$ we have

$$E\left(\frac{1}{\|X\|^2}\right) = \frac{1}{\sigma^2} E\left(\frac{1}{p-2+2K}\right), \tag{2.1}$$

where $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ is a Poisson random variable with parameter $\frac{\|\theta\|^2}{2\sigma^2}$.

Lemma 2.1 ([23])**.** *Let X be a $N(\upsilon, \sigma^2)$ real random variable and let $f : \mathbb{R} \to \mathbb{R}$ be an indefinite integral of the Lebesgue measurable function, f' essentially the derivative of f. Suppose also that $E|f'(X)| < +\infty$. Then*

$$E\left[\left(\frac{X-\upsilon}{\sigma}\right) f(X)\right] = E(f'(X)).$$

Now, let $X \sim N_p(\theta, \sigma^2 I_p)$ where σ^2 is unknown and estimated by S^2 ($S^2 \sim \sigma^2 \chi^2_n$). And let the balanced loss function defined as: for any estimator δ of θ

$$L_\omega(\delta, \theta) = \omega\|\delta - \delta_0\|^2 + (1-\omega)\|\delta - \theta\|^2, \tag{2.2}$$

where $0 \leq \omega < 1$ and δ_0 is the MLE. We associate to this balanced loss function the risk function defined by

$$R_\omega(\delta, \theta) = E(L_\omega(\delta, \theta)). \tag{2.3}$$

In this model, it is clear that the MLE is $\delta_0 = X$, its risk function is $(1-\omega)p\sigma^2$.

Indeed, $R_\omega(X, \theta) = \omega E(\|X - X\|^2) + (1-\omega) E(\|X - \theta\|^2)$, where $X \sim N_p(\theta, \sigma^2 I_p)$, then $\frac{X-\theta}{\sigma} \sim N_p(0, I_p)$, thus $\frac{\|X-\theta\|^2}{\sigma^2} \sim \chi^2_p$. Hence, $E(\|X-\theta\|^2) = E(\sigma^2 \chi^2_p) = \sigma^2 p$.

It is well known that δ_0 is minimax and inadmissible for $p \geq 3$, thus any estimator which dominates it is also minimax.

3. Minimaxity

3.1. **James-Stein estimator.** Consider the estimator

$$\delta_a = \left(1 - a\frac{S^2}{\|X\|^2}\right) X = X - a\frac{S^2}{\|X\|^2} X, \tag{3.1}$$

where a is a real parameter.

Proposition 3.1. *Under the balanced loss function L_ω, we have:*

$$R_\omega(\delta_a,\theta) = (1-\omega)p\sigma^2 + [a^2\sigma^2 n(n+2) - 2a(1-\omega)\sigma^2 n(p-2)]E\left(\frac{1}{p-2+2K}\right),$$

where $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ is a Poisson random variable with parameter $\frac{\|\theta\|^2}{2\sigma^2}$.

Proof.

$$R_\omega(\delta_a,\theta) = \omega E(\|\delta_a - X\|^2) + (1-\omega)E(\|\delta_a-\theta\|^2).$$

From the independence between two random variables S^2 and $\|X\|^2$, we obtain

$$\begin{aligned} E(\|\delta_a - X\|^2) &= E\left(\| - a\frac{S^2}{\|X\|^2}X\|^2\right) \\ &= a^2E(S^2)E\left(\frac{1}{\|X\|^2}\right) \\ &= a^2E((\sigma^2\chi^2_n)^2)E\left(\frac{1}{\|X\|^2}\right) \\ &= a^2\sigma^2 n(n+2)E\left(\frac{1}{p-2+2K}\right), \end{aligned}$$

where $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ is a Poisson random variable with parameter $\frac{\|\theta\|^2}{2\sigma^2}$ and the last equality according to the formula (2.1) and the fact that $E((\chi^2_n)^2) = n(n+2)$. Now,

$$\begin{aligned} E(\|\delta_a - \theta\|^2) =& E\left(\|X - a\frac{S^2}{\|X\|^2}X - \theta\|^2\right) \\ =& E(\|X-\theta\|^2) + a^2E(S^2)^2E\left(\frac{1}{\|X\|^2}\right) \\ & - 2aE(S^2)E\left(\left\langle X-\theta, \frac{1}{\|X\|^2}X\right\rangle\right). \end{aligned}$$

As

$$E\left(\left\langle X-\theta, \frac{1}{\|X\|^2}X\right\rangle\right) = E\left[\sum_{i=1}^{p}\left(y_i - \frac{\theta_i}{\sigma}\right)\frac{y_i}{\|y\|^2}\right],$$

where for any $i = 1,\ldots,p$, $y_i = \frac{x_i}{\sigma} \sim N\left(\frac{\theta_i}{\sigma},1\right)$ and by using Lemma 2.1, we get

$$\begin{aligned} E\left[\left\langle X-\theta, \frac{1}{\|X\|^2}X\right\rangle\right] &= \sum_{i=1}^{p} E\left(\frac{\partial}{\partial y_i}\frac{1}{\sum_{j=1}^{p} y_j^2}y_i\right) \\ &= \sum_{i=1}^{p} E\left[\frac{1}{\|y\|^2} - \frac{2y_i^2}{\|y\|^4}\right] \\ &= (p-2)E\left(\frac{1}{\|y\|^2}\right) \end{aligned}$$

$$= (p-2)E\left(\frac{1}{p-2+2K}\right),$$

where $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ is a Poisson random variable with parameter $\frac{\|\theta\|^2}{2\sigma^2}$ and the last equality comes from formula (2.1). Thus,

$$\begin{aligned}R_\omega(\delta_a,\theta) =&\omega a^2\sigma^2 n(n+2)E\left(\frac{1}{p-2+2K}\right)\\&+(1-\omega)\left[p\sigma^2+a^2\sigma^2 n(n+2)E\left(\frac{1}{p-2+2K}\right)\right]\\&-2a(1-\omega)\sigma^2 n(p-2)E\left(\frac{1}{p-2+2K}\right)\\&=(1-\omega)p\sigma^2+[a^2\sigma^2 n(n+2)-2a(1-\omega)\sigma^2 n(p-2)]E\left(\frac{1}{p-2+2K}\right).\square\end{aligned}$$

Using Proposition 3.1, we note that under the balanced loss function L_ω, a sufficient condition so that δ_a dominating the MLE X is

$$a \geq 0 \quad \text{and} \quad a(n+2)-2(1-\omega)(p-2) \leq 0,$$

which is equivalent to

$$0 \leq a \leq \frac{2(1-\omega)(p-2)}{n+2}. \tag{3.2}$$

From Proposition 3.1 and the convexity of risk function $R_\omega(\delta_a,\theta)$ on a, one can easily show that the optimal value of a that minimizes the risk function $R_\omega(\delta_a,\theta)$ is

$$\alpha = \frac{(1-\omega)(p-2)}{n+2}.$$

For $a=\alpha$, we obtain the James-Stein estimator

$$\delta_{JS}=\delta_\alpha=\left(1-\alpha\frac{S^2}{\|X\|^2}\right)X=\left(1-\frac{(1-\omega)(p-2)}{n+2}\frac{S^2}{\|X\|^2}\right)X. \tag{3.3}$$

It follows from Proposition 3.1 that the risk function of δ_{JS} is given by

$$R_\omega(\delta_{JS},\theta)=(1-\omega)p\sigma^2-(1-\omega)^2(p-2)^2\frac{n}{n+2}\sigma^2E\left(\frac{1}{p-2+2K}\right), \tag{3.4}$$

where $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$.

From the formula (3.4), it is easy to see that $R_\omega(\delta_{JS},\theta) \leq R_\omega(X,\theta)$, then the James-Stein estimator δ_{JS} dominates the MLE X, and thus it is minimax.

