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STABILITY AND ULTIMATE BOUNDEDNESS OF SOLUTIONS OF CERTAIN THIRD ORDER NONLINEAR RECTANGULAR MATRIX DIFFERENTIAL EQUATIONS

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Abstract. We present in this paper the qualitative study of solutions of certain third order nonlinear matrix differential equations where the unknown function X is matrix-valued. The properties of solutions were investigated using Lyapunov's direct method by employing the use of suitable Lyapunov functionals obtained from the differential equations describing the system satisfying certain requirements for establishing the stability and boundedness of solutions of the system considered. An example is given to demonstrate the significance of the results obtained as well as analysis through geometric graphs describing the dynamics of the system's solutions. The results obtained are novel and will significantly enhance and extend the results of those mentioned in the literature.

1. Introduction

We investigate the stability and boundedness of solutions of matrix differential equations

$$\dddot{X} + A\ddot{X} + \Psi(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}), \tag{1.1}$$

where $X : \mathbb{R} \to \widetilde{M}$ is the unknown function, $A \in \mathcal{N}$ is a symmetric matrix with constant values, $\Psi, H : \widetilde{M} \to \widetilde{M}$ and $P : \mathbb{R} \times \widetilde{M} \times \widetilde{M} \times \widetilde{M} \to \widetilde{M}$; $\widetilde{M}$ is an $n \times m$ and $\mathcal{N}$ an $n \times n$ matrices, $\mathbb{R}$ the real line $-\infty < t < +\infty$.

The study of characteristics of solutions to differential equations is majorly about deducting essential qualities of solutions of the differential equations without actually

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solving them. For many years, the characteristics of solutions to nonlinear differential equations of the third order have been studied by many mathematicians and have gotten several interesting results for some various and special cases of $n = m = 1$ and $m = 1$ in equation (1.1) (see [1–5, 7–15, 17, 18, 20, 21, 23–26, 28–32], respectively.

In the relevant literature, we observe that works in the area of **matrix** differential equation are not as active as they were in scalar and vector differential equations. Hence, results for the nonlinear differential equation in which the unknown function X is **matrix-valued** (so that $X : \mathbb{R} \to \mathbb{R}^{n\times n}$) are relatively scarce (see [19, 22] and [27]). For example, in 1976, Tejumola [27] discussed the asymptotic stability, ultimate boundedness and presence of periodic solutions of second order matrix differential equation here $\ddot{X} + A\dot{X} + H(X) = P(t, X, \dot{X})$, where A is an $n \times n$ symmetric matrix with constant values. X, $H(X)$ and $P(t, X, \dot{X})$ being continuous $n \times n$ matrices in the real domain. He introduced some standard matrix notations which were widely used. That is, the continuous $n \times n$ matrix function $H(X)$ with $n^2 \times n^2$ generalized Jacobian matrix denoted by $JH(X)$ and the constant $n \times n$ matrix A. He also proved two lemmas which are vital to the proof of the stated theorems. The obtained results are a generalization of an earlier result of [10]. If $X \in \mathbb{R}^n$, the special case for which $n = m$ and $\Psi(\dot{X}) = B(\dot{X})$ in (1.1) for the equation $\dddot{X} + A\ddot{X} + B\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X})$, where A, B are $n \times n$ symmetric matrices with constant values, with X, $H(X)$ and $P(t, X, \dot{X})$ being continuous $n \times n$ matrices in the real domain has been studied by Omeike [19] for the ultimate boundedness of solutions of a certain third order nonlinear matrix differential equations. In the same vein, Omeike and Afuwape [22] proved the ultimate boundedness results of the same equation under some specified conditions on the nonlinear terms. The result obtained here is a rectangular matrix analogue of the results obtained in [19, 22] and an extension of the matrix result achieved in [27]. This means that if $n = m$ in (1.1), the result obtained in this study reduces to the results obtained in [19] and [22] which are square matrix equations and which themselves are matrix analogues of the vector equations in [3] and [12].

The investigations in Olutimo [16] are related to [27] and provided the extensions of some of the results of [27] to (1.1), where X is a rectangular matrix (i.e., $X : \mathbb{R} \to \mathbb{R}^{n\times m}$) and A, B are $n \times n$ symmetric matrices with constant values. X, $H(X)$, and $P(t, X, \dot{X}, \ddot{X})$ being continuous $n \times m$ matrices in the real domain. The present investigation is based on [16] where $X : \mathbb{R} \to \mathbb{R}^{n\times m}$, $n \neq m$ and $B\dot{X} = \Psi(\dot{X})$ in (1.1). Based on our review of the literature, no research derived from [16] was discovered. Moreover, the results for which the unknown function X is not a square matrix were left open in [27]. Tejumola in [27] remarked: *"Our present investigation is of explanatory nature, efforts are being made to expand its scope to cover the situation for which the unknown function X is not necessarily a square matrix. Our results in this direction will be announced elsewhere."* To our knowledge, results in this path do not exist. In this case, we shall give augmentation of some results of [16] to certain third order matrix differential equations (1.1). In addition, matrix differential equations contribute appreciably to the study and plan of complex dynamic systems across various fields, giving valuable insights into the dynamic behaviour of interconnected elements. Systems of this type occur in response and stability of electrical and coupled circuits (see [6] and [27]). In particular, the intuitive idea of qualitative properties of solutions of rectangular matrix differential equations is of practical importance in analyzing the layout of control systems, models of cross interactions between competing species, and spread of diseases in a community with different traits as well in image compression and processing of tasks. Also, the results obtained in this work will be comparable in generality to the results obtained in [3,12,16,22,26,27] and some results existing in the literature. A numerical instance is provided to demonstrate the importance and relevance of the results achieved as well as provide a graphical analysis to corroborate our discoveries regarding the behaviour of solutions of the rectangular matrix equation (1.1).

2. Representation and Definition

We shall use the following standard matrix representation in this study. For $X \in \widetilde{M}$, X^T and x_{ij}, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, represent the transpose and the elements of X, respectively with $(x_{ij})(y_{jk})$, $k = 1, 2, \ldots, n$, the matrix product XY^T of $X, Y \in \widetilde{M}$. $X_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ with $X^j = col(x_{1j}, x_{i2}, \ldots, x_{nj})$ signify the ith row and jth column of X, respectively, and $\underline{X} = (X_1, X_2, \ldots, X_n)^T$ being nm column vector having n rows of X. Now let us represent $\mathbf{JH(X)}$ the $nm \times nm$ generalized the matrix representing the partial derivatives is the matrix linked to the Jacobian determinant $\frac{\partial(H_1,H_2,\ldots,H_n)}{\partial(X_1,X_2,\ldots,X_n)}$ at X when using the function $H : \widetilde{M} \to \widetilde{M}$. Also, $\mathbf{J}\Psi(X)$ the $nm \times nm$ generalized matrix representing the partial derivatives is the matrix linked to the Jacobian determinant $\frac{\partial(\Psi_1,\Psi_2,\ldots,\Psi_n)}{\partial(X_1,X_2,\ldots,X_n)}$ at X when using the function $\Psi : \widetilde{M} \to \widetilde{M}$.

For matrix $A \in \mathcal{N}$ with constant values, we assign an $nm \times nm$ matrix $\widehat{\mathbf{A}}$ having nm diagonal $m \times m$ matrices $(a_{ij} I_m)$ (I_m is the unit $m \times m$ matrix) and so that $(a_{ij} I_m)$ belongs to the ith-n row and jth-n column of $\widehat{\mathbf{A}}$. $\widehat{\mathbf{A}}$ is a $l \times l$ matrix where $l = mn$. In the particular instance in which A is a 2×2 matrix, X is a 2×3 matrix, $\widehat{\mathbf{A}}$ is the 6×6 matrix

$$\begin{pmatrix} a_{11} I_3 & a_{12} I_3 \\ a_{21} I_3 & a_{22} I_3 \end{pmatrix}.$$

For any given $X, Y \in \mathcal{M}$, $\langle X, Y\rangle = \operatorname{trace} XY^T$. $\|X - Y\|^2 = \langle X - Y, X - Y\rangle$ defines a norm on $\mathcal{M}$. $\|X\| = |\underline{X}|_{nm}$, where $|\cdot|_{nm}$ refers to the standard Euclidean norm in $\mathbb{R}^{nm}$ and $\underline{X} \in \mathbb{R}^{nm}$ is defined as mentioned earlier.

