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NEW INTEGRAL EQUATIONS FOR THE MONIC HERMITE POLYNOMIALS

KARIMA ALI KHELIL[1], RIDHA SFAXI[2], AND AMMAR BOUKHEMIS[1]

Abstract. In this article, we are study the question of existence of integral equation for the monic $\mathcal{H}$ermite polynomials H_n, where the intervening real function does not depend on the index n, well-known by the linear functional $\mathscr{W}_x$ given by its moments $H_n(x) = \langle \mathscr{W}_x, t^n \rangle$, $n \geq 0$, $|x| < \infty$. Also, we obtain some properties of the zeros of this intervening function. Furthermore, we obtain an integral representation of the Dirac mass δ_x, for every real number x.

1. Introduction

Given two sequences $\{B_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ of normalized polynomials with real coefficients, with one real variable x and where $\deg B_n = \deg Q_n = n$, for every integer $n \geq 0$. The problem of integral equation between these two polynomial sequences consists in finding a real function $u(\cdot, t)$ defined in $I \times \mathbb{R}$, where $I \subset \mathbb{R} =]-\infty, +\infty[$, and satisfying the condition:

$$\int_{-\infty}^{\infty} u(x,t)t^n \, \mathrm{d}t < \infty, \quad n \geq 0, \, x \in I,$$

such that

$$B_n(x) = \int_{-\infty}^{\infty} u(x,t) Q_n(t) \, \mathrm{d}t, \quad n \geq 0, \, x \in I.$$

When $Q_n(x) = x^n$, for all integer $n \geq 0$, i.e.,

$$B_n(x) = \int_{-\infty}^{\infty} u(x,t)t^n \, \mathrm{d}t, \quad n \geq 0, \, x \in I,$$

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we recognize the usual integral representation of the polynomial sequence $\{B_n\}_{n\geq 0}$, called here by *the canonical-integral representation of* $\{B_n\}_{n\geq 0}$. When $Q_n(x) = B_n(x)$, for all integer $n \geq 0$, i.e.,

$$B_n(x) = \int_{-\infty}^{\infty} u(x,t) B_n(t)\,\mathrm{d}t, \quad n \geq 0,\ x \in I,$$

it is appropriate to say that it is an *auto-integral representation* of $\{B_n\}_{n\geq 0}$.

In fact, this kind of integral equation is of great relevance in the theory of orthogonal polynomials as well as the moment theory and their applications, [8, 9, 3, 15]. For this reason-in the past as nowadays has attracted the attention of many authors; see, for instance, [5, 6, 7, 12, 4, 1, 10, 11]. Based on the principle that *the terms of any sequence of complex numbers are the moments of a unique linear functional on polynomials*, the study of such linear functionals accurate some hypergeometric properties of such sequences, [2, 13, 14].

In this work, we are interested by the normalized Hermite polynomial sequence $\{H_n\}_{n\geq 0}$. Recall that $\{H_n\}_{n\geq 0}$ is orthogonal with respect to a linear functional on polynomials, namely $\mathscr{H}$ and well-known by its integral representation on the real line [10]

$$\langle \mathscr{H}, p\rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} p\,(t)\, e^{-t^2}\,\mathrm{d}t, \quad p \in \mathbb{P},$$

where $\mathbb{P}$ is the vector space of polynomials in one variable with real coefficients and $\mathbb{P}'$ its algebraic dual space. Notice that $\langle u, p\rangle$ is the action of a linear functional $u \in \mathbb{P}'$ on $p \in \mathbb{P}$ and by $(u)_n := \langle u, t^n\rangle$, $n \geq 0$, the moments of u with respect to the canonical sequence $\{t^n\}_{n\geq 0}$. For any u in $\mathbb{P}'$, any q in $\mathbb{P}$ and any complex numbers a, b, c with $a \neq 0$, recall that $Du = u'$, qu, $h_a u$ and $\tau_b u$, be respectively, the derivative, the left multiplication, the homothetic and the translation of the linear functionals defined by duality [9]:

$$\begin{aligned}
\langle u', f\rangle &:= -\langle u, f'\rangle\,, \\
\langle qu, f\rangle &:= \langle u, qf\rangle\,, \\
\langle h_a u, f\rangle &:= \langle u, h_a f\rangle = \langle u, f\,(ax)\rangle\,, \\
\langle \tau_{-b} u, f\rangle &:= \langle u, \tau_b f\rangle = \langle u, f\,(x-b)\rangle\,, \quad f \in \mathbb{P}.
\end{aligned}$$

The linear functional $\mathscr{H}$ is normalized, i.e., $(\mathscr{H})_0 = 1$. It satisfies the following Pearson equation [10]:

$$\mathscr{H}' + 2x\mathscr{H} = 0_{\mathbb{P}'}.$$

The moments of $\mathscr{H}$ are given by

$$(\mathscr{H})_n = \frac{n!}{2^{n+1}\Gamma(\frac{n}{2}+1)}\Big(1 + (-1)^n\Big), \quad n \geq 0.$$