3.2. **Positive-Part of James-Stein estimator.** We consider the positive-part of James-Stein estimator defined by

$$\delta_{JS}^{+} = \left(1 - \alpha \frac{S^2}{\|X\|^2}\right)^{+} X = \left(1 - \alpha \frac{S^2}{\|X\|^2}\right) X \mathbb{I}_{\alpha \frac{S^2}{\|X\|^2} \leq 1}, \tag{3.5}$$

where $\left(1 - \alpha \frac{S^2}{\|X\|^2}\right)^{+} = \max\left(0, 1 - \alpha \frac{S^2}{\|X\|^2}\right)$. We recall that

$$\delta_{JS}^{-} = \left(1 - \alpha \frac{S^2}{\|X\|^2}\right)^{-} X = \left(\alpha \frac{S^2}{\|X\|^2} - 1\right) X \mathbb{I}_{\alpha \frac{S^2}{\|X\|^2} \geq 1}, \tag{3.6}$$

where $\mathbb{I}_{\alpha \frac{S^2}{\|X\|^2} \geq 1}$ is the indicating function of the set $\left(\alpha \frac{S^2}{\|X\|^2} \geq 1\right)$.

We note that the positive-part of James-Stein estimator δ_{JS}^{+} has the form (3.1), corresponding to $a^{+} = \min\left\{\frac{(1-\omega)(p-2)}{n+2}, \frac{S^2}{\|X\|^2}\right\}$. Since a^{+} satisfies the relation (3.2), δ_{JS}^{+} dominates the MLE X under the balanced loss function L_ω, thus δ_{JS}^{+} is minimax.

3.3. **Dominating the positive-part of James-Stein estimator to James-Stein estimator.** It is well known that the positive-part of James-Stein estimator dominates the James-Stein estimator for the standard case where $\omega = 0$ (see Baranchick [1]). In this part, we show that this property remains valid for any $0 < \omega < 1$.

Theorem 3.1. *Under the balanced loss function L_ω, the positive-part of James-Stein estimator δ_{JS}^{+} dominates the James-Stein estimator δ_{JS}.*

Proof.

$$R_\omega(\delta_{JS}^{+}, \theta) = \omega E(\|\delta_{JS}^{+} - X\|^2) + (1-\omega) E(\|\delta_{JS}^{+} - \theta\|^2)$$

and

$$R_\omega(\delta_{JS}, \theta) = \omega E(\|\delta_{JS} - X\|^2) + (1-\omega) E(\|\delta_{JS} - \theta\|^2).$$

Baranchick [1] has showed that $E(\|\delta_{JS}^{+} - \theta\|^2) \leq E(\|\delta_{JS} - \theta\|^2)$ for $p \geq 3$ and all $(\theta, \sigma) \in (\mathbb{R}^p \times \mathbb{R}^{+})$. Then δ_{JS}^{+} dominates δ_{JS} under the balanced loss function L_ω, if and only if $E(\|\delta_{JS}^{+} - X\|^2) - E(\|\delta_{JS} - X\|^2) \leq 0$. Now,

$$\begin{aligned}
E(\|\delta_{JS}^{+} - X\|^2) =& E(\|\delta_{JS}^{+} - \delta_{JS} + \delta_{JS} - X\|^2) \\
=& E(\|\delta_{JS}^{+} - \delta_{JS}\|^2) + E(\|\delta_{JS} - X\|^2) + 2E[\langle \delta_{JS}^{+} - \delta_{JS}, \delta_{JS} - X\rangle] \\
=& E(\|\delta_{JS}^{-}\|^2) + E(\|\delta_{JS} - X\|^2) + 2E[\langle \delta_{JS}^{-}, \delta_{JS} - X\rangle] \\
=& E\left[\left\|\left(\alpha \frac{S^2}{\|X\|^2} - 1\right) \mathbb{I}_{\alpha \frac{S^2}{\|X\|^2} \geq 1} X\right\|^2\right] + E(\|\delta_{JS} - X\|^2) \\
&+ 2E\left[\left\langle \left(\alpha \frac{S^2}{\|X\|^2} - 1\right) \mathbb{I}_{\alpha \frac{S^2}{\|X\|^2} \geq 1} X, -\alpha \frac{S^2}{\|X\|^2} X\right\rangle\right] \\
=& E\left[\left(\alpha^2 \frac{S^4}{\|X\|^2} + \|X\|^2 - 2\alpha S^2\right) \mathbb{I}_{\alpha \frac{S^2}{\|X\|^2} \geq 1}\right] + E(\|\delta_{JS} - X\|^2)
\end{aligned}$$

$$- 2E\left[\left(\alpha^2\frac{S^4}{\|X\|^2} - \alpha S^2\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right].$$

Then

$$\begin{aligned}
&E(\|\delta^+_{JS} - X\|^2) - E(\|\delta_{JS} - X\|^2)\\
=&E\left[\left(\alpha^2\frac{S^4}{\|X\|^2} + \|X\|^2 - 2\alpha S^2\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right] - 2E\left[\left(\alpha^2\frac{S^4}{\|X\|^2} - \alpha S^2\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right]\\
=&E\left[\left(\|X\|^2 - \alpha^2\frac{S^4}{\|X\|^2}\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right]\\
=&E\left[\left(\frac{1}{\|X\|^2}(\|X\|^2 - \alpha S^2)(\|X\|^2 + \alpha S^2)\right)\mathbb{I}_{(\|X\|^2-\alpha S^2)\leq 0}\right]\\
\leq&0.
\end{aligned}$$

□

4. Limits of Risks Ratios

4.1. **Bounds and limit of the risks ratio of James-Stein estimator.** In this part, we study the limit of risks ratio of the James-Stein estimator δ_{JS} to the MLE X, when the dimension p tends to infinity and the sample size n is fixed on one hand, and on the other hand when p and n tend simultaneously to infinity. The following lemma gives a lower and an upper bounds of the ratio $R_\omega(\delta_{JS}, \theta)/R_\omega(X, \theta)$, which helps us to calculate the limit of risks ratio.

Lemma 4.1. *Assume the estimator δ_{JS} given in* (3.3). *Under the balanced loss function L_ω, we have*

$$1 - \frac{n(1-\omega)(p-2)}{(n+2)(p+\frac{\|\theta\|^2}{\sigma^2})} \leq \frac{R_\omega(\delta_{JS}, \theta)}{R_\omega(X, \theta)} \leq 1 - \frac{n(1-\omega)(p-2)^2}{(n+2)p(p-2+\frac{\|\theta\|^2}{\sigma^2})}.$$

Proof. From Lemma 2.1 of Hamdaoui and Benmansour [12], we have

$$\frac{1}{p-2+\frac{\|\theta\|^2}{\sigma^2}} \leq E\left(\frac{1}{p-2+2K}\right) \leq \frac{p}{(p-2)(p+\frac{\|\theta\|^2}{\sigma^2})}.$$

Using the formula (3.4), we obtain the desired result. □

Theorem 4.1. *Assume the estimator δ_{JS} given in* (3.1), *if* $\lim_{p\to\infty}\frac{\|\theta\|^2}{p\sigma^2} = c$ $(c > 0)$, *then*

i) $\lim_{p\to\infty}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)} = \frac{\left(1-(1-\omega)\frac{n}{n+2}\right)+c}{1+c}$;

ii) $\lim\limits_{\substack{p\to\infty\\ n\to\infty}}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)} = \frac{\omega+c}{1+c}$.