3. Preliminary Results

We shall use the following results to prove our theorems.

Lemma 3.1. *Assume that matrices* $\widehat{\boldsymbol{A}}$ *and* $\mathbf{JH}(\mathbf{X})$ *are symmetric and commute with respect to* $X \in \widetilde{M}$ *and* $H(0) = 0$. *Then,*

$$\langle H(X), AX\rangle = \int_0^1 \underline{X}^T \widehat{\boldsymbol{A}} \boldsymbol{JH}(\sigma X)\underline{X} d\sigma. \tag{3.1}$$

Proof. Since $H(0) = 0$ and each $h_{ij} \in \mathcal{C}'(\widetilde{M})$, $i, j = 1, 2, \ldots, n$, we have the following:

$$h_{ij}(X) = \int_0^1 \frac{d}{d\sigma} h_{ij}(\xi) d\sigma = \int_0^1 \sum_{k,l=1}^{n,m} \frac{\partial h_{ij}(\xi)}{\partial(\sigma x)_{kl}} x_{kl} d\sigma, \quad \xi = \sigma X. \tag{3.2}$$

But, by definition

$$\langle H(X), AX\rangle = \text{trace}\left\{ h_{ij}(X)\left(\sum_{r=1}^{n} a_{ir}x_{rj}\right)^T\right\},$$

so that, in the light of Equation (3.2),

$$\langle H(X), AX\rangle = \int_0^1 \sum_{i,j=1}^{n,m}\sum_{k,l=1}^{n,m} \frac{\partial h_{ij}(\xi)}{\partial x_{kl}} x_{kl} \sum_{k=1}^{n} a_{ik}x_{kj} d\sigma.$$

The representation (3.1) follows from the definitions of $\widehat{\mathbf{A}}$ and $\mathbf{JH}(\mathbf{X})$ and the fact that $\widehat{\mathbf{A}}$ is symmetric. □

Lemma 3.2. *Consider* $\mathbf{JH}(\mathbf{X})$ *being symmetric for any* $X \in \widetilde{M}$ *with* $H(0) = 0$. *Then,*

$$\frac{d}{dt}\int_0^1 \langle H(\sigma X), X\rangle d\sigma = \langle H(X), \dot{X}\rangle, \quad \textit{for all } X \in \widetilde{M}. \tag{3.3}$$

Proof. We know that

$$\frac{d}{dt}\int_0^1 \langle H(\sigma X), X\rangle d\sigma = \int_0^1 \langle H(\sigma X), \dot{X}\rangle d\sigma + \int_0^1 \left\langle \frac{d}{dt} H(\sigma X), X\right\rangle d\sigma. \tag{3.4}$$

Observe from equation (3.2) that:

$$\frac{d}{dt} H(\sigma X) = \left(\sigma \sum_{k,l=1}^{n,m} \frac{\partial h_{ij}(\xi)}{\partial x_{kl}} \dot{x}_{kl}\right), \quad \text{where } \xi = \sigma X,$$

from which it follows, by the definition of the inner product, that

$$\left\langle \frac{d}{dt} H(\sigma X), X\right\rangle = \sigma \sum_{i,j=1}^{n,m}\left(\sum_{k,l=1}^{n,m} \frac{\partial h_{ij}(\xi)}{\partial x_{kl}} \dot{x}_{kl}\right) x_{ij} = \sigma \sum_{i,j=1}^{n,m}\left(\sum_{k,l=1}^{n,m} \frac{\partial h_{kl}(\xi)}{\partial x_{ij}} \dot{x}_{kl}\right) x_{ij},$$

since $\mathbf{JH(X)}$ is symmetric. Therefore, by interchanging the order of summation and replacing k, l by i and j, respectively, we have that:

$$\left\langle \frac{d}{dt} H(\sigma X), X\right\rangle = \sigma \sum_{i,j=1}^{n,m}\left(\sum_{k,l=1}^{n,m} \frac{\partial h_{ij}(\xi)}{\partial x_{kl}} x_{kl}\right) \dot{x}_{ij} = \left\langle \sigma\frac{d}{d\sigma} H(\sigma X), \dot{X}\right\rangle. \tag{3.5}$$

Since

$$\frac{d}{d\sigma}h_{ij}(\xi) = \sum_{k,l=1}^{n,m} \frac{\partial h_{ij}(\xi)}{\partial x_{kl}} x_{kl}, \quad \text{by (3.2)},$$

integrating (3.5) by parts, we have

$$\int_0^1 \sigma \frac{d}{d\sigma} H(\sigma X) d\sigma = H(X) - \int_0^1 H(\sigma X) d\sigma.$$

The integral (3.5) equals

$$\langle H(X), \dot{X} \rangle - \int_0^1 \langle H(\sigma X) d\sigma, \dot{X} \rangle,$$

and substituting into (3.4), the result (3.3) is obtained. □

Remark 3.1. Lemmas 1 and 2 respectively of [27] is now included in Lemma 3.1 and Lemma 3.2 if $n = m$.

Lemma 3.3. *Set* $\Psi(0) = 0$ *and presume that* $\boldsymbol{J}\Psi(Y)$ *is symmetric for any* $Y \in \widetilde{M}$. *Then,*

$$\langle \Psi(Y), Y \rangle = \int_0^1 \{\underline{Y}^T [\boldsymbol{J}\Psi(\tau Y)] \underline{Y}\} d\tau.$$

Proof. The proof proceeds by making use of the result

$$\Psi(Y) = \int_0^1 \mathbf{J}\Psi(\tau Y) d\tau,$$

for $Y \in \widetilde{M}$, which is obtained by integrating the equality

$$\frac{d}{d\sigma}\Psi(\tau Y) = \mathbf{J}\Psi(\tau Y) Y,$$

that is,

$$\psi_{ij}(y) = \int_0^1 \frac{d}{d\sigma}\psi_{ij}(\rho) d\sigma = \int_0^1 \sum_{k,l=1}^{n,m} \frac{\partial \psi_{ij}(\rho)}{\partial(\sigma y)_{kl}} y_{kl} d\sigma, \quad \rho = \sigma y,$$

with respect to σ, taking into account that $\Psi(0) = 0$ and each $\psi_{ij} \in \mathcal{C}'(\widetilde{M})$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. □

We express (1.1) as

$$\begin{aligned} \dot{X} &= Y, \\ \dot{Y} &= Z, \\ \dot{Z} &= -AZ - \Psi(Y) - H(X) + P(t, X, Y, Z), \end{aligned} \tag{3.6}$$

where an $n \times m$ matrix X is the unknown function, A is an $n \times n$ symmetric matrix with constant values, $\Psi(Y)$, $H(X)$ and P are continuous $n \times m$ matrices in the real domain.

4. Stability of Solutions

Here, we investigate the stability of solutions of equation (1.1), where $P = 0$ in equation (3.6).

The following result will establish the stability of solutions of (1.1).