This leads to the following integral representation of the moments of $\mathscr{H}$

$$\frac{n!}{2^{n+1}\Gamma(\frac{n}{2}+1)}\Big(1 + (-1)^n\Big) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^n e^{-t^2}\,\mathrm{d}t, \quad n \geq 0.$$

The normalized Hermite polynomial H_n can be represented in terms of a definite integral containing the real variable x as parameter [8]

$$H_n(x) = \frac{e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} (-it)^n e^{-t^2+2itx}\, \mathrm{d}t, \quad n \geq 0,\ |x| < \infty.$$

Equivalently,

$$H_n(x) = \int_0^{\infty} h_n(t,x)\ t^n\, \mathrm{d}t, \quad n \geq 0,\ |x| < \infty,$$

where the intervening real function $h_n(t,\cdot)$ depends on the integer n, and given by

$$h_n(t,x) = \frac{2}{\sqrt{\pi}}\ e^{x^2-t^2} \cos\left(2tx + n\frac{\pi}{2}\right).$$

The polynomial H_n satisfies the following integral equation [8]

$$H_n(x) = \frac{(-i)^n}{\sqrt{2\pi}} e^{\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}+itx} H_n(t)\, \mathrm{d}t, \quad n \geq 0,\ |x| < \infty.$$

Equivalently,

$$H_n(x) = \int_0^{\infty} r_n(t,x)\ H_n(t)\, \mathrm{d}t, \quad n \geq 0,\ |x| < \infty,$$

where the real function $r_n(t,\cdot)$ depends on the integer n, and given by

$$r_n(t,x) = \frac{1}{\sqrt{2}} h_n\left(\frac{t}{\sqrt{2}}, \frac{x}{\sqrt{2}}\right).$$

The main purpose of this work is to give two new integral equations for the polynomial sequence $\{H_n\}_{n\geq 0}$, where the intervening real functions do not depend on the integer n. In summary, we are going to establish the following.

– *The canonical-integral representation*:

$$H_n(x) = \int_{-\infty}^{\infty} U(t-x) t^n\, \mathrm{d}t, \quad n \geq 0,\ |x| < \infty,$$

where

$$U(t) = S^{-1} e^{t^2} \int_{|t|}^{\infty} e^{-y^2} e^{-y^{\frac{1}{4}}} \sin y^{\frac{1}{4}}\, \mathrm{d}y,$$
$$S = \int_{-\infty}^{\infty} e^{\xi^2} \int_{|\xi|}^{\infty} e^{-y^2} e^{-y^{\frac{1}{4}}} \sin y^{\frac{1}{4}}\, \mathrm{d}y\, \mathrm{d}\xi > 0.$$

– *The auto-integral representation*:

$$H_n(x) = \int_{-\infty}^{\infty} V(t-x) H_n(t)\, \mathrm{d}t, \quad n \geq 0,\ |x| < \infty,$$

where

$$V(t) = \begin{cases} \dfrac{e^{-t^{\frac{1}{4}}} \sin(t^{\frac{1}{4}})}{\pi t}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

2. New Canonical-Integral Representation of $\{H_n\}_{n\geq 0}$

First, let us recall some properties of $\{H_n\}_{n\geq 0}$, [8, 10].

-*The Taylor expansion*:

$$H_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k n!}{2^{2k} k!(n-2k)!} x^{n-2k}, \quad n \geq 0.$$

-*The symmetry property*:

$$H_n(-x) = (-1)^n H_n(x), \quad n \geq 0. \tag{2.1}$$

-*The Appel property*:

$$H'_n(x) = nH_{n-1}(x), \quad n \geq 0,\ H_{-1}(x) = 0.$$

-*The three-terms-recurrence relation*:

$$\begin{cases} H_{-1}(x) = 0, \quad H_0(x) = 1, \\ H_{n+1}(x) = xH_n(x) - \dfrac{n}{2} H_{n-1}(x), \quad n \geq 0. \end{cases} \tag{2.2}$$

Next, let $\mathscr{W}_x$ be the linear functional on polynomials and given by its moments

$$H_n(x) = \langle \mathscr{W}_x, t^n \rangle, \quad n \geq 0,\ |x| < \infty. \tag{2.3}$$

From (2.1) and (2.3), we show that

$$\mathscr{W}_{-x} = h_{-1}(\mathscr{W}_x), \quad |x| < \infty.$$

From (2.2) and (2.3), the linear functional $\mathscr{W}_x$ satisfies

$$(\mathscr{W}_x)_0 = 1, \quad \mathscr{W}'_x - 2(t-x)\mathscr{W}_x = 0, \quad |x| < \infty. \tag{2.4}$$

Lemma 2.1. *For any real number x, the following properties hold:*

$$\mathscr{W}_x = \tau_x \mathscr{W}_0, \tag{2.5}$$

$$H_n(x) = \langle \mathscr{W}_0, (t+x)^n \rangle, \quad n \geq 0, \tag{2.6}$$

where $\mathscr{W}_0$ is symmetric (i.e., $h_{-1}(\mathscr{W}_0) = \mathscr{W}_0$), normalized (i.e., $(\mathscr{W}_0)_0 = 1$) and satisfying the Pearson equation $\mathscr{W}'_0 - 2t\mathscr{W}_0 = 0$.