Proof. i) Using Lemma 4.1 and under the condition $\lim\limits_{p\to\infty}\frac{\|\theta\|^2}{p\sigma^2} = c$, we have

$$\lim_{p\to\infty}\frac{R_\omega(\delta_{JS}, \theta)}{R_\omega(X, \theta)} \leq 1 - (1-\omega)\frac{n}{n+2}\lim_{p\to\infty}\left[\frac{(p-2)^2}{p}\frac{\frac{1}{p}}{\frac{p-2}{p}+\frac{\|\theta\|^2}{p\sigma^2}}\right]$$

$$= 1 - (1-\omega)\frac{n}{n+2}\lim_{p\to\infty}\left[\frac{(p-2)^2}{p^2}\frac{1}{\frac{p-2}{p}+\frac{\|\theta\|^2}{p\sigma^2}}\right]$$
$$= 1 - (1-\omega)\frac{n}{n+2}\frac{1}{1+c}$$
$$= \frac{\left(1-(1-\omega)\frac{n}{n+2}\right)+c}{1+c}$$

and

$$\lim_{p\to\infty}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)} \geq 1-(1-\omega)\frac{n}{n+2}\lim_{p\to\infty}\left[\frac{\frac{p-2}{p}}{\frac{p}{p}+\frac{\|\theta\|^2}{p\sigma^2}}\right]$$
$$= 1-(1-\omega)\frac{n}{n+2}\frac{1}{1+c}$$
$$= \frac{\left(1-(1-\omega)\frac{n}{n+2}\right)+c}{1+c}.$$

ii) From Lemma 4.1 and under the condition $\lim_{p\to\infty}\frac{\|\theta\|^2}{p\sigma^2} = c$, we obtain

$$\lim_{\substack{p\to\infty\\ n\to\infty}}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)} \leq 1-(1-\omega)\lim_{\substack{p\to\infty\\ n\to\infty}}\left[\frac{n}{n+2}\frac{(p-2)^2}{p}\frac{\frac{1}{p}}{\frac{p-2}{p}+\frac{\|\theta\|^2}{p\sigma^2}}\right]$$
$$= 1-(1-\omega)\frac{1}{1+c} = \frac{\omega+c}{1+c}$$

and

$$\lim_{\substack{p\to\infty\\ n\to\infty}}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)} \geq 1-(1-\omega)\lim_{\substack{p\to\infty\\ n\to\infty}}\left[\frac{n}{n+2}\frac{\frac{p-2}{p}}{\frac{p}{p}+\frac{\|\theta\|^2}{p\sigma^2}}\right]$$
$$= 1-(1-\omega)\frac{1}{1+c} = \frac{\omega+c}{1+c}.$$ □

Remark 4.1. As $0 \leq \omega < 1$, then

$$\frac{1-\frac{n}{n+2}+c}{1+c} \leq (1-(1-\omega)\frac{\frac{n}{n+2}+c}{1+c} < 1$$

and $c/(1+c) \leq (\omega+c)/(1+c) < 1$, thus for p tends to infinity and n is fixed, or for p and n tend simultaneously to infinity, the limit of risks ratio of James-Stein estimator δ_{JS} to the MLE X, is less than 1. Therefore, Theorem 4.1 show the stability of minimaxity property of James-Stein estimator δ_{JS} for the large values of n and p.

4.2. **Bounds and limit of the risks ratio of the positive-part of James-Stein estimator.** The results for the positive-part of James-Stein estimator δ^+_{JS} are similar to those for the ordinary James-Stein estimator δ_{JS}, although the calculations are a bit more difficult. In the following proposition, we give the explicit formula of the risk function of δ^+_{JS}.

Proposition 4.1. *The risk function of the Positive-Part of James-Stein estimator δ^+_{JS} under the balanced loss function L_ω, is*

$$\begin{aligned} R_\omega(\delta^+_{JS},\theta) =& R_\omega(\delta_{JS},\theta) \\ &+ E\left[\left(\|X\|^2 - \alpha^2\frac{S^4}{\|X\|^2} + 2(1-\omega)\sigma^2(p-2)\alpha\frac{S^2}{\|X\|^2} - p\sigma^2\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right]. \end{aligned}$$

Proof.

$$\begin{aligned} R_\omega(\delta^+_{JS},\theta) =& \omega E(\|\delta^+_{JS} - X\|^2) + (1-\omega)E(\|\delta^+_{JS} - \theta\|^2) \\ =& \omega E(\|\delta^+_{JS} - \delta_{JS} + \delta_{JS} - X\|^2) + (1-\omega)E(\|\delta^+_{JS} - \delta_{JS} + \delta_{JS} - \theta\|^2) \\ =& \omega E[\|\delta^+_{JS} - \delta_{JS}\|^2 + \|\delta_{JS} - X\|^2 + 2\langle\delta^+_{JS} - \delta_{JS}, \delta_{JS} - X\rangle] \\ &+ (1-\omega)E[\|\delta^+_{JS} - \delta_{JS}\|^2 + \|\delta_{JS} - \theta\|^2 + 2\langle\delta^+_{JS} - \delta_{JS}, \delta_{JS} - X + X - \theta\rangle] \\ =& [\omega E(\|\delta_{JS} - X\|^2) + (1-\omega)E(\|\delta_{JS} - \theta\|^2)] + E[\|\delta^+_{JS} - \delta_{JS}\|^2] \\ &+ 2E[\langle\delta^+_{JS} - \delta_{JS}, \delta_{JS} - X\rangle + 2(1-\omega)\langle\delta^+_{JS} - \delta_{JS}, X - \theta\rangle] \\ =& R_\omega(\delta_{JS},\theta) + E[\|\delta^+_{JS} - \delta_{JS}\|^2] + 2E[\langle\delta^+_{JS} - \delta_{JS}, \delta_{JS} - X\rangle] \\ &+ 2(1-\omega)E[\langle\delta^+_{JS} - \delta_{JS}, X - \theta\rangle]. \end{aligned}$$

Now, we compute the expectations in the right side hand of the last equality.

$$\begin{aligned} E[\|\delta^+_{JS} - \delta_{JS}\|^2] &= E[\|\delta^-_{JS}\|^2] \\ &= E\left[\left\|\left(\alpha\frac{S^2}{\|X\|^2} - 1\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}X\right\|^2\right] \\ &= E\left[\left(\alpha^2\frac{S^4}{\|X\|^4} + 1 - 2\alpha\frac{S^2}{\|X\|^2}\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\|X\|^2\right] \\ &= E\left[\left(\alpha^2\frac{S^4}{\|X\|^2} + \|X\|^2 - 2\alpha S^2\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right], \end{aligned} \tag{4.1}$$

$$\begin{aligned} E[\langle\delta^+_{JS} - \delta_{JS}, \delta_{JS} - X\rangle] &= E[\langle\delta^-_{JS}, \delta_{JS} - X\rangle] \\ &= E\left[\left\langle\left(\alpha\frac{S^2}{\|X\|^2} - 1\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}X, -\alpha\frac{S^2}{\|X\|^2}X\right\rangle\right] \\ &= -E\left[\left(\alpha^2\frac{S^4}{\|X\|^2} - \alpha S^2\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right], \end{aligned} \tag{4.2}$$

and by using Lemma 2.1 of Shao and Strawdermen [21], we have

$$\begin{aligned} E[\langle\delta^+_{JS} - \delta_{JS}, X - \theta\rangle] &= E\left[\left\langle\left(\alpha\frac{S^2}{\|X\|^2} - 1\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}X, X - \theta\right\rangle\right] \\ &= \sigma^2 E\left[\left((p-2)\alpha\frac{S^2}{\|X\|^2} - p\right)\mathbb{I}_{\alpha\frac{S^2}{\|X\|^2}\geq 1}\right]. \end{aligned} \tag{4.3}$$

According to the formulas (4.1), (4.2) and (4.3) we get the desired result. □

In the Theorem 3.1 we showed that $R_\omega(\delta^+_{JS},\theta) \leq R_\omega(\delta_{JS},\theta))$ for $p \geq 3$ and all $(\theta,\sigma) \in (\mathbb{R}^p \times \mathbb{R}^+)$, then the upper bound given in Lemma 4.1 plays the role of the upper bound of $R_\omega(\delta^+_{JS},\theta)/R_\omega(X,\theta)$. Thus for calculate the limit of risks ratio $R_\omega(\delta^+_{JS},\theta)/R_\omega(X,\theta)$, it suffices to determine a lower bound. The following proposition gives a lower bound of risks ratio $R_\omega(\delta^+_{JS},\theta)/R_\omega(X,\theta)$.