Theorem 4.1. *Let us assume H satisfies a condition for the existence and uniqueness of solutions of (3.6) with $H(0) = 0$ and for any $X, Y \in \widetilde{M}$.*

(i) *The matrices $\widehat{\boldsymbol{A}}$, $\boldsymbol{J}\Psi(Y)$, and $\boldsymbol{JH}(\boldsymbol{X})$ exhibit symmetry and are positively definite. $\widehat{\boldsymbol{A}}$, $\boldsymbol{J}\Psi(Y)$, and $\boldsymbol{JH}(\boldsymbol{X})$ commute pairwise and are associative.*

(ii) *The eigenvalues $\lambda_i(\widehat{\boldsymbol{A}})$ of $\widehat{\boldsymbol{A}}$, $\lambda_i(\boldsymbol{J}\Psi(Y))$ of $\boldsymbol{J}\Psi(Y)$ and $\lambda_i(\boldsymbol{JH}(\boldsymbol{X}))$ of $\boldsymbol{JH}(\boldsymbol{X})$, $i = 1, 2, \ldots, nm$, satisfy:*

$$0 < \delta_a \leq \lambda_i(\widehat{\boldsymbol{A}}) \leq \triangle_a,$$
$$0 < \delta_\psi \leq \lambda_i(\boldsymbol{J}\Psi(Y)) \leq \triangle_\psi,$$
$$0 < \delta_h \leq \lambda_i(\boldsymbol{JH(X)}) \leq \triangle_h,$$

with δ_a, δ_ψ, δ_h, $\triangle_a$, $\triangle_\psi$, $\triangle_h$ being finite constants.

Then, every solution of equation (3.6) satisfies

$$\|X(t)\|^2 \to 0, \quad \|Y(t)\|^2 \to 0 \quad \text{and} \quad \|Z(t)\|^2 \to 0, \quad \text{as} \quad t \to +\infty.$$

Proof. For the proof of Theorem 4.1, we use the following function $V = V(X, Y, Z)$ as specified by

$$2V = 2V_a + 2V_b, \tag{4.1}$$

where

$$\begin{aligned} 2V_a =& 2\int_0^1 \langle H(\xi X), X\rangle d\xi + 2\varrho \int_0^1 \langle \Psi(\tau Y), Y\rangle d\tau + \varrho\langle Z, Z\rangle \\ &+ 2\varrho\langle Y, H(X)\rangle + 2\langle Y, Z\rangle + \langle AY, Y\rangle \end{aligned}$$

and

$$\begin{aligned} 2V_b =& 2\delta_a \int_0^1 \langle H(\xi X), X\rangle d\xi + 2\int_0^1 \langle \Psi(\tau Y), Y\rangle d\tau + \langle AY, AY\rangle + \langle Z, Z\rangle \\ &+ \eta\delta_a\delta_\psi^2\langle X, X\rangle + 2\eta\delta_a^2\delta_\psi\langle X, Y\rangle + 2\eta\delta_a\delta_\psi\langle X, Z\rangle + 2\delta_a\langle Y, Z\rangle \\ &+ 2\langle Y, H(X)\rangle - \eta\delta_a\delta_\psi\langle Y, Y\rangle, \end{aligned}$$

where

$$\frac{1}{\delta_a} < \varrho < \frac{\delta_\psi}{\triangle_h} \tag{4.2}$$

and

$$\begin{aligned} \eta < \min\Big\{ &\frac{1}{\delta_a}, \frac{\delta_a}{\delta_\psi}, \frac{2(1+\delta_a)\delta_\psi - 2(1+\varrho)\triangle_h - \delta_a^2(\triangle_a - \delta_a)^2}{2[(1+\varrho)\delta_a - (1+\delta_a)][\delta_a^2\delta_\psi + \delta_a\delta_\psi\delta_h^{-1}(\triangle_\psi - \delta_\psi)^2]}, \\ &\frac{(1+\varrho)\delta_a - (1+\delta_a)}{2\delta_a\delta_\psi\delta_h^{-1}(\triangle_a - \delta_a)^2}\Big\}. \end{aligned} \tag{4.3}$$

It is clear from (4.1), that $V(0,0,0) = 0$.

V_a can be re-written as

$$\begin{aligned}2V_a =&2\varrho\int_0^1 \langle \Psi(\tau Y), Y\rangle d\tau + 2\varrho\langle Y, H(X)\rangle + \langle Y, AY\rangle + 2\int_0^1 \langle H(\xi X), X\rangle d\xi\\ &+ \varrho\langle Z + \varrho^{-1}Y, Z + \varrho^{-1}Y\rangle - \varrho^{-1}\langle Y, Y\rangle.\end{aligned}$$

For each term of the above expression, it is clear that

$$2\varrho\int_0^1 \langle \Psi(\tau Y), Y\rangle d\tau + 2\varrho\langle Y, H(X)\rangle.$$

By Lemma 3.3,

$$2\int_0^1 \langle \Psi(\tau Y), Y\rangle d\tau = 2\int_0^1\int_0^1 \tau_1\langle \mathbf{J}\Psi(\tau_1\tau_2 Y)\underline{Y}, \underline{Y}\rangle d\tau_1 d\tau_2$$

and

$$2\langle Y, H(X)\rangle = 4\int_0^1\int_0^1 \tau_1\langle .Y, H(X)\rangle d\tau_1 d\tau_2.$$

So, that

$$\begin{aligned}&2\varrho\int_0^1 \langle \Psi(\tau Y), Y\rangle d\tau + 2\varrho\langle Y, H(X)\rangle\\ =&2\varrho\int_0^1\int_0^1 \tau_1\left\{\langle \mathbf{J}\Psi(\tau_1\tau_2 Y)\underline{Y}, \underline{Y}\rangle + \langle Y, H(X)\rangle\right\} d\tau_1 d\tau_2.\end{aligned}$$

It should be noted that matrix $\mathbf{J}\Psi$ is as assumed in condition (i) of Theorem 4.1. Hence, $\mathbf{J}\Psi^{\frac{1}{2}}$ and $\mathbf{J}\Psi^{-\frac{1}{2}}$ do exist which are non-singular and symmetric for all $Y \in \widetilde{M}$. So, we have

(4.4)

$$\varrho\langle \mathbf{J}\Psi Y, Y\rangle d\tau + \varrho\langle Y, H(X)\rangle = \sum_{i=1}^{m} \varrho|\mathbf{J}\Psi^{\frac{1}{2}}Y^i + \mathbf{J}\Psi^{-\frac{1}{2}}H(X^i)|_n^2 - \varrho\{\underline{X}^T[\mathbf{J}\Psi^{-1}\mathbf{J}\mathbf{H}^2]\underline{X}\},$$

where $\mathbf{J}\Psi$ stands for $\mathbf{J}\Psi(\tau_1\tau_2 Y)$ and $\mathbf{JH}$ for $\mathbf{JH(X)}$. Thus,

$$\begin{aligned}2V_a =&2\int_0^1 \langle H(\xi X), X\rangle d\tau\\ &- 2\varrho\int_0^1 \tau_1\int_0^1 \{\underline{X}^T[\mathbf{J}\Psi^{-1}(\tau_1\tau_2 Y)\mathbf{JH}(\tau_1 X)]\mathbf{JH}(\tau_1\tau_2 X)\underline{X}\}d\tau_1\tau_2\\ &+ \varrho\langle Z + \varrho^{-1}Y, Z + \varrho^{-1}Y\rangle + \langle AY, Y\rangle - \varrho^{-1}\langle Y, Y\rangle\\ &+ 2\int_0^1 \tau_1\int_0^1 \varrho\sum_{i=1}^{m}|\mathbf{J}\Psi^{\frac{1}{2}}Y^i + \mathbf{J}\Psi^{-\frac{1}{2}}H(X^i)|_n^2 d\tau_1\tau_2.\end{aligned} \tag{4.5}$$