Proof. Let x be a fixed real number. We have $(\tau_{-x}\mathscr{W}_x)_0 = (\mathscr{W}_x)_0 = H_0(x) = 1$. If we take (2.4) into account, we can write

$$\begin{aligned} \langle (\tau_{-x}\mathscr{W}_x)' - 2t(\tau_{-x}\mathscr{W}_x), p(t) \rangle &= - \langle \mathscr{W}_x, p'(t-x) \rangle - 2 \langle \mathscr{W}_x, (t-x)p(t-x) \rangle \\ &= \langle \mathscr{W}'_x, p(t-x) \rangle - 2 \langle (t-x)\mathscr{W}_x, p(t-x) \rangle \\ &= \langle \mathscr{W}'_x - 2(t-x)\mathscr{W}_x, p(t-x) \rangle \\ &= 0, \quad p \in \mathbb{P}. \end{aligned}$$

So, the normalized linear functional $\tau_{-x}\mathscr{W}_x$ satisfies: $(\tau_{-x}\mathscr{W}_x)' - 2t(\tau_{-x}\mathscr{W}_x) = 0$. The fact that $\mathscr{W}_0$ is the unique normalized linear functional satisfying the Pearson equation $\mathscr{W}'_0 - 2t\mathscr{W}_0 = 0$, yields $\tau_{-x}\mathscr{W}_x = \mathscr{W}_0$ and then $\mathscr{W}_x = \tau_x\mathscr{W}_0$.

Finally, (2.6) follows in a straightforward way from (2.3) and (2.5). $\square$

2.1. **An integral representation of $\mathscr{W}_x$.** At first, we start by giving an integral representation of $\mathscr{W}_0$ as follows

$$\langle \mathscr{W}_0, p\rangle = \int_{-\infty}^{\infty} U(t)p\,(t)\;\mathrm{d}t, \quad p \in \mathbb{P}, \tag{2.7}$$

where we assume that the function U is absolutely continuous on the real line and decaying as fast as its derivative U'.

By an easy integration by parts, we obtain

$$\begin{aligned} 0 = \langle \mathscr{W}_0' - 2t\mathscr{W}_0, p\rangle &= -\langle \mathscr{W}_0, p'(t) + 2tp(t)\rangle = -\int_{-\infty}^{\infty} U(t)\Big(p'(t) + 2tp(t)\Big)\,\mathrm{d}t \\ &= -\left[U(t)p(t)\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \Big(U'(t) - 2tU(t)\Big)p(t)\,\mathrm{d}t, \quad p \in \mathbb{P}. \end{aligned}$$

The following condition:

$$\lim_{t\to\pm\infty} U(t)p(t) = 0, \quad p \in \mathbb{P}, \tag{2.8}$$

leads to

$$\int_{-\infty}^{\infty} \Big(U'(t) - 2tU(t)\Big)p(t)\,\mathrm{d}t = 0, \quad p \in \mathbb{P}. \tag{2.9}$$

This implies

$$U'(t) - 2tU(t) = \lambda f(t), \tag{2.10}$$

where $\lambda \neq 0$ is arbitrary and the function f is locally integrable, with rapid decay, and representing the null function, i.e.,

$$\int_{-\infty}^{\infty} t^n f(t)\,\mathrm{d}t = 0, \quad n \geq 0.$$

Conversely, if U is a solution of (2.10) verifying the hypothesis above and the condition:

$$\int_{-\infty}^{\infty} U(t)\,\mathrm{d}t \neq 0, \tag{2.11}$$

then (2.8) and (2.9) are fulfilled and (2.7) defines a linear functional $\mathscr{W}_0$, which is a solution of the Pearson equation $\mathscr{W}_0' - 2t\mathscr{W}_0 = 0$. Putting

$$f\,(t) = -\operatorname{sgn}(t)s(|t|), \quad t \in]-\infty, +\infty[,$$

where s is the Stieltjes function [10, 1, 11],

$$s(t) = \begin{cases} 0, & t \leq 0, \\ e^{-t^{\frac{1}{4}}}\sin t^{\frac{1}{4}}, & t > 0. \end{cases}$$