Proposition 4.2. *For all* $p \geq 3$, *we have the following lower bound of the risks ratio* $\frac{R_\omega(\delta^+_{JS},\theta)}{R_\omega(X,\theta)}$

$$\begin{aligned}\frac{R_\omega(\delta^+_{JS},\theta)}{R_\omega(X,\theta)} \geq & \frac{R_\omega(\delta_{JS},\theta)}{(1-\omega)p\sigma^2} + \frac{p+\lambda}{(1-\omega)p}\int_0^{+\infty}\mathbb{P}\left(\chi^2_n \geq \frac{u}{\alpha}\right)\chi^2_{p+4}(\lambda,du)\\ &- \frac{4}{p}\int_0^{+\infty}\mathbb{P}\left(\chi^2_n \geq \frac{u}{\alpha}\right)\chi^2_{p-2}(\lambda,du)\\ &- \frac{(p-2)n}{(1-\omega)p(n+2)}\int_0^{+\infty}\mathbb{P}\left(\chi^2_{n+4} \geq \frac{u}{\alpha}\right)\chi^2_{p-2}(\lambda,du).\end{aligned} \tag{4.4}$$

Proof. As $\frac{\|X\|^2}{\sigma^2} \sim \chi^2_p(\lambda)$ and $\frac{S^2}{\sigma^2} \sim \chi^2_n$, where $\lambda = \frac{\|\theta\|^2}{\sigma^2}$, we have

$$\begin{aligned}\sigma^2 E\left(\|X\|^2\mathbb{I}_{\frac{\alpha S^2}{\|X\|^2}\geq 1}\right) =&\sigma^2 E\left(\chi^2_p(\lambda)\mathbb{I}_{\chi^2_n \geq \frac{\chi^2_p(\lambda)}{\alpha}}\right)\\ =&\sigma^2\int_0^{+\infty}\left(\int_{\frac{u}{\alpha}}^{+\infty}\chi^2_n(0,dt)\right)u\chi^2_p(\lambda,du)\\ =&\sigma^2 p\int_0^{+\infty}\mathbb{P}\left(\chi^2_n \geq \frac{u}{\alpha}\right)\chi^2_{p+2}(\lambda,du)\\ &+\sigma^2\lambda\int_0^{+\infty}\mathbb{P}\left(\chi^2_n \geq \frac{u}{\alpha}\right)\chi^2_{p+4}(\lambda,du).\end{aligned}$$

The last equality is obtained by using the formula (6.1) of Lemma 6.1 Appendix with $h(u) = \int_{\frac{u}{\alpha}}^{+\infty}\chi^2_n(0,dt)$. As the function $\mathbb{P}\left(\chi^2_n \geq \frac{u}{\alpha}\right)$ is non increasing on u and using the formula (6.2) of Lemma 6.2, we obtain

$$\sigma^2 E\left(\|X\|^2\mathbb{I}_{\frac{\alpha S^2}{\|X\|^2}\geq 1}\right) \geq q\sigma^2(p+\lambda)\int_0^{+\infty}\mathbb{P}\left(\chi^2_n \geq \frac{u}{\alpha}\right)\chi^2_{p+4}(\lambda,du), \tag{4.5}$$

$$\begin{aligned}\sigma^2 E\left\{\left(2(p-2)\frac{\alpha S^2}{\|X\|^2} - 2p\right)\mathbb{I}_{\frac{\alpha S^2}{\|X\|^2}\geq 1}\right\} \geq & -4\sigma^2 E\left(\mathbb{I}_{\frac{\alpha S^2}{\|X\|^2}\geq 1}\right)\\ =& -4\sigma^2\int_0^{+\infty}\left(\int_{\frac{u}{\alpha}}^{+\infty}\chi^2_n(0,dt)\right)\chi^2_p(\lambda,du)\\ =& -4\sigma^2\int_0^{+\infty}\mathbb{P}\left(\chi^2_n \geq \frac{u}{\alpha}\right)\chi^2_p(\lambda,du)\\ \geq & -4\sigma^2\int_0^{+\infty}\mathbb{P}\left(\chi^2_n \geq \frac{u}{\alpha}\right)\chi^2_{p-2}(\lambda,du).\end{aligned} \tag{4.6}$$

The last inequality comes from formula (6.1). Now,

$$E\left(-\frac{\alpha^2 S^4}{\|X\|^2}\mathbb{I}_{\frac{\alpha S^2}{\|X\|^2}\geq 1}\right) = -\sigma^2\alpha^2\int_0^{+\infty}\left(\int_{\frac{u}{\alpha}}^{+\infty} t^2\chi_n^2(0,dt)\right)\frac{1}{u}\chi_p^2(\lambda,du)$$
$$\geq -\frac{\sigma^2\alpha}{n+2}\int_0^{+\infty}\left(\int_{\frac{u}{\alpha}}^{+\infty} t^2\chi_n^2(0,dt)\right)\chi_{p-2}^2(\lambda,du).$$

The last inequality comes from formula (6.1), taking $h(u)=\frac{1}{u}\int_{\frac{u}{\alpha}}^{+\infty} t^2\chi_n^2(0,dt)$. However, using formula (6.1) again, we get

$$\int_{\frac{u}{\alpha}}^{+\infty} t^2\chi_n^2(0,dt) = n\int_{\frac{u}{\alpha}}^{+\infty} t\chi_{n+2}^2(0,dt)$$
$$= n(n+2)\int_{\frac{u}{\alpha}}^{+\infty}\chi_{n+4}^2(0,dt)$$
$$= n(n+2)\mathbb{P}\left(\chi_{n+4}^2\geq\frac{u}{\alpha}\right),$$

thus, we have

$$E\left(-\frac{\alpha^2 S^4}{\|X\|^2}\mathbb{I}_{\frac{\alpha S^2}{\|X\|^2}\geq 1}\right)\geq -\sigma^2\alpha n\int_0^{+\infty}\mathbb{P}\left(\chi_{n+4}^2\geq\frac{u}{\alpha}\right)\chi_{p-2}^2(\lambda,du), \tag{4.7}$$

combining to the formulas (4.5), (4.6) and (4.7), we get the desired result. □

Theorem 4.2. *Assume the estimator* δ_{JS}^+ *given in* (3.5), *if* $\lim_{p\to\infty}\frac{\|\theta\|^2}{p\sigma^2}=c$ $(c>0)$, *then*

i) $\lim\limits_{p\to\infty}\frac{R_\omega(\delta_{JS}^+,\theta)}{R_\omega(X,\theta)}=\frac{(1-(1-\omega)\frac{n}{n+2})+c}{1+c}$;

ii) $\lim\limits_{\substack{p\to\infty\\ n\to\infty}}\frac{R_\omega(\delta_{JS}^+,\theta)}{R_\omega(X,\theta)}=\frac{\omega+c}{1+c}$.