From (4.5), the expression

$$\begin{aligned}&2\int_0^1 \langle H(\tau X), X\rangle d\tau - 2\varrho\int_0^1 \tau_1\int_0^1 \{\underline{X}^T[\mathbf{J}\Psi^{-1}(\tau_1\tau_2 Y)\mathbf{JH}(\tau_1 X)]\mathbf{JH}(\tau_1\tau_2 X)\underline{X}\}d\tau_1\tau_2\\ =&2\int_0^1 \tau_1\int_0^1 \{\underline{X}^T[\widehat{\mathbf{I}} - \varrho\mathbf{J}\Psi^{-1}(\tau_1\tau_2 Y)\mathbf{JH}(\tau_1 X)]\mathbf{JH}(\tau_1\tau_2 X)\underline{X}\}d\tau_1\tau_2\end{aligned}$$

$$\geq (1 - \varrho\delta_\psi^{-1}\triangle_h)\delta_h|\underline{X}|_{nm}^2.$$

Also, we give the estimate for this expression in equation (4.5)

$$\varrho\langle Z + \varrho^{-1}Y, Z + \varrho^{-1}Y\rangle = \sum_{i=1}^{m} |Z^i + \varrho^{-1}Y^i|_n^2.$$

Also,

$$\langle AY, Y\rangle - \varrho^{-1}\langle Y, Y\rangle = \langle(\widehat{\mathbf{A}} - \varrho^{-1}\widehat{\mathbf{I}})\underline{Y}, \underline{Y}\rangle = \{\underline{Y}^T(\widehat{\mathbf{A}} - \varrho^{-1}\widehat{\mathbf{I}})\underline{Y}\} \geq (\delta_a - \varrho^{-1})|\underline{Y}|_{nm}^2.$$

Combining all the estimates of V_a we obtain

$$\begin{aligned} 2V_a &\geq (1 - \varrho\delta_\psi^{-1}\triangle_h)\delta_h|\underline{X}|_{nm}^2 + (\delta_a - \varrho^{-1})|\underline{Y}|_{nm}^2 + \sum_{i=1}^{m} |Z^i + \varrho^{-1}Y^i|_n^2 \\ &\geq (1 - \varrho\delta_\psi^{-1}\triangle_h)\delta_h\|X\|^2 + (\delta_a - \varrho^{-1})\|Y\|^2 + \|Z + \varrho^{-1}Y\|^2. \end{aligned} \tag{4.6}$$

Similarly, we re-arrange $2V_b$ to get

$$\begin{aligned} 2V_b =& \sum_{i=1}^{m} |Z^i + \delta_a Y^i + \eta\delta_a\delta_\psi X^i|_n^2 + \langle AY, AY\rangle \\ &+ 2\int_0^1 \langle\Psi(\tau Y), Y\rangle d\tau - \delta_\psi\langle Y, Y\rangle + \eta\delta_a\delta_\psi^2(1 - \eta\delta_a)\langle X, X\rangle \\ &+ 2\delta_a\int_0^1 \langle H(\tau X), X\rangle d\tau - \delta_\psi^{-1}\langle H(X), H(X)\rangle \\ &+ \delta_a(\delta_a - \eta\delta_\psi)\langle Y, Y\rangle + \delta_\psi\langle Y + \delta_\psi^{-1}H(X), Y + \delta_\psi^{-1}H(X)\rangle. \end{aligned}$$

For this function it is easy to see term by term that

$$\begin{aligned} \langle AY, AY\rangle - \delta_a^2\langle Y, Y\rangle = \langle(\widehat{\mathbf{A}}^2 - \delta_a^2\widehat{\mathbf{I}})\underline{Y}, \underline{Y}\rangle = \underline{Y}^T(\widehat{\mathbf{A}}^2 - \delta_a^2\widehat{\mathbf{I}})\underline{Y} = (\delta_a^2 - \delta_a^2)|\underline{Y}|_{nm}^2 \\ = (\delta_a^2 - \delta_a^2)\|Y\|^2 > 0. \end{aligned}$$

By Lemma 3.3, we obtain for the following expression

$$\begin{aligned} 2\int_0^1 \tau\int_0^1 \langle[\mathbf{J}\Psi(\tau Y) - \delta_\psi\widehat{\mathbf{I}}]\underline{Y}, \underline{Y}\rangle d\tau d\sigma =& 2\int_0^1\int_0^1 \tau\{\underline{Y}^T[\mathbf{J}\Psi(\tau Y) - \delta_\psi\widehat{\mathbf{I}}]\underline{Y}\}d\tau d\sigma \\ =& \underline{Y}^T(\delta_\psi\widehat{\mathbf{I}} - \delta_\psi\widehat{\mathbf{I}})\underline{Y} \\ \geq& (\delta_\psi - \delta_\psi)|\underline{Y}|_{nm}^2 \\ =& (\delta_\psi - \delta_\psi)\|Y\|^2 \\ \geq& 0. \end{aligned}$$

Moreover, the expression gives

$$\eta\delta_a\delta_\psi^2(1 - \eta\delta_a)\langle X, X\rangle = \underline{X}^T(\eta\delta_a\delta_\psi^2(1 - \eta\delta_a)\underline{X} \geq \eta\delta_a\delta_\psi^2(1 - \eta\delta_a)|\underline{X}|_{nm}^2.$$

Also, we get the estimate for this

$$\delta_a(\delta_a - \eta\delta_\psi)\langle Y, Y\rangle = \underline{Y}^T\delta_a(\delta_a - \eta\delta_\psi)\underline{Y} \geq \delta_a(\delta_a - \eta\delta_\psi)|\underline{Y}|_{nm}^2.$$

Furthermore, this expression yields

$$\begin{aligned}
&2\delta_a \int_0^1 \langle H(\tau X), X\rangle d\tau - \delta_\psi^{-1}\langle H(X), H(X)\rangle \\
=&2\int_0^1 \tau_1 \int_0^1 \{\underline{X}^T[\delta_a\widehat{\mathbf{I}} - \delta_\psi^{-1}\mathbf{JH}(\tau_1 X)]\mathbf{JH}(\tau_1\tau_2 X)\underline{X}\}d\tau_1\tau_2 \\
\geq&(\delta_a - \delta_\psi^{-1}\triangle_h)\delta_h|\underline{X}|^2_{nm}
\end{aligned}$$

and

$$\delta_\psi\langle Y + \delta_\psi^{-1}H(X), Y + \delta_\psi^{-1}H(X)\rangle = \delta_\psi \sum_{i=1}^m |Y^i + \delta_\psi^{-1}H(X^i)|^2_n.$$

Combining the estimates of V_b, we have:

$$\begin{aligned}
2V_b \geq&\eta\delta_a\delta_\psi^2(1-\eta\delta_a)|\underline{X}|^2_{nm} + (\delta_a - \delta_\psi^{-1}\triangle_h)\delta_h|\underline{X}|^2_{nm} \\
&+ \delta_a(\delta_a - \eta\delta_\psi)|\underline{Y}|^2_{nm} + \delta_\psi\sum_{i=1}^m |Y^i + \delta_\psi^{-1}H(X^i)|^2_n \\
&+ \sum_{i=1}^m |Z^i + \delta_a Y^i + \eta\delta_a\delta_\psi X^i|^2_n.
\end{aligned} \tag{4.7}$$

Thus, combining estimates (4.6)–(4.7) in Equation (4.1), we obtain

$$\begin{aligned}
2V \geq&(1-\varrho\delta_\psi^{-1}\triangle_h)\delta_h|\underline{X}|^2_{nm} + \eta\delta_a\delta_\psi^2(1-\eta\delta_a)|\underline{X}|^2_{nm} + (\delta_a - \delta_\psi^{-1}\triangle_h)\delta_h|\underline{X}|^2_{nm} \\
&+ (\delta_a - \varrho^{-1})|\underline{Y}|^2_{nm} + \delta_a(\delta_a - \eta\delta_\psi)|\underline{Y}|^2_{nm} + \sum_{i=1}^m |Z^i + \varrho^{-1}Y^i|^2_n \\
&+ \delta_\psi\sum_{i=1}^m |Y^i + \delta_\psi^{-1}H(X^i)|^2_n + \sum_{i=1}^m |Z^i + \delta_a Y^i + \eta\delta_a\delta_\psi X^i|^2_n.
\end{aligned} \tag{4.8}$$