In view of the fact that $\int_0^{\infty} t^n s(t)\,\mathrm{d}t = 0$, $n \geq 0$, we get

$$\begin{aligned}\int_{-\infty}^{\infty} t^n f(t)\,\mathrm{d}t &= -\int_{-\infty}^{\infty} t^n \operatorname{sgn}(t) s(|t|)\,\mathrm{d}t = \int_{-\infty}^{0} t^n s(-t)\,\mathrm{d}t + \int_0^{\infty} t^n s(t)\,\mathrm{d}t \\ &= (-1)^n \int_0^{\infty} t^n s(t)\,\mathrm{d}t + \int_0^{\infty} t^n s(t)\,\mathrm{d}t = \Big(1+(-1)^n\Big)\int_0^{\infty} t^n s(t)\,\mathrm{d}t \\ &= 0, \quad n \geq 0.\end{aligned}$$

Let U be the function defined on the real line and given by,

$$U(t) = \lambda e^{t^2} \int_{|t|}^{\infty} e^{-y^2} s(y)\,\mathrm{d}y, \quad t \in]-\infty, +\infty[. \tag{2.12}$$

An easy computation shows that $U'(t) = 2tU(t) - \lambda s\,(t)$ for every $t \geq 0$, $U'(t) = 2tU(t) + \lambda s\,(-t)$ for every $t < 0$.

Equivalently,

$$U'(t) - 2tU(t) = \lambda f\,(t)\,, \quad t \in]-\infty, +\infty[.$$

For $|t|$ large, we have

$$|U(t)| \leq |\lambda|\, e^{t^2} \int_{|t|}^{\infty} e^{-y^2} e^{-y^{\frac{1}{4}}}\,\mathrm{d}y \leq |\lambda|\, e^{-\frac{1}{2}|t|^{\frac{1}{4}}} e^{t^2} \int_{|t|}^{\infty} e^{-y^2}\,\mathrm{d}y \leq o\left(e^{-\frac{1}{2}|t|^{\frac{1}{4}}}\right), \quad |t| \to \infty,$$

by the fact that,

$$\begin{aligned}\lim_{|t|\to\infty} e^{t^2} \int_{|t|}^{\infty} e^{-y^2}\,\mathrm{d}y &= \lim_{x\to\infty} e^{x^2} \int_x^{\infty} e^{-y^2}\,\mathrm{d}y = \lim_{x\to\infty} \frac{\int_x^{\infty} e^{-y^2}\,\mathrm{d}y}{e^{-x^2}} = \lim_{x\to\infty} \frac{e^{-x^2}}{2xe^{-x^2}} = \lim_{x\to\infty} \frac{1}{2x} \\ &= 0.\end{aligned}$$

Hence, the condition (2.8) holds. Clearly, $U \in L^1]-\infty, +\infty[$. Condition (2.11) can be written as follows:

$$\int_{-\infty}^{\infty} U(t)\,\mathrm{d}t = \lambda S \neq 0,$$

where after reverse the order of integration, we get

$$\begin{aligned}S &= 2\int_0^{\infty} U(t)\,\mathrm{d}t = 2\int_0^{\infty} e^{t^2} \int_t^{\infty} e^{-y^2} s(y)\,\mathrm{d}y\,\mathrm{d}t \\ &= 2\int_0^{\infty} e^{-y^2} \left(\int_0^{y} e^{t^2}\,\mathrm{d}t\right) e^{-y^{\frac{1}{4}}} \sin y^{\frac{1}{4}}\,\mathrm{d}y,\end{aligned}$$

and by making the change of the variable $x = y^{\frac{1}{4}}$, it follows that

$$S = 8\int_0^{\infty} y^3 e^{-y} F(y^4) \sin y\,\mathrm{d}y,$$

where $F(z) = e^{-z^2} \int_0^{z} e^{t^2}\,\mathrm{d}t$, $z \in \mathbb{C}$, is the Dawson function (called also the Dawson integral), [8]. The Dawson function is an entire function for all $z \in \mathbb{C}$ and remains bounded for all real number z. Recall that the Dawson function satisfies [8]

$$F(0) = 0, \quad F'(z) = -2zF(z) + 1, \quad z \in \mathbb{C}, \tag{2.13}$$

$$F(z) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k z^{2k+1}}{1 \cdot 3 \cdots (2k+1)}, \quad |z| < \infty,$$

$$F(z) \simeq \frac{1}{2z}, \quad |z| \to \infty,$$

$$F(-z) = -F(z), \quad z \in \mathbb{C},$$

$$0 \le F(y) \le F_{\max} = 0,541\ldots, \quad y \ge 0, \tag{2.14}$$

where $F_{\max} = F(x_{\max})$, with $x_{\max} = 0,942\ldots$ Notice that $x_{\max}$ is the only critical point of F on the interval $[0, +\infty[$. The following result contains simple but fundamental properties which will be useful in the sequel.