Proof. In the one hand, from Theorem 3.1, we showed that $R_\omega(\delta_{JS}^+,\theta)\leq R_\omega(\delta_{JS},\theta)$ for $p\geq 3$ and all $(\theta,\sigma)\in(\mathbb{R}^p\times\mathbb{R}^+)$ and using Theorem 4.1, we have

$$\lim_{p\to+\infty}\frac{R_\omega(\delta_{JS}^+,\theta)}{R_\omega(X,\theta)}\leq\frac{(1-(1-\omega)\frac{n}{n+2})+c}{1+c} \tag{4.8}$$

and

$$\lim_{\substack{p\to\infty\\ n\to\infty}}\frac{R_\omega(\delta_{JS}^+,\theta)}{R_\omega(X,\theta)}\leq\frac{\omega+c}{1+c}. \tag{4.9}$$

In the other hand, when p tends to infinity and n is fixed, we have $\alpha=\frac{(1-\omega)(p-2)}{n+2}$ tending to $+\infty$. According to the Lebesque's Theorem by taking for example, the increasing sequel with p $\left(f_p(u)=\int_{\frac{u}{\alpha}}^{+\infty}\chi_n^2(0,dt)=\mathbb{P}\left(\chi_n^2\geq\frac{u}{\alpha}\right)\right)$ and the fact that

$$\lim_{p\to+\infty}\mathbb{P}\left(\chi_n^2\geq\frac{u}{\alpha}\right)=\mathbb{P}\left(\chi_n^2\geq 0\right)=1,\quad\text{for all } n\geq 1,$$

we obtain

$$\lim_{p\to+\infty}\int_0^{+\infty}\mathbb{P}\left(\chi_n^2\geq\frac{u}{\alpha}\right)\chi_{p+4}^2(\lambda,du)=1. \tag{4.10}$$

In the case where p and n tend simultaneously to infinity, we have

$$\mathbb{P}\left(\chi_n^2\geq\frac{u}{\alpha}\right)=\mathbb{P}\left(\sum_{i=1}^{n}y_i^2\geq\frac{u(n+2)}{p-2}\right)=\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}y_i^2\geq\frac{u}{p-2}+\frac{2u}{n(p-2)}\right),$$

where $y_1,y_2,\ldots,y_n$ are independent Gaussian random variables centered and reduced. Then by the strong law of large numbers, we have

$$\begin{aligned}\lim_{\substack{p\to\infty\\n\to\infty}}\mathbb{P}\left(\chi_n^2\geq\frac{u}{\alpha}\right)&=\lim_{\substack{p\to\infty\\n\to\infty}}\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}y_i^2\geq\frac{u}{p-2}+\frac{2u}{n(p-2)}\right)\\&=\lim_{\substack{p\to\infty\\n\to\infty}}\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}y_i^2\geq 0\right)\\&=\mathbb{P}(1\geq 0)=1.\end{aligned}$$

Thus,

$$\lim_{\substack{p\to\infty\\n\to\infty}}\int_0^{+\infty}\mathbb{P}\left(\chi_n^2\geq\frac{u}{\alpha}\right)\chi_{p+4}^2(\lambda,du)=\int_0^{+\infty}\chi_{p+4}^2(\lambda,du)=1. \tag{4.11}$$

Using Proposition 4.2, formulas (4.10) and (4.11) and the condition

$$\lim_{p\to\infty}\frac{\|\theta\|^2}{p\sigma^2}=\lim_{p\to\infty}\frac{\lambda}{p}=c,$$

leads to

$$\begin{aligned}\lim_{p\to+\infty}\frac{R_\omega(\delta_{JS}^+,\theta)}{R_\omega(X,\theta)}&\geq\lim_{p\to+\infty}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}+\lim_{p\to+\infty}\left[\frac{p+\lambda}{(1-\omega)p}-\frac{4}{p}-\frac{(p-2)n}{(1-\omega)p(n+2)}\right]\\&=\lim_{p\to+\infty}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}+\frac{1-\frac{n}{n+2}}{1-\omega}+\frac{c}{1-\omega}\\&\geq\lim_{p\to+\infty}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}\end{aligned}$$

and

$$\begin{aligned}\lim_{\substack{p\to\infty\\n\to\infty}}\frac{R_\omega(\delta_{JS}^+,\theta)}{R_\omega(X,\theta)}&\geq\lim_{\substack{p\to\infty\\n\to\infty}}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}+\lim_{\substack{p\to\infty\\n\to\infty}}\left[\frac{p+\lambda}{(1-\omega)p}-\frac{4}{p}-\frac{(p-2)n}{(1-\omega)p(n+2)}\right]\\&=\lim_{\substack{p\to\infty\\n\to\infty}}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}+\frac{c}{1-\omega}\\&\geq\lim_{\substack{p\to\infty\\n\to\infty}}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}.\end{aligned}$$

It follows from Theorem 4.1 that

$$\lim_{p\to+\infty}\frac{R_\omega(\delta_{JS}^+,\theta)}{R_\omega(X,\theta)}\geq\frac{(1-(1-\omega)\frac{n}{n+2})+c}{1+c} \tag{4.12}$$

and

$$\lim_{\substack{p\to\infty\\ n\to\infty}} \frac{R_\omega(\delta^+_{JS},\theta)}{R_\omega(X,\theta)} \geq \frac{\omega+c}{1+c}. \tag{4.13}$$

Combining formulas (4.8), (4.9), (4.12) and (4.13) we get the desired result. □

5. Simulation Results

First, we illustrate graphically the risks ratios of the James-Stein estimator δ_{JS} and the positive-part of James-Stein estimator δ^+_{JS} to the MLE X as a function of $\lambda = \|\theta\|^2/(2\sigma^2)$ for various values of n, p and ω. Secondly, we give the tables that show the values of risks ratios of the James-Stein estimator δ_{JS} and the positive-part of James-Stein estimator δ^+_{JS} to the MLE X according to divers values of $\lambda = \|\theta\|^2/(2\sigma^2)$ but this time we fix n and p and vary ω.

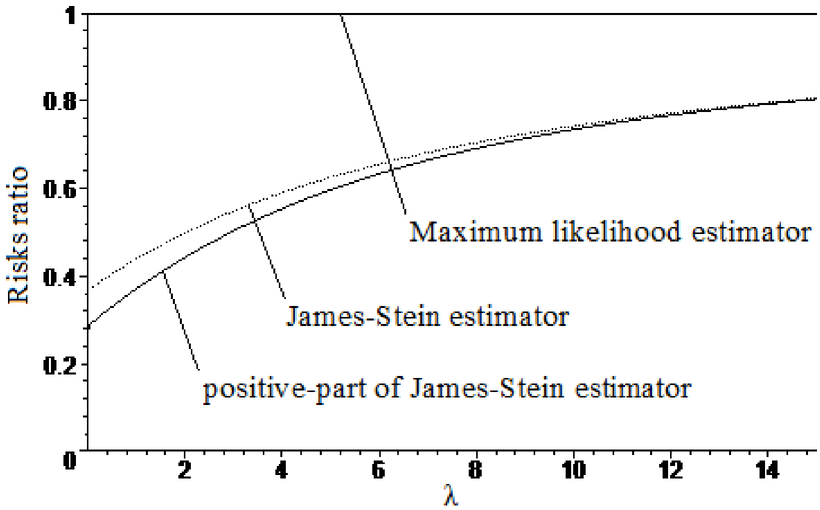

Figure 1. The graphs of risks ratios $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$ and $R_\omega(\delta^+_{JS},\theta)/R_\omega(X,\theta)$ as functions of λ for $n = 30$, $p = 8$ and $\omega = 0.1$

Figures 1–6 show that the risks ratios of the James-Stein estimator δ_{JS} and the positive-part of James-Stein estimator δ^+_{JS} to the MLE X are less than 1, thus the estimators δ_{JS} and δ^+_{JS} dominate the MLE X for large values of n and p. We also observe that the gain increases if ω is near to 0 and decreases if ω is near to 1. Tables 1-6 illustrate this note.

In Table 1 and Table 2, we give the values of ratios $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$ and $R_\omega(\delta^+_{JS},\theta)/R_\omega(X,\theta)$ for $n = 50$ and $p = 10$ and $n = 100$ and $p = 10$, respectively for divers values of $\lambda = (\|\theta\|^2)/(2\sigma^2)$ and ω.