That is,

$$\begin{aligned}
2V \geq& \delta_h(1-\varrho\delta_\psi^{-1}\triangle_h)\|X\|^2 + \eta\delta_a\delta_\psi^2(1-\eta\delta_a)\|X\|^2 + \delta_h(\delta_a - \delta_\psi^{-1}\triangle_h)\|X\|^2 \\
&+ (\delta_a - \varrho^{-1})\|Y\|^2 + \delta_a(\delta_a - \eta\delta_\psi)\|Y\|^2 + \|Z + \varrho^{-1}Y\|^2 \\
&+ \delta_\psi\|Y + \delta_\psi^{-1}H(X)\|^2 + \|Z + \delta_a Y + \eta\delta_a\delta_\psi X\|^2,
\end{aligned}$$

where

$$(\delta_\psi - \triangle_h\delta_a^{-1}) > 0 \quad \text{and} \quad \triangle_h\, \delta_\psi^{-1}(\delta_a - \triangle_h\delta_\psi^{-1}) > 0, \quad \text{by (4.2)}.$$

Thus, it is very clear from the terms in equation (4.8) there is a constant $D_1 > 0$ very small so that:

$$V \geq D_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2), \tag{4.9}$$

for every $X, Y, Z \in \widetilde{M}$. The above estimates are valid since

$$\sum_{i=1}^m |X^i|^2_n = \sum_{i=1}^m |X_i|^2_n = |\underline{X}|^2_{nm} = \|X\|^2, \quad \text{for any } X \in \widetilde{M}.$$

Consider $(X(t), Y(t), Z(t))$ as arbitrary solutions of the system in (3.6). We now differentiate the function $V(t) = (X(t), Y(t), Z(t))$ defined in (4.1) with respect to t along the system (3.6) and using Lemma 3.2, yields

$$\begin{aligned}\dot{V}(t) = &-\frac{1}{2}\eta\delta_a\delta_\psi\int_0^1 \underline{X}^T\mathbf{JH}(\tau X)\underline{X}d\tau - \{\underline{Y}^T[(1+\delta_a)\mathbf{J}\Psi(Y)\\ &-(1+\varrho)\mathbf{JH(X)} - \eta\delta_a^2\delta_\psi\widehat{\mathbf{I}}]\underline{Y}\}\\ &-\frac{1}{2}\{\underline{Z}^T[(1+\varrho)\widehat{\mathbf{A}} - (1+\delta_a)\widehat{\mathbf{I}}]\underline{Z}\}\\ &-\frac{1}{4}\eta\delta_a\delta_\psi\left\{\int_0^1 \underline{X}^T\mathbf{JH}(\tau X)\underline{X} + 4\langle(\widehat{\mathbf{A}} - \delta_a\widehat{\mathbf{I}})X, Z\rangle\right\}d\tau\\ &-\frac{1}{4}\eta\delta_a\delta_\psi\left\{\int_0^1 \underline{X}^T\mathbf{JH}(\tau X)\underline{X} + 4\langle(\mathbf{J}\Psi(\tau Y) - \delta_\psi\widehat{\mathbf{I}})X, Y\rangle\right\}d\tau\\ &-\frac{1}{2}\left\{\{\underline{Z}^T[(1+\varrho)\widehat{\mathbf{A}} - (1+\delta_a)\widehat{\mathbf{I}}]\underline{Z}\} + 2\langle(\widehat{\mathbf{A}} - \delta_a\widehat{\mathbf{I}})\widehat{\mathbf{A}}Y, Z\rangle\right\}.\end{aligned}$$

Following the same reasoning in (4.4), it can be observed that

$$\begin{aligned}&\underline{X}^T\mathbf{JH}(\tau X)\underline{X} + 4\langle(\widehat{\mathbf{A}} - \delta_a\widehat{\mathbf{I}})X, Z\rangle\\ =&\sum_{i=1}^m |\mathbf{JH}^{\frac{1}{2}}X^i + 2\mathbf{JH}^{-\frac{1}{2}}(\widehat{\mathbf{A}} - \delta_a\widehat{\mathbf{I}})Z^i|_n^2 - \{\underline{Z}^T[2(\widehat{\mathbf{A}} - \delta_a\widehat{\mathbf{I}})\mathbf{JH}^{-\frac{1}{2}}]^2\underline{Z}\}.\end{aligned}$$

Also,

$$\begin{aligned}&\underline{X}^T\mathbf{JH}(\tau X)\underline{X} + 4\langle(\mathbf{J}\Psi(Y) - \delta_\psi\widehat{\mathbf{I}})X, Y\rangle\\ =&\sum_{i=1}^m |\mathbf{JH}^{\frac{1}{2}}X^i + 2\mathbf{JH}^{-\frac{1}{2}}(\mathbf{J}\Psi(Y) - \delta_\psi\widehat{\mathbf{I}})Y^i|_n^2 - \{\underline{Y}^T[2(\mathbf{J}\Psi(Y) - \delta_\psi\widehat{\mathbf{I}})\mathbf{JH}^{-\frac{1}{2}}]^2\underline{Y}\},\end{aligned}$$

where $\mathbf{JH} = \mathbf{JH}(X)$ and

$$\begin{aligned}&\underline{Z}^T[(1+\varrho)\widehat{\mathbf{A}} - (1+\delta_a)\widehat{\mathbf{I}}]\underline{Z} + 2\langle(\widehat{\mathbf{A}} - \delta_a\widehat{\mathbf{I}})\widehat{\mathbf{A}}Y, Z\rangle\\ =&\sum_{i=1}^m \left|[(1+\varrho)\widehat{\mathbf{A}} - (1+\delta_a)\widehat{\mathbf{I}}]^{\frac{1}{2}}Z^i + [(1+\varrho)\widehat{\mathbf{A}} - (1+\delta_a)\widehat{\mathbf{I}}]^{-\frac{1}{2}}(\widehat{\mathbf{A}} - \delta_a\widehat{\mathbf{I}})\widehat{\mathbf{A}}Y^i\right|_n^2\\ &-\{\underline{Y}^T[(1+\varrho)\widehat{\mathbf{A}} - (1+\delta_a)\widehat{\mathbf{I}}]^{-1}(\widehat{\mathbf{A}} - \delta_a\widehat{\mathbf{I}})^2\widehat{\mathbf{A}}^2\underline{Y}\}.\end{aligned}$$

Thus,

$$\begin{aligned}\dot{V}(t) \leq &-\frac{1}{2}\eta\delta_a\delta_\psi\int_0^1 \underline{X}^T\mathbf{JH}(\tau X)\underline{X}d\tau - \{\underline{Y}^T[(1+\delta_a)\mathbf{J}\Psi(Y)\\ &-(1+\varrho)\mathbf{JH(X)} - \eta\delta_a^2\delta_\psi\widehat{\mathbf{I}}]\underline{Y}\}\\ &-\frac{1}{2}\{\underline{Z}^T[(1+\varrho)\widehat{\mathbf{A}} - (1+\delta_a)\widehat{\mathbf{I}}]\underline{Z}\}\\ &+\frac{1}{4}\eta\delta_a\delta_\psi\int_0^1 \{\underline{Z}^T[2(\widehat{\mathbf{A}} - \delta_a\widehat{\mathbf{I}})\mathbf{JH}^{-\frac{1}{2}}]^2\underline{Z}\}d\tau\\ &+\frac{1}{4}\eta\delta_a\delta_\psi\int_0^1 \underline{Y}^T[2(\mathbf{J}\Psi(Y) - \delta_\psi\widehat{\mathbf{I}})\mathbf{JH}^{-\frac{1}{2}}]^2\underline{Y}d\tau\end{aligned}$$

$$+\frac{1}{2}\{\underline{Y}^T[(1+\varrho)\widehat{\mathbf{A}}-(1+\delta_a)\widehat{\mathbf{I}}]^{-1}(\widehat{\mathbf{A}}-\delta_a\widehat{\mathbf{I}})^2\widehat{\mathbf{A}}^2\underline{Y}\}.$$