Lemma 2.2. *The Dawson function satisfies:*

$$F(y) < \frac{1}{2y} \text{ if and only if } 0 < y < x_{\max},$$

$$F(y) > \frac{1}{2y} \text{ if and only if } y > x_{\max},$$

$$F(y) = \frac{1}{2y} \text{ if and only if } y = x_{\max}.$$

Proof. The proof is an immediate consequence of (2.13) and (2.14). ☐

We can write

$$S = \int_0^\infty G(y) \sin y \, \mathrm{d}y, \tag{2.15}$$

where

$$G(y) = 8y^3 e^{-y} F(y^4), \quad y \ge 0. \tag{2.16}$$

From (2.16) and (2.14), we obtain

$$0 \le G(y) \le 8F_{\max} y^3 e^{-y}, \quad y \ge 0.$$

Directly, $G(0) = 0$ and $\lim_{y\to\infty} G(y) = 0$, which implies that G has a maximum for $y = \overline{y} > 0$, satisfying $G'(\overline{y}) = 0$, i.e.,

$$F(\overline{y}^4) = \frac{4\overline{y}^4}{8\overline{y}^8 + \overline{y} - 3}.$$

Notice that the function G is decreasing on the interval $[\overline{y}, +\infty[$.

Lemma 2.3. *We have* $\overline{y} \le 3$.

Proof. If we suppose that $\overline{y} > 3$, then $F(\overline{y}^4) < \frac{1}{2\overline{y}^4}$. By Lemma 2.2, this yields $\overline{y}^4 < x_{\max} = 0,942...$, i.e., $\overline{y} < (0,942\ldots)^{\frac{1}{4}} < 3$. This is a contradiction. ☐

Furthermore, the following technical lemma will be needed.

Lemma 2.4 ([1])**.** *Consider the following integral: $S=\int_0^\infty G(x)\sin x\,\mathrm{d}x$, where the function $G:[0,+\infty[\to[0,+\infty[$ is continuous on $[0,+\infty[$, decreasing on $[2\pi,+\infty[$. Suppose that $\int_0^{2\pi} G(y)\sin y\,\mathrm{d}y>0$, then $S>0$.*

The function G given by (2.16) satisfies the condition of the previous lemma. Indeed, G is a nonnegative function on $[0,+\infty[$ and decreasing on $[2\pi,+\infty[$. In order to show that S, given by (2.15), is positive, it suffices to prove that $\displaystyle\int_0^{2\pi} G(y)\sin y\,\mathrm{d}y>0$. Equivalently,

$$\int_0^{\pi} G(y)\sin y\,\mathrm{d}y > -\int_\pi^{2\pi} G(y)\sin y\,\mathrm{d}y.$$

In view of Lemma 2.2, the fact that $G\geq 0$, $\sin y\geq 0$, for all $y\in[0,\pi]$, $x_{\max}^{\frac{1}{4}}<\dfrac{\pi}{2}$ and $\sin y\geq \dfrac{2}{\pi}y$ for all $y\in\left[0,\dfrac{\pi}{2}\right]$, we obtain

$$\begin{aligned}\int_0^{\pi} G(y)\sin y\,\mathrm{d}y &\geq \int_{x_{\max}^{\frac{1}{4}}}^{\pi} y^3e^{-y}\sin y\ F(y^4)\,\mathrm{d}y \geq \int_{x_{\max}^{\frac{1}{4}}}^{\pi} \frac{y^3e^{-y}\sin y}{2y^4}\,\mathrm{d}y\\ &\geq \frac{1}{2}\int_{x_{\max}^{\frac{1}{4}}}^{\frac{\pi}{2}} e^{-y}\frac{\sin y}{y}\,\mathrm{d}y \geq \frac{1}{2}\frac{2}{\pi}\int_{x_{\max}^{\frac{1}{4}}}^{\frac{\pi}{2}} e^{-y}\,\mathrm{d}y\\ &\geq \frac{1}{\pi}\left(e^{-x_{\max}^{\frac{1}{4}}}-e^{-\frac{\pi}{2}}\right).\end{aligned}$$

Then, we have

$$\int_0^{\pi} G(y)\sin y\,\mathrm{d}y \geq \frac{1}{\pi}\left(e^{-x_{\max}^{\frac{1}{4}}}-e^{-\frac{\pi}{2}}\right)\simeq 0,0263. \tag{2.17}$$

On the other hand, we have

$$-\int_\pi^{2\pi} G(y)\sin y\,\mathrm{d}y = -\int_\pi^{2\pi} y^3e^{-y}\sin yF(y^4)\,\mathrm{d}y \leq -F(\pi^4)\int_\pi^{2\pi} y^3e^{-y}\sin y\,\mathrm{d}y.$$

By integration by parts and an easy computation we find

$$-\int_\pi^{2\pi} y^3e^{-y}\sin y\,\mathrm{d}y = \frac{1}{2}e^{-2\pi}\pi\Big(6+12\pi+8\pi^2+e^{\pi}(3+3\pi+\pi^2)\Big)\simeq 1,8731$$

and

$$F(\pi^4)=e^{-\pi^8}\int_0^{\pi^4} e^{t^2}\,\mathrm{d}t \leq e^{-\pi^8}\int_0^{\pi^4} e^{\pi^4 t}\,\mathrm{d}t = \frac{1-e^{-\pi^8}}{\pi^4}\simeq 0,010266,$$

then

$$-\int_\pi^{2\pi} G(y)\sin y\,\mathrm{d}y \leq 1,8731\cdot 0,010266\simeq 0,01922. \tag{2.18}$$