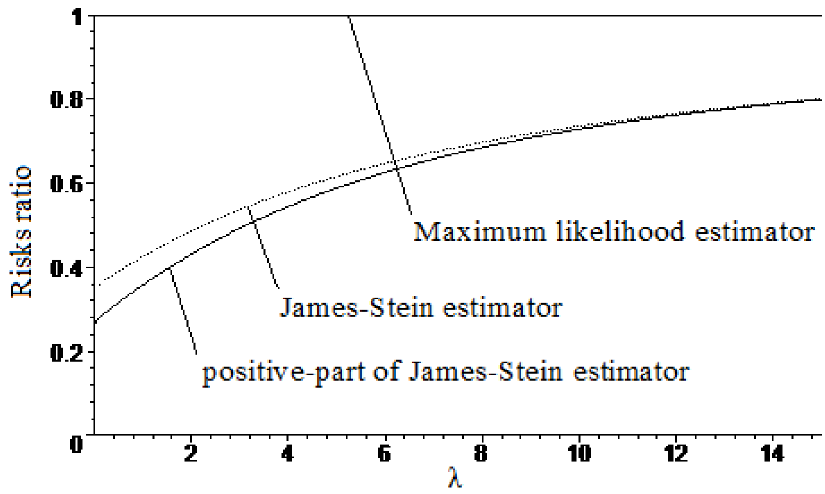

FIGURE 2. The graphs of risks ratios $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$ and $R_\omega(\delta^+_{JS},\theta)/R_\omega(X,\theta)$ as functions of λ for $n = 50,\ p = 8$ and $\omega = 0.1$

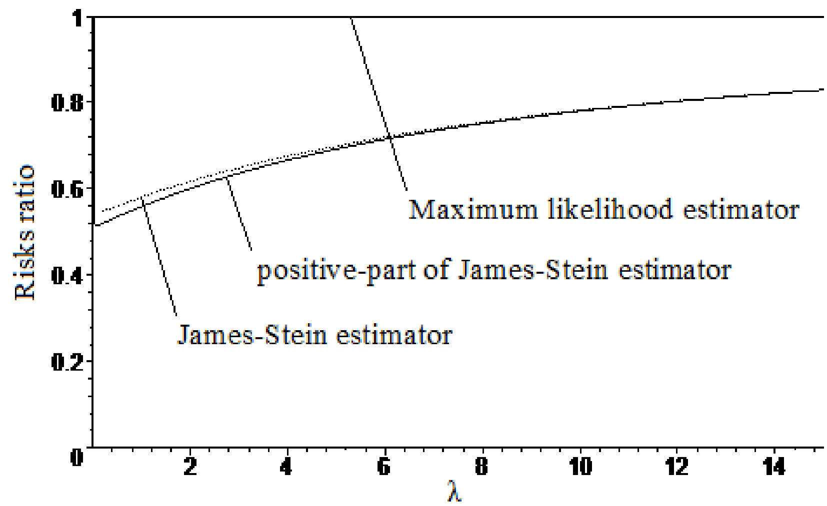

FIGURE 3. The graphs of risks ratios $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$ and $R_\omega(\delta^+_{JS},\theta)/R_\omega(X,\theta)$ as functions of λ for $n = 50,\ p = 10$ and $\omega = 0.4$

From Tables 1–2, first, for any values of ω and $\lambda = \|\theta\|^2/(2\sigma^2)$, the ratio $R_\omega(\delta^+_{JS},\theta)/R_\omega(X,\theta)$ is less than the ratio $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$, which shows that the positive-part of James-Stein estimator δ^+_{JS} dominates the James-Stein estimator δ_{JS}. Secondly, on the one hand, if ω and $\lambda = \|\theta\|^2/(2\sigma^2)$ are small, the ratios are close to 0 than 1, and therefore the gain is very important. On the other hand, as much as ω goes to 1, the gain will be small and the risks ratios are almost equal. In the case

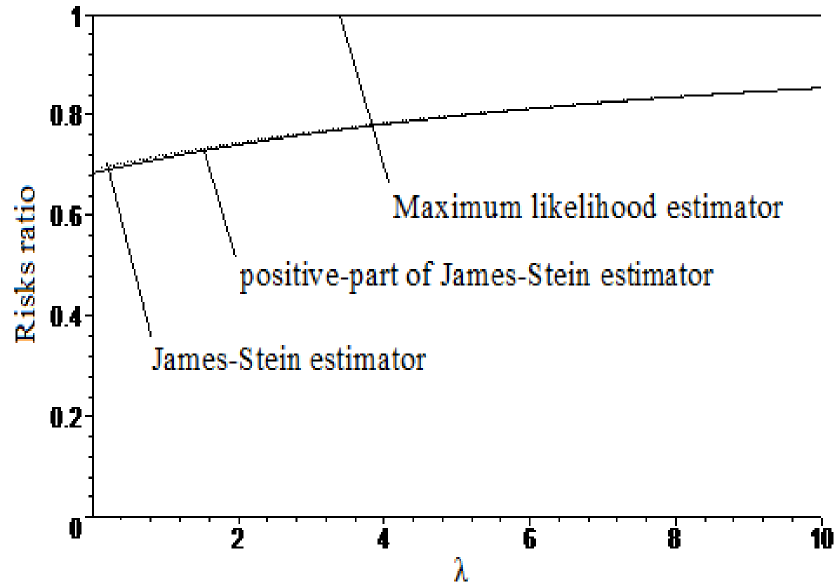

FIGURE 4. The graphs of risks ratios $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$ and $R_\omega(\delta^{+}_{JS},\theta)/R_\omega(X,\theta)$ as functions of λ for $n = 50,\ p = 10$ and $\omega = 0.6$

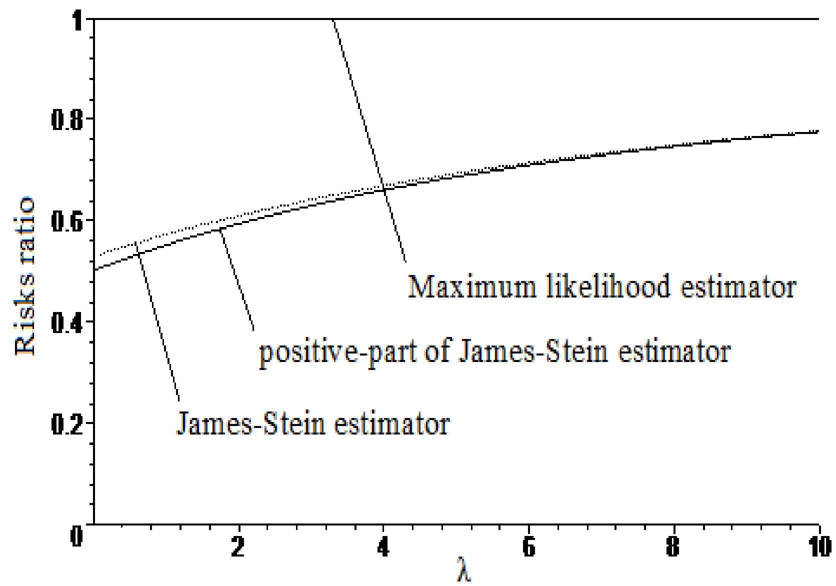

FIGURE 5. The graphs of risks ratios $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$ and $R_\omega(\delta^{+}_{JS},\theta)/R_\omega(X,\theta)$ as functions of λ for $n = 100,\ p = 10$ and $\omega = 0.4$

ω is near to 1 and λ is large, the gain is almost equal to zero and the risks ratios are the same.