Note that,

$$\int_0^1 \{\underline{Z}^T[2(\widehat{\mathbf{A}}-\delta_a\widehat{\mathbf{I}})\mathbf{J}\mathbf{H}^{-\frac{1}{2}}]^2\underline{Z}\}d\tau = 4\int_0^1 \{\underline{Z}^T[\mathbf{J}\mathbf{H}^{-1}(\widehat{\mathbf{A}}-\delta_a\widehat{\mathbf{I}})^2]\underline{Z}\}d\tau$$

and

$$\int_0^1 \underline{Y}^T[2(\mathbf{J}\Psi(Y)-\delta_\psi\widehat{\mathbf{I}})\mathbf{J}\mathbf{H}^{-\frac{1}{2}}]^2\underline{Y}d\tau = 4\int_0^1 \underline{Y}^T[\mathbf{J}\mathbf{H}^{-1}(\mathbf{J}\Psi(Y)-\delta_\psi\widehat{\mathbf{I}})^2]\underline{Y}d\tau.$$

It follows that

$$\begin{aligned}\dot V(t)\le &-2^{-1}\eta\delta_a\delta_\psi\int_0^1 \underline{X}^T\mathbf{J}\mathbf{H}(\tau X)\underline{X}d\tau\\ &-\int_0^1\Big\{\underline{Y}^T[(1+\delta_a)\mathbf{J}\Psi(Y)-(1+\varrho)\mathbf{J}\mathbf{H}(\mathbf{X})\\ &-\eta\delta_a^2\delta_\psi\widehat{\mathbf{I}}-\eta\delta_a\delta_\psi\mathbf{J}\mathbf{H}^{-1}(\mathbf{J}\Psi(Y)-\delta_\psi\widehat{\mathbf{I}})^2\\ &-2^{-1}[(1+\varrho)\widehat{\mathbf{A}}-(1+\delta_a)\widehat{\mathbf{I}}]^{-1}(\widehat{\mathbf{A}}-\delta_a\widehat{\mathbf{I}})^2\widehat{\mathbf{A}}^2]\underline{Y}\Big\}d\tau\\ &-2^{-1}\int_0^1\Big\{\underline{Z}^T[(1+\varrho)\widehat{\mathbf{A}}-(1+\delta_a)\widehat{\mathbf{I}}-2\eta\delta_a\delta_\psi\mathbf{J}\mathbf{H}^{-1}(\widehat{\mathbf{A}}-\delta_a\widehat{\mathbf{I}})^2]\underline{Z}\Big\}\,d\tau.\end{aligned}$$

Using the hypothesis (ii) of Theorem 4.1 and following the same reasoning in [24, Lemma 1] and [26, Lemma 2], to get

$$\begin{aligned}\dot V(t)\le &-2^{-1}\eta\delta_a\delta_\psi\delta_h|\underline{X}|^2_{nm}\\ &-\Big[(1+\delta_a)\delta_\psi-(1+\varrho)\triangle_h-\eta\delta_a^2\delta_\psi-\eta\delta_a\delta_\psi\delta_h^{-1}(\triangle_\psi-\delta_\psi)^2\\ &-2^{-1}\Big[((1+\varrho)\delta_a-(1+\delta_a))^{-1}(\triangle_a-\delta_a)^2\delta_a^2\Big]\Big]|\underline{Y}|^2_{nm}\\ &-2^{-1}\Big[(1+\varrho)\delta_a-(1+\delta_a)-2\eta\delta_a\delta_\psi\delta_h^{-1}(\triangle_a-\delta_a)^2\Big]|\underline{Z}|^2_{nm}.\end{aligned}$$

If we choose η, such that it satisfies (4.3), then we obtain

$$\dot V(t)\le -\delta_1|\underline{X}|^2_{nm}-\delta_2|\underline{Y}|^2_{nm}-\delta_3|\underline{Z}|^2_{nm}, \tag{4.10}$$

where

$$\begin{aligned}\delta_1=&2^{-1}\eta\delta_a\delta_\psi\delta_h,\\ \delta_2=&(1+\delta_a)\delta_\psi-(1+\varrho)\triangle_h-\eta\delta_a^2\delta_\psi-\eta\delta_a\delta_\psi\delta_h^{-1}(\triangle_\psi-\delta_\psi)^2\\ &-2^{-1}\Big[((1+\varrho)\delta_a-(1+\delta_a))^{-1}(\triangle_a-\delta_a)^2\delta_a^2\Big],\\ \delta_3=&2^{-1}\Big[(1+\varrho)\delta_a-(1+\delta_a)-\eta\delta_a\delta_\psi\delta_h^{-1}(\triangle_a-\delta_a)^2\Big].\end{aligned}$$

The above estimates are valid since

$$\sum_{i=1}^m|X^i|_n^2=\sum_{i=1}^m|X_i|_n^2=|\underline{X}|^2_{nm}=\|X\|^2,\quad \text{for any } X\in\widetilde{M}.$$

Thus, $\dot{V}(t) \leq 0$. Now, using $\frac{d}{dt}V(X,Y,Z) = 0$ with (3.6), it is evident that $X = Y = Z = 0$. Thus, the conditions stated in Theorem 4.1 are met.

Therefore, the trivial solution of (3.6) exhibits asymptotic stability. □

4.1. **Numerical Example.** Let us consider (1.1):

$$\dddot{X} + A\ddot{X} + \Psi(\dot{X}) + H(X) = 0, \quad X \in \widetilde{M}, \tag{4.11}$$

$\widetilde{M}$ being the set of all matrices with dimensions $n \times m$ over the real numbers.

We consider the equivalent system of equation (4.11) in equation (3.6). Let us take for $n = 2$ and $m = 3$ with

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \quad \Psi(Y) = \begin{pmatrix} 4y_1 + \frac{0.01y_1}{1+y_1^2} & 5y_2 + \frac{0.1y_2}{1+y_2^2} & 5y_3 + \frac{0.01y_3}{1+y_3^2} \\ 6y_4 + \frac{0.1y_4}{1+y_4^2} & 4y_5 + \frac{y_5}{1+y_5^2} & 5y_6 + \frac{0.001y_6}{1+y_6^2} \end{pmatrix}$$

and

$$H(X) = \begin{pmatrix} 0.1\tan^{-1}x_1 + 0.01x_1 & 0.1x_2 & 0.01\tan^{-1}x_3 + 0.1x_3 \\ 0.2x_4 & \tan^{-1}x_5 + 0.1x_5 & 0.11x_6 \end{pmatrix}.$$

Thus,

$$X = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & y_2 & y_3 \\ y_4 & y_5 & y_6 \end{pmatrix}, \quad Z = \begin{pmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \end{pmatrix}.$$

By the notation,

$$\widehat{\mathbf{A}} = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$\mathbf{JH(X)} = \begin{pmatrix} \frac{0.1}{1+x_1^2} + 0.01 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{0.01}{1+x_3^2} + 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{1+x_5^2} + 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.11 \end{pmatrix}$$

and

$$\mathbf{J}\Psi(Y) = \mathbf{J}\Psi(Y_p) + \mathbf{J}\Psi(Y_q),$$

where

$$\mathbf{J}\Psi(Y_p)=\begin{pmatrix} 4-\frac{0.02y_1^2}{(1+y_1^2)^2}+\frac{0.01}{1+y_1^2} & 0 & 0 & 0 & 0 & 0\\ 0 & 5-\frac{0.2y_2^2}{(1+y_2^2)^2}+\frac{0.1}{1+y_2^2} & 0 & 0 & 0 & 0\\ 0 & 0 & 5-\frac{0.02y_3^2}{(1+y_3^2)^2}+\frac{0.01}{1+y_3^2} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with

$$\mathbf{J}\Psi(Y_q)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 6-\frac{0.2y_4^2}{(1+y_4^2)^2}+\frac{0.1}{1+y_4^2} & 0 & 0\\ 0 & 0 & 0 & 0 & 4-\frac{2y_5^2}{(1+y_5^2)^2}+\frac{1}{1+y_5^2} & 0\\ 0 & 0 & 0 & 0 & 0 & 5-\frac{0.002y_6^2}{(1+y_6^2)^2}+\frac{0.001}{1+y_6^2} \end{pmatrix}.$$

Clearly, $\widehat{\mathbf{A}}$, $\mathbf{J}\Psi(Y)$ are symmetric. $\widehat{\mathbf{A}}$, $\mathbf{J}\Psi(Y)$ and $\mathbf{JH(X)}$ are associative and commute pairwise.