From (2.17) and (2.18), we deduce that

$$\int_0^{\pi} G(y)\sin y\,\mathrm{d}y > -\int_\pi^{2\pi} G(y)\sin y\,\mathrm{d}y.$$

Proposition 2.1. *The normalized Hermite polynomial H_n has the following integral representations:*

$$H_n(x) = \int_{-\infty}^{\infty} U(t-x)t^n \, \mathrm{d}t, \quad n \geq 0, \ |x| < \infty, \tag{2.19}$$

where

$$U(t) = S^{-1} e^{t^2} \int_{|t|}^{\infty} e^{-y^2} e^{-y^{\frac{1}{4}}} \sin y^{\frac{1}{4}} \, \mathrm{d}y,$$
$$S = \int_{-\infty}^{\infty} e^{\xi^2} \int_{|\xi|}^{\infty} e^{-y^2} e^{-y^{\frac{1}{4}}} \sin y^{\frac{1}{4}} \, \mathrm{d}y \, \mathrm{d}\xi > 0.$$

Proof. It is a straightforward consequence of Lemma 2.1 and 2.4, and (2.12). ☐

2.2. **On the zeros of the function U.** By the change of the variable $y = x^4$, the function U given by (2.19), can by written as

$$U(t) = 4S^{-1} e^{t^2} V(|t|^{\frac{1}{4}}), \quad |t| < \infty, \tag{2.20}$$

where

$$V(t) = \int_t^{\infty} x^3 e^{-x^8 - x} \sin x \, \mathrm{d}x = \int_0^{\infty} (x+t)^3 e^{-(x+t)^8 - x - t} \sin(x+t) \, \mathrm{d}x, \quad t \geq 0.$$

Clearly, the function U is even and their zeros are exactly those of the function $t \mapsto V(|t|^{\frac{1}{4}})$. Observe that we have

$$V(k\pi) = (-1)^k I_k, \quad k \geq 0,$$

where

$$I_k = \int_0^{\infty} G_k(x) \sin x \, \mathrm{d}x$$

and

$$G_k(x) = G_0(x + k\pi) = (x + k\pi)^3 e^{-(x+k\pi)^8 - (x+k\pi)}.$$

Lemma 2.5. *For every integer $k \geq 0$, we have $I_k > 0$.*

Proof. Let $h(x) = -8x^8 - x + 3$ for all $x \geq 0$. So, $h'(x) = -64x^7 - 1 < 0$ for all $x \geq 0$ and h is decreasing on $[0, +\infty[$. The function h is a bijection from $[0, +\infty[$ to $]-\infty, 3]$. Directly, there exists a unique solution $\theta \in [0, +\infty[$ solution of the equation: $h(x) = 0$, where $x \geq 0$. By the intermediate value theorem, we can see that $\frac{1}{2} < \theta < 1$, since $h(\frac{1}{2}) = \frac{9}{2} > 0$ and $h(1) = -6 < 0$. So, $h(x) < 0$, for all $x \in]\theta, +\infty[$, and $h(x) > 0$ for all $x \in [0, \theta[$. It is clear that $G_0'(x) = x^2 e^{-x^8 - x} h(x)$ for all $x \geq 0$. Thus, G_0 is decreasing on $[\theta, +\infty[$. The fact that $\theta < 1$ allows us to say that:

- the function G_0 is decreasing on the interval $[\pi, +\infty[$;
- the function G_k is decreasing on the interval $[0, +\infty[$ for every $k \geq 1$.

For every fixed integer $k \geq 1$, we have

$$I_k = \lim_{n \to \infty} \int_0^{2n\pi} G_k(x) \sin x \, \mathrm{d}x.$$

Clearly,

$$\int_0^{2n\pi} G_k(x)\sin x\,\mathrm{d}x = \sum_{l=0}^{n-1}\int_0^{\pi}\Big(G_k\big(x+2l\pi\big)-G_k\big(x+(2l+1)\pi\big)\Big)\sin x\,\mathrm{d}x,$$

for every integer $n \geq 1$. Since $\sin x > 0$ on $]0,\pi[$, and all the functions G_k, $k \geq 1$, are decreasing on $[0,+\infty[$, we have

$$\int_0^{\pi}\Big(G_k\big(x+2l\pi\big)-G_k\big(x+(2l+1)\pi\big)\Big)\sin x\,\mathrm{d}x > 0, \quad l \geq 0.$$

Accordingly, it follows that

$$I_k \geq \int_0^{\pi}\Big(G_k(x)-G_k(x+\pi)\Big)\sin x\,\mathrm{d}x > 0, \quad k \geq 1.$$