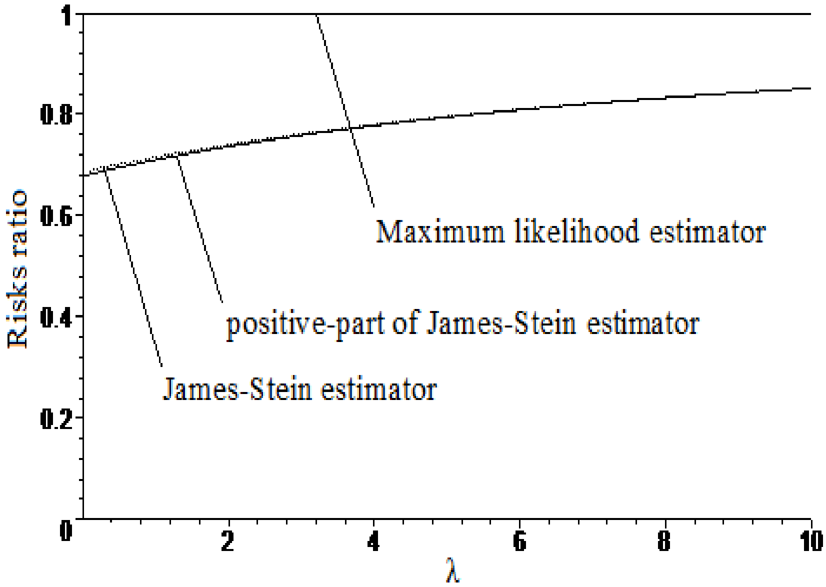

FIGURE 6. The graphs of risks ratios $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$ and $R_\omega(\delta^+_{JS},\theta)/R_\omega(X,\theta)$ as functions of λ for $n = 100,\ p = 10$ and $\omega = 0.6$

CONCLUSION

In this work, we established the minimaxity of the James-Stein estimator δ_{JS} and the positive-part of James-Stein estimator δ^+_{JS} of a multivariate normal mean distribution $X \sim N_p(\theta, \sigma^2 I_p)$ under the balanced loss function. If the limit of the ratio $\|\theta\|^2/p$ is a constant $c > 0$, the risks ratios $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$ and $R_\omega(\delta^+_{JS},\theta)/R_\omega(X,\theta)$ tend to the values less than 1, thus we ensured the stability of the minimaxity property of the James-Stein estimator δ_{JS} and the positive-part of James-Stein estimator δ^+_{JS} even if the dimension of the parameter spaces p and the sample size n tend to infinity. An extension of this work is to obtain the similar results in the case where the model has a symmetrical spherical distribution.

6. APPENDIX

Lemma 6.1 (Bock [8]). *Let $X \sim N_p(\theta, I_p)$ where $X = (X_1, \ldots, X_p)^\top$ and $\theta = (\theta_1, \ldots, \theta_p)^\top$, then for any measurable function $h : [0, +\infty[\to \mathbb{R}$*

$$E\left(h\left(\|X\|^2\right)X_i^2\right) = E\left[h\left(\chi^2_{p+2}\left(\|\theta\|^2\right)\right)\right] + \theta_i^2 E\left[h\left(\chi^2_{p+4}\left(\|\theta\|^2\right)\right)\right].$$

Moreover,

$$\begin{aligned} E\left(h\left(\|X\|^2\right)\|X\|^2\right) &= E\left[\chi^2_p\left(\|\theta\|^2\right)h\left(\chi^2_p\left(\|\theta\|^2\right)\right)\right] \\ &= pE\left[h\left(\chi^2_{p+2}\left(\|\theta\|^2\right)\right)\right] + \|\theta\|^2 E\left[h\left(\chi^2_{p+4}\left(\|\theta\|^2\right)\right)\right]. \end{aligned} \tag{6.1}$$

Lemma 6.2 ([3]). *Let f is a real function. If for $p \geq 3$, $E_{\chi^2_p(\lambda)}[f(U)]$ exists, then*

TABLE 1. The values of risks ratios $R_\omega(\delta_{JS},\theta)/R_\omega(X,\theta)$ and $R_\omega(\delta_{JS}^{+},\theta)/R_\omega(X,\theta)$ as functions of λ for $n=50$ and $p=10$

λ	risks ratio	$\omega=0.1$	$\omega=0.2$	$\omega=0.3$	$\omega=0.4$	$\omega=0.6$	$\omega=0.8$	$\omega=0.9$
0.4	δ_{JS}	0.3105	0.3871	0.4637	0.5403	0.6936	0.8468	0.9234
	δ_{JS}^{+}	0.2416	0.3671	0.4261	0.5015	0.6854	0.8462	0.9223
1	δ_{JS}	0.3715	0.4414	0.5112	0.5810	0.7207	0.8603	0.9302
	δ_{JS}^{+}	0.3124	0.3949	0.4775	0.5590	0,7148	0.8600	0.9302
2	δ_{JS}	0.4260	0.4898	0.5536	0.6173	0.7449	0.8724	0.9362
	δ_{JS}^{+}	0.3766	0.4522	0.5272	0.6007	0.7408	0.8722	0.9362
3	δ_{JS}	0,4728	0,5314	0,5900	0,6485	0,7657	0,8828	0,9414
	δ_{JS}^{+}	0,4321	0,5014	0,5696	0,6360	0,7628	0,8827	0,9414
5	δ_{JS}	0,5486	0,5987	0,6489	0,6991	0,7994	0,8997	0,9498
	δ_{JS}^{+}	0,5220	0,5802	0,6370	0,6922	0,7980	0,8996	0,9498
10	δ_{JS}	0,6719	0,7084	0,7448	0,7813	0,8542	0,9271	0,9635
	δ_{JS}^{+}	0,6640	0,7034	0,7420	0,7798	0,8540	0,9271	0,9635
15	δ_{JS}	0,7443	0,7727	0,8011	0,8296	0,8864	0,9432	0,9716
	δ_{JS}^{+}	0,7422	0,7715	0,8005	0,8293	0,8863	0,9432	0,9716
20	δ_{JS}	0,7912	0,8144	0,8376	0,8608	0,9072	0,9536	0,9768
	δ_{JS}^{+}	0,7907	0,8141	0,8375	0,8607	0,9072	0,9536	0,9768
25	δ_{JS}	0,8238	0,8434	0,8630	0,8825	0,9217	0,9608	0,9804
	δ_{JS}^{+}	0,8237	0,8433	0,8629	0,8825	0,9217	0,9608	0,9804
30	δ_{JS}	0,8477	0,8646	0,8816	0,8985	0,9323	0,9662	0,9831
	δ_{JS}^{+}	0,8477	0,8646	0,8815	0,8985	0,9323	0,9662	0,9831

a) *if f is monotone non-increasing we have*

$$E_{\chi^2_{p+2}(\lambda)}\left[f(U)\right] \le E_{\chi^2_p(\lambda)}\left[f(U)\right]; \tag{6.2}$$

b) *if f is monotone non-decreasing we have*

$$E_{\chi^2_{p+2}(\lambda)}\left[f(U)\right] \ge E_{\chi^2_p(\lambda)}\left[f(U)\right]. \tag{6.3}$$

Acknowledgements. The authors are extremely grateful to the editor and the referees for carefully reading the paper and for valuable comments and helpful remarks and suggestions which greatly improved the presentation of this paper. This research is supported by the Thematic Research Agency in Science and Technology (ATRST-Algeria).

TABLE 2. The values of risks ratios $R_\omega(\delta_{JS}, \theta)/R_\omega(X, \theta)$ and $R_\omega(\delta^+_{JS}, \theta)/R_\omega(X, \theta)$ as functions of λ for $n = 100$ and $p = 10$