A simple calculation (with the earlier notations), it is clear that:

$$\delta_a = 2 \leq \lambda_i(\widehat{\mathbf{A}}) \leq 4 = \triangle_a, \quad i = 1,2,3,4,5,6.$$

Thus,

$$\delta_\psi = 4 \leq \lambda_i(\mathbf{J}\Psi(Y)) \leq 6.1 = \triangle_\psi, \quad i = 1,2,3,4,5,6,$$

and

$$\delta_h = 0.1 \leq \lambda_i(\mathbf{JH(X)}) \leq 1.1 = \triangle_h \quad i = 1,2,3,4,5,6,$$

and since, by (4.2),

$$\frac{1}{2} < \varrho < \frac{20}{11},$$

we choose $\varrho = \frac{6}{5}$ so that

$$\eta < \min\{0.5, 1, 0.0009, 0.002\},$$

with the fulfillment of the conditions of Theorem 4.1, the solutions of (3.6) exhibit asymptotic stability.

Remark 4.1. For the case $m = 1$ (that is in $\mathbb{R}^n$), Theorem 4.1 reduces to Corollary 1 in [10] and [27], with obvious modifications.

Remark 4.2. If specialized to case $n = m = 1$ (that is in $\mathbb{R}$) and $h(x) = cx$, equation (1.1) reduces to the scalar differential equation with constant coefficients:

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0.$$

5. Boundedness of Solutions

Here, we investigate the boundedness of solutions of (1.1), where $P \neq 0$ in the equivalent system (3.6).

The following is our boundedness result for (1.1).

Theorem 5.1. *Assuming all the conditions of Theorem 4.1 are met and P satisfies:*

$$\|P(t,X,Y,Z)\| \leq \theta_1(t) + \theta_2(t)(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{\frac{\nu}{2}} + \delta_0(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{\frac{1}{2}}, \tag{5.1}$$

for all $t > 0$, uniformly in (X,Y,Z), where ν, $0 \leq \nu < 1$, and $\delta_0 \geq 0$ are constants and the continuous functions $\theta_1(t)$, $\theta_2(t)$.

There are constants $\triangle_0$, $\triangle_1$, such that if $\delta_0 \leq \triangle_0$, then every solution $X(t)$ of (1.1) ultimately satisfies

$$\|X(t)\|^2 \leq \triangle_1, \quad \|\dot{X}(t)\|^2 \leq \triangle_1, \quad \|\ddot{X}(t)\|^2 \leq \triangle_1,$$

for all sufficiently large t.

To prove Theorem 5.1 we use the matrix scalar function defined in (4.1).

The result below readily follows from (4.1).

Lemma 5.1. *Assuming the satisfaction of all conditions of Theorem 4.1, there exist constants D_1 and D_2 such that:*

$$D_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \leq V(X,Y,Z) \leq D_2(\|X\|^2 + \|Y\|^2 + \|Z\|^2), \tag{5.2}$$

for any given $X, Y, Z \in \widetilde{M}$.

Proof. It should be noted that $V_a + V_b$ is now the expression in (4.8).

The left side of (5.2) in Lemma 5.1 is established in (4.9) if we can find $D_1 \geq 0$ very small so that:

$$V \geq D_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2), \quad \text{for } X, Y, Z \in \widetilde{M}.$$

Also the right side of (5.2) of Lemma 5.1 follows by the same reasoning in [3, 10, 24] and [26] if we choose

$$D_2 = \max\{2\triangle_h + \delta_a\triangle_h + \eta\delta_a\delta_\psi^2 + \eta\delta_a^2\delta_\psi + \eta\delta_a\delta_\psi, \\ 2\delta_a + \varrho\triangle_\psi + 1 + \varrho\triangle_h + \delta_a^2 + \triangle_\psi + \eta\delta_a^2\delta_\psi + \triangle_h + \eta\delta_a\delta_\psi, 2 + \varrho + \eta\delta_a\delta_\psi\}.$$

The above estimates are valid since

$$\sum_{i=1}^{m} |X^i|_n^2 = \sum_{i=1}^{m} |X_i|_n^2 = |\underline{X}|_{nm}^2 = \|X\|^2, \quad \text{for any } X \in \widetilde{M}.$$

The proof of Lemma 5.1 is now concluded. □

We also require the following lemma.

Lemma 5.2. *Assuming the satisfaction of all the conditions in Theorem* 4.1*, consider solutions* $X(t), Y(t), Z(t)$ *be solutions of* (3.6) *with* $V(t) = V(X(t), Y(t), Z(t))$*. Constants* $\triangle_0$*,* D_3 *and* D_4 *exist such that if* δ_0 *in* (5.1) *satisfies* $\delta_0 \leq \triangle_0$*, then*

$$\dot{V}(t) \leq -D_3Q^2 + D_4(\theta_1(t)Q + \theta_2(t)Q^{1+\nu}), \quad Q \equiv (\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{\frac{1}{2}}. \tag{5.3}$$

Proof. Given $\dot{V}(t)_{(3.6)}$, for $P = 0$ in (4.10), now for $P \neq 0$ in (1.1), along any solutions of (3.6) we have

$$\begin{aligned}\dot{V}(t) \leq &- \delta_1|\underline{X}|^2_{nm} - \delta_2|\underline{Y}|^2_{nm} - \delta_3|\underline{Z}|^2_{nm} \\ &+ \langle \eta\delta_a^2\delta_\psi X + (1+\delta_a)Y + (1+\varrho)Z, P(t,X,Y,Z)\rangle.\end{aligned}$$

That is,

$$\begin{aligned}\dot{V}(t) \leq &- \delta_1|\underline{X}|^2_{nm} - \delta_2|\underline{Y}|^2_{nm} - \delta_3|\underline{Z}|^2_{nm} \\ &+ \{\eta\delta_a^2\delta_\psi\|X\| + (1+\delta_a)\|Y\| + (1+\varrho)\|Z\|\}\|P(t,X,Y,Z)\|.\end{aligned}$$

If $P(t, X, Y, Z)$ satisfies (5.1), we get

$$\begin{aligned}\dot{V}(t) \leq &- \delta_1\|X\|^2 - \delta_2\|Y\|^2 - \delta_3\|Z\|^2 \\ &+ (\eta\delta_a^2\delta_\psi\|X\| + (1+\delta_a)\|Y\| + (1+\varrho)\|Z\|)\Big[(\theta_1(t)+ \\ &+ \theta_2(t)\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right)^{\frac{\nu}{2}} + \delta_0\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right)^{\frac{1}{2}}\Big].\end{aligned}$$

Thus,

$$\dot{V}(t) \leq - \delta_1\|X\|^2 - \delta_2\|Y\|^2 - \delta_3\|Z\|^2 + \delta_4Q\theta_1(t) + \delta_4\theta_2(t)Q^{\nu+1} + \delta_4\delta_0Q^2,$$

where $\delta_4 = \max\{\eta\delta_a^2\delta_\psi, (1+\delta_a), (1+\varrho)\}$.