For $k = 0$, let's note first that G_0 is nonnegative and continuous on $[0,+\infty[$ and decreasing on $[2\pi,+\infty[$. By Lemma 2.4, in order to show that $I_0 > 0$, it suffices to show that $\int_0^{2\pi} G_0(x)\sin x\,\mathrm{d}x > 0$. Equivalently,

$$\int_0^{\pi} G_0(x)\sin x\,\mathrm{d}x > -\int_{\pi}^{2\pi} G_0(x)\sin x\,\mathrm{d}x. \tag{2.21}$$

On the one hand, we have

$$\int_0^{\pi} G_0(x)\sin x\,\mathrm{d}x = \int_0^{\theta} G_0(x)\sin x\,\mathrm{d}x + \int_{\theta}^{\pi} G_0(x)\sin x\,\mathrm{d}x. \tag{2.22}$$

By the fact that $G_0(x)\sin x \geq 0$ for every $x \in [0,\pi]$, the function G_0 is decreasing on the interval $[\theta,\pi]$, we can write

$$\int_{\theta}^{\pi} G_0(x)\sin x\,\mathrm{d}x \geq G_0(\pi)\int_{\theta}^{\pi}\sin x\,\mathrm{d}x = G_0(\pi)\big(1+\cos\theta\big),$$

but, $\theta \in]0,\frac{\pi}{2}[$, then

$$\int_{\theta}^{\pi} G_0(x)\sin x\,\mathrm{d}x \geq G_0(\pi) = \pi^3 e^{-\pi^8-\pi}.$$

Since $\theta \in]\frac{\pi}{2},\pi[\subset]0,\pi[$, we get

$$\int_0^{\theta} G_0(x)\sin x\,\mathrm{d}x \geq e^{-\theta^8}\int_0^{\theta} x^3e^{-x}\sin x\,\mathrm{d}x \geq e^{-1}\int_0^{\frac{\pi}{2}} x^3e^{-x}\sin x\,\mathrm{d}x,$$

by an easy computation, we obtain

$$\int_0^{\frac{\pi}{2}} x^3e^{-x}\sin x\,\mathrm{d}x = \frac{35\sin(\frac{\pi}{2})-19\cos(\frac{\pi}{2})}{16\sqrt{e}},$$

and hence,

$$\int_0^{\theta} G_0(x)\sin x\,\mathrm{d}x \geq \frac{35\sin\frac{\pi}{2}-19\cos\frac{\pi}{2}}{16e\sqrt{e}} = \vartheta, \tag{2.23}$$

where $\vartheta \approx 0.0014752$.

From (2.22) and (2.23), we get

$$\int_0^{\pi} G_0(x)\sin x\,\mathrm{d}x \geq \eta_1,$$

where $\eta_1 = \vartheta + \pi^3 e^{-\pi^8-\pi}$.

On the other hand, since $\sin x \leq 0$, for all $x \in [\pi, 2\pi]$, we obtain

$$-\int_{\pi}^{2\pi} G_0(x)\sin x \,\mathrm{d}x = -\int_{\pi}^{2\pi} x^3 e^{-x^8-x}\sin x \,\mathrm{d}x \leq -e^{-\pi^8}\int_{\pi}^{2\pi} x^3 e^{-x}\sin x \,\mathrm{d}x,$$

by an easy computation, we get

$$-\int_{\pi}^{2\pi} x^3 e^{-x}\sin x \,\mathrm{d}x = \frac{\pi}{2}e^{-2\pi}\Big(6 + 12\pi + 8\pi^2 + e^{\pi}(3 + 3\pi + \pi^2)\Big) \approx 1.8731,$$

and hence,

$$-\int_{\pi}^{2\pi} G_0(x)\sin x \,\mathrm{d}x \leq \eta_2,$$

where $\eta_2 = \beta e^{-\pi^8}$ and $\beta \approx 1.8731$.

Since $\eta_1 > \eta_2$, the condition (2.21) holds. Thus, $I_0 > 0$. $\square$

Proposition 2.2. *The function U, given by* (2.20)*, has the following properties.*

i) *The function U is even and all its zeros are placed symmetrically with respect to the origin.*
ii) *For every integer $k \geq 0$, $\operatorname{sgn} U((k\pi)^4) = (-1)^k$.*
iii) *For every integer $k \geq 0$, there exists a unique solution $\xi_k \in \big](k\pi)^4, ((k+1)\pi)^4\big[$ solution of the equation $U(x) = 0$, where $x \in [(k\pi)^4, ((k+1)\pi)^4]$.*

Proof. The property given by i) is immediate, by taking (2.20) into account.