λ	risks ratio	$\omega = 0.1$	$\omega = 0.2$	$\omega = 0.3$	$\omega = 0.4$	$\omega = 0.6$	$\omega = 0.8$	$\omega = 0.9$
0.4	δ_{JS}	0.2970	0.3751	0.4532	0.5313	0.6876	0.8438	0.9219
	δ^+_{JS}	0.2092	0.3207	0.4025	0.5040	0.6800	0.8433	0.9219
1	δ_{JS}	0,3592	0,4304	0,5016	0,5728	0,7152	0,8576	0,9288
	δ^+_{JS}	0,3020	0,3857	0,4694	0,5519	0,7098	0,8573	0,9288
2	δ_{JS}	0,4147	0,4798	0,5448	0,6098	0,7399	0,8699	0,9350
	δ^+_{JS}	0,3671	0,4439	0,5198	0,5942	0,7361	0,8697	0,9350
3	δ_{JS}	0,4625	0,5222	0,5819	0,6416	0,7611	0,8805	0,9403
	δ^+_{JS}	0,4235	0,4937	0,5627	0,6300	0,7585	0,8804	0,9403
5	δ_{JS}	0,5397	0,5909	0,6420	0,6932	0,7954	0,8977	0,9489
	δ^+_{JS}	0,5146	0,5735	0,6309	0,6868	0,7942	0,8977	0,9489
10	δ_{JS}	0,6655	0,7027	0,7398	0,7770	0,8513	0,9257	0,9628
	δ^+_{JS}	0,6582	0,6982	0,7373	0,7757	0,8511	0,9257	0,9628
15	δ_{JS}	0,7393	0,7683	0,7972	0,8262	0,8841	0,9421	0,9710
	δ^+_{JS}	0,7375	0,7672	0,7967	0,8260	0,8841	0,9421	0,9710
20	δ_{JS}	0,7871	0,8108	0,8344	0,8581	0,9054	0,9527	0,9763
	δ^+_{JS}	0,7867	0,8105	0,8343	0,8580	0,9054	0,9527	0,9763
25	δ_{JS}	0,8203	0,8403	0,8603	0,8802	0,9202	0,9601	0,9800
	δ^+_{JS}	0,8203	0,8403	0,8603	0,8802	0,9202	0,9601	0,9800
30	δ_{JS}	0,8447	0,8620	0,8792	0,8965	0,9310	0,9655	0,9827
	δ^+_{JS}	0,8447	0,8620	0,8792	0,8965	0,9310	0,9655	0,9827

REFERENCES

[1] A. J. Baranchik, *Multiple Regression and Estimation of the Mean of a Multivariate Normal Distribution*, Technical Report **51**(1), Stanford University, Stanford, 1964.

[2] A. Benkhaled and A. Hamdaoui, *General classes of shrinkage estimators for the multivariate normal mean with unknown variancee: Minimaxity and limit of risks ratios*, Kragujevac J. Math. **46**(2) (2020), 193–213.

[3] D. Benmansour and A. Hamdaoui, *Limit of the ratio of risks of James-Stein estimators with unknown variance*, Far East Journal of Theoretical Statistics **36**(1) (2011), 31–53.

[4] D. Benmansour and T. Mourid, *Etude d'une classe d'estimateurs avec rétrécisseur de la moyenne d'une loi gaussienne*, Ann. I.S.U.P. **51** (2007), 83–106.

[5] J. Berger, *Admissible minimax estimation of a multivariate normal mean with arbitrary quadratic loss*, Ann. Statist. **4**(1) (1976), 223–226. `https://doi.org/10.1214/aos/1176343356`

[6] J. Berger and R. Wolpert, *Estimating the mean function of a Gaussian process and the Stein effect*, J. Multivariate Anal. **13**(3) (1983), 401–424. `https://doi.org/10.1016/0047-259X(83)90018-0`

[7] P. K. Bhattacharya, *Estimating the mean of a multivariate normal population with general quadratic loss function*, The Annals of Mathematical Statistics **37**(6) (1966), 1819–1824. https://doi.org/10.1214/aoms/1177699174

[8] M. E. Bock, *Minimax estimators of the mean of a multivariate normal distribution*, Ann. Statist. **3**(1) (1975), 209–218. https://doi.org/10.1214/aos/1176343009

[9] G. Casella and J. T. Hwang, *Limit expressions for the risk of James-Stein estimators*, Canad. J. Statist. **10**(4) (1982), 305–309. https://doi.org/10.2307/3556196

[10] N. S. Farsipour and A. Asgharzadeh, *Estimation of a normal mean relative to balanced loss functions*, Statist. Papers **45**(2) (2004), 279–286. https://doi.org/10.1007/BF02777228

[11] A. Hamdaoui and A. Benkhaled and N. Mezouar, *Minimaxity and limits of risks ratios of shrinkage estimators of a multivariate normal mean in the bayesian case*, Stat. Optim. Inf. Comput. **8**(2) (2020), 507–520. https://doi.org/10.19139/soic-2310-5070-735

[12] A. Hamdaoui and D. Benmansour, *Asymptotic properties of risks ratios of shrinkage estimators*, Hacet. J. Math. Stat. **44**(5) (2015), 1181–1195. https://doi.org/10.15672/HJMS.2014377624

[13] A. Hamdaoui and N. Mezouar, *Risks ratios of shrinkage estimators for the multivariate normal mean*, Journal of Mathematics and Statistics **13**(2) (2017), 77–87. https://doi.org/10.3844/jmssp.2017.77.87

[14] A. Hamdaoui, N. Mezouar, D. Benmansour and D. Bouguenna, *Examples of shrinkage estimators of the mean, dominating the maximum likelihood estimator in larges dimension*, IOSR Journal of Mathematics **12**(3) (2016), 14–28. https://doi.org/10.9790/5728-1203041428

[15] B. E. Hansen, *The risk of James-Stein and lasso shrinkage*, Econometric Rev. **35** (2016), 1465–1470. https://doi.org/10.1080/07474938.2015.1092799

[16] W. James and C. Stein, *Estimation with quadratic loss*, in: *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume* 1*: Contributions to the Theory of Statistics*, University of California Press, Berkeley, 1961, 361–379.

[17] T. F. Li and W. H. Kio, *Generalized James-Stein estimators*, Comm. Statist. Theory Methods **11**(20) (1982), 2249–2257. https://doi.org/10.1080/03610928208828386

[18] D. V. Lindley, *Discussion on Professor Stein's paper*, J. R. Stat. Soc. Ser. B. Stat. Methodol. **24** (1962), 285–287.

[19] K. Selahattin, S. Sadullah, M. R. Özkale and H. Güler, *More on the restricted ridge regression estimation*, J. Stat. Comput. Simul. **81**(11) (2011), 1433–1448. https://doi.org/10.1080/00949655.2010.491480

[20] K. Selahattin and D. Issam, *The optimal extended balanced loss function estimators*, J. Comput. Appl. Math. **345** (2019), 86–98. https://doi.org/10.1016/j.cam.2018.06.021

[21] P. Shao and W. E. Strawderman, *Improving on the James-Stein positive-part estimator of the multivariate normal mean vector for the case of common unknown variances*, Ann. Statist. **22**(3) (1994), 1517–1539.

[22] C. Stein, *Inadmissibilty of the usual estimator for the mean of a multivariate normal distribution*, in: *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Volume* 1*: Contributions to the Theory of Statistics*, University of California Press, Berkeley, 1956, 197–206.

[23] C. Stein, *Estimation of the mean of a multivariate normal distribution*, Ann. Statist. **9**(6) (1981), 1135–1151. https://doi.org/10.1214/aos/1176345632

[24] X. Xie, S. C. Kou and L. Brown, *Optimal shrinkage estimators of mean parameters in family of distribution with quadratic variance*, Ann. Statist. **44**(2) (2016), 564–597. https://doi.org/10.1214/15-AOS1377

[1]Department of Mathematics,
University of Sciences and Technology, Mohamed Boudiaf, Oran,
Laboratory of Statistics and Random Modelisations of University Abou Bekr Belkaid (LSMA), Tlemcen
El Mnaouar, BP 1505, Bir El Djir 31000, Oran, Algeria
Email address: abdenour.hamdaoui@yahoo.fr, abdenour.hamdaoui@unv-usto.dz

[2]Department of Biology,
University of Mascara,
Laboratory of Geomatics, Ecology and Environment (LGEO2E), Mascara University
Sidi Said, Mascara, 29000 Algeria
Email address: benkhaled08@yahoo.fr

[3]Department of Mathematics,
University of Sciences and Technology, Mohamed Boudiaf, Oran,
Laboratory of Analysis and Application of Radiation (LAAR), USTO-MB
El Mnaouar, BP 1505, Bir El Djir 31000, Oran, Algeria
Email address: mekki.terbeche@gmail.com