By (2.20), $\operatorname{sgn} U(t) = \operatorname{sgn} V(t^{\frac{1}{4}})$ for all $t \geq 0$. Since, $V'(x) = -t^3 e^{-t^8-t}\sin(t)$ for all $t \geq 0$, then $\operatorname{sgn} V'(t) = (-1)^{k+1}$ for all $t \in \big]k\pi, (k+1)\pi\big[$ and all integer $k \geq 0$. We have already seen that $\operatorname{sgn} V(k\pi) = (-1)^k$ for all integer $k \geq 0$. Then, for every integer $k \geq 0$, there exists a unique $\tau_k \in \big]k\pi, (k+1)\pi\big[$ solution to the equation $V(x) = 0$, where $x \in [k\pi, (k+1)\pi]$. In view of (2.20), for every integer $k \geq 0$, we infer that $\operatorname{sgn} U((k\pi)^4) = (-1)^k$, and there exists a unique $\xi_k = \tau_k^4 \in \big](k\pi)^4, ((k+1)\pi)^4\big[$ solution of the equation $U(x) = 0$, where $x \in [(k\pi)^4, ((k+1)\pi)^4]$. Thus, ii) and iii) hold. $\square$

3. An Auto-Integral Representation of the Normalized Hermite Polynomials

Recall that the Stieltjes integral formula is given by [10]

$$\int_0^{\infty} x^{p-1} e^{-ax}\sin mx \,\mathrm{d}x = \frac{\Gamma(p)}{(a^2+m^2)^{\frac{p}{2}}}\sin p\theta, \tag{3.1}$$

for any positive real numbers p, q, m, with $\sin\theta = \dfrac{m}{r}$, $\cos\theta = \dfrac{a}{r}$, $0 < \theta < \frac{\pi}{2}$ and $r = \sqrt{a^2+m^2}$.

From (3.1) taking with $\theta = \frac{\pi}{4}$, it comes that $a = m = 1$, and

$$\int_0^\infty x^{p-1} e^{-x} \sin x \, \mathrm{d}x = \frac{\Gamma(p)}{2^{\frac{p}{2}}} \sin \frac{p\pi}{4}, \quad p > 0.$$

In particular, for $p = 4(n+1)$, we get

$$\int_0^\infty x^{4n+3} e^{-x} \sin x \, \mathrm{d}x = 0, \quad n \geq 0,$$

and the transformation $x = t^{\frac{1}{4}}$, yields,

$$\int_0^\infty t^n e^{-t^{\frac{1}{4}}} \sin t^{\frac{1}{4}} \, \mathrm{d}t = 0, \quad n \geq 0. \tag{3.2}$$

On the other hand, by (3.1) and the recursion property of the Gamma function, to known $\Gamma(z+1) = z\Gamma(z)$, for all $z \in \mathbb{C}$ such that $z \neq -n$ for every integer $n \geq 0$, and $\Gamma(1) = 1$, we can write

$$\int_0^\infty x^{p-1} e^{-ax} \sin mx \, \mathrm{d}x = \frac{\Gamma(p+1)}{(a^2+m^2)^{\frac{p}{2}}} \frac{\sin p\theta}{p}, \quad p > 0,$$

and by letting $p \to 0^+$, we get

$$\int_0^\infty \frac{e^{-ax} \sin mx}{x} \, \mathrm{d}x = \theta.$$

For $\theta = \frac{\pi}{4}$, $m = a = 1$, the transformation $x = t^{\frac{1}{4}}$, gives us

$$\int_0^\infty \frac{e^{-t^{\frac{1}{4}}} \sin t^{\frac{1}{4}}}{\pi t} \, \mathrm{d}t = 1,$$

and by taking (3.2) into account, we obtain

$$\int_{-\infty}^\infty t^n W(t) \, \mathrm{d}t = \delta_{n,0}, \quad n \geq 0,$$

where

$$W(t) = \begin{cases} \dfrac{e^{-t^{\frac{1}{4}}} \sin t^{\frac{1}{4}}}{\pi t}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

This leads to the following integral representation of the Dirac mass δ_0,

$$\langle \delta_0, p \rangle = \int_{-\infty}^\infty W(t) p(t) \, \mathrm{d}t = p(0), \quad p \in \mathbb{P},$$

and more general to an integral representation of the Dirac mass δ_x, for every real number x,

$$\langle \delta_x, p \rangle = \int_{-\infty}^\infty W(t-x) p(t) \, \mathrm{d}t = p(x), \quad p \in \mathbb{P}.$$

Consequently, the following auto-integral representation of the normalized $\mathcal{H}$ermite polynomial H_n holds

$$H_n(x) = \int_{-\infty}^\infty W(t-x) H_n(t) \, \mathrm{d}t, \quad n \geq 0.$$

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[1]Department of Mathematics,
University of Badji Mokhtar,
Annaba, Algeria
Email address: aboukhemis@yahoo.com
Email address: kalikhelil@gmail.com

[2]Coll. Edu. of Girls, Scientific Sections,
University of Hafr-Al-Batin,
Hafr-Al-Batin, Saudi Arabia
Email address: ridhasfaxi@gmail.com