Let $\triangle_0$ be now fixed as

$$\triangle_0 = \frac{1}{2}\delta_4^{-1}\min\{\delta_1, \delta_2, \delta_3\} > 0.$$

Then, for $\delta_0 \leq \triangle_0$, we shall have from the above inequality for V that

$$\dot{V}(t) \leq -\delta_5Q^2 + \delta_4\{\theta_1(t)Q + \theta_2(t)Q^{\nu+1}\},$$

where $\delta_5 = \delta_0\triangle_0$. Thus, we obtain (5.3) with $D_3 = \delta_5$ and $D_4 = \delta_4$. The estimates above are valid since

$$\sum_{i=1}^{m}|X^i|^2_n = \sum_{i=1}^{m}|X_i|^2_n = |\underline{X}|^2_{nm} = \|X\|^2,$$

for any $X \in \widetilde{M}$.

We conclude the proof of Theorem 5.1 by using inequalities (5.2) and (5.3) and by adapting the reasoning presented in [16], we can easily conclude this part of the proof, therefore, we skip it. Hence, Theorem 5.1 then follows as pointed out earlier. □

5.1. **Numerical Example.** We consider the non-homogeneous form of (4.11) in Example 4.1 as

$$\dddot{X} + A\ddot{X} + \Psi(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}), \quad X \in \widetilde{M}.$$

Suppose we choose

$$P(t, X, Y, Z) = \begin{pmatrix} \frac{1}{1+t^2+x^2+y^2+z^2} & \frac{1}{1+t^2+x^2+y^2+z^2} & \frac{1}{1+t^2+x^2+y^2+z^2} \\ \frac{1}{1+t^2+x^2+y^2+z^2} & \frac{1}{1+t^2+x^2+y^2+z^2} & \frac{1}{1+t^2+x^2+y^2+z^2} \end{pmatrix}.$$

We have that

$$\begin{aligned} \|P(t, X, Y, Z)\| &= \frac{6}{1+t^2}\left(\sum_{i=1}^{3}|X_i|_n^2 + \sum_{i=1}^{3}|Y_i|_n^2 + \sum_{i=1}^{3}|Z_i|_n^2\right) \\ &\leq 6\left(|\underline{X}|_6^2 + |\underline{Y}|_6^2 + |\underline{Z}|_6^2\right) \leq 6\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right) \leq 6. \end{aligned}$$

Remark 5.1. If $n = m$ (that is $\mathbb{R}^{n\times n}$) and $\Psi(\dot{X}) = B(\dot{X})$ in (1.1), Theorem 5.1 reduces to Theorem 1 in [22]. That is, a direct generalization of [22] and [19].

Remark 5.2. For the case $m = 1$ (that is in $\mathbb{R}^n$) in equation (1.1), this result is a matrix analogue of a result of [3, 12] and [26] with obvious modifications.

6. Conclusion

This study gives an insight into the qualitative behaviour of solutions of third order rectangular matrix differential equations. The use of Lyapunov's direct method provides an effective approach to analyze and establish sufficient conditions on stability and ultimate boundedness of solutions of rectangular matrix differential equations as well as provides a valuable tool for the wider study of dynamical systems whose state variables are valued in rectangular array. Numerical simulations and analysis were given in system (4.11) which satisfies all the conditions of Theorem 4.1, Theorem 5.1 and inequalities (4.2) and (4.3). These new results significantly improve those present in existing literature as well as contribute to the qualitative aspects of the theory of matrix differential equations thus providing for the development of more general formulations.

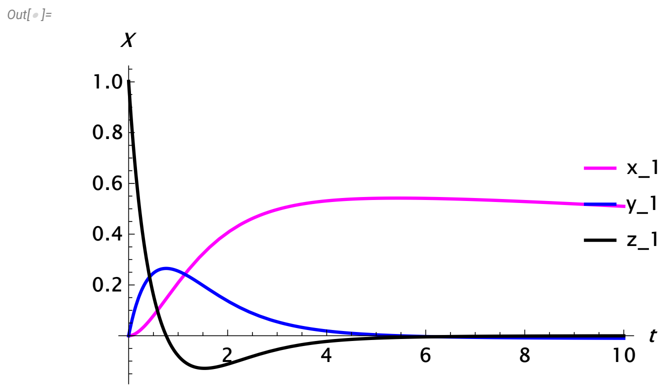

FIGURE 1. The plot of triple $(x_1(t), y_1(t), z_1(t))$ where $x_1(t)$ (in pink), $y_1(t)$ (in blue) and $z_1(t)$ (black) respectively of system (4.11) meeting the conditions of Theorem 4.1

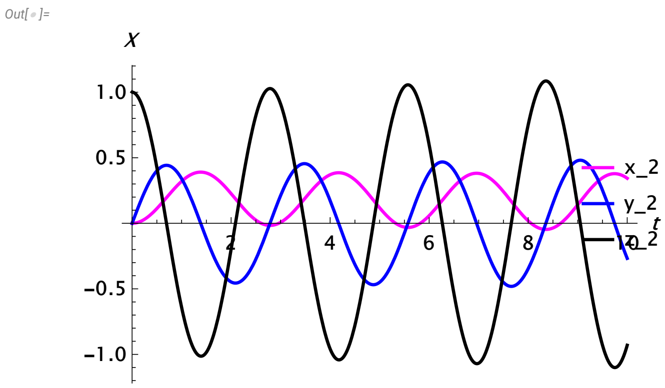

FIGURE 2. The plot of triple $(x_2(t), y_2(t), z_2(t))$ where $x_2(t)$ (in pink), $y_2(t)$ (in blue) and $z_2(t)$ (black) respectively of system (4.11) meeting the conditions of Theorem 4.1 and Theorem 5.1

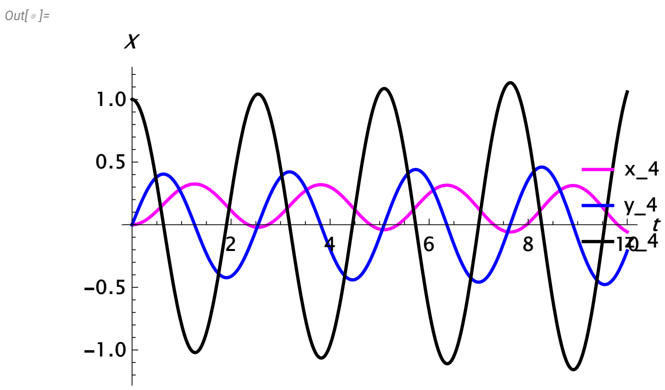

FIGURE 4. The plot of triple $(x_4(t), y_4(t), z_4(t))$ where $x_4(t)$ (in pink), $y_4(t)$ (in blue) and $z_4(t)$ (black) respectively of system (4.11) meeting the conditions of Theorem 4.1

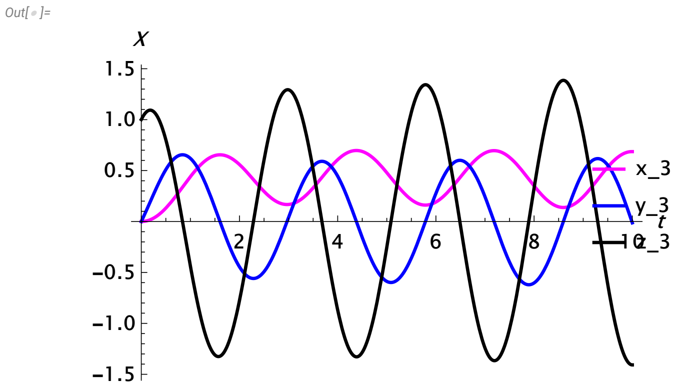

FIGURE 3. The plot of triple $(x_3(t), y_3(t), z_3(t))$ where $x_3(t)$ (in pink), $y_3(t)$ (in blue) and $z_3(t)$ (black) respectively of system (4.11) meeting the conditions of Theorem 4.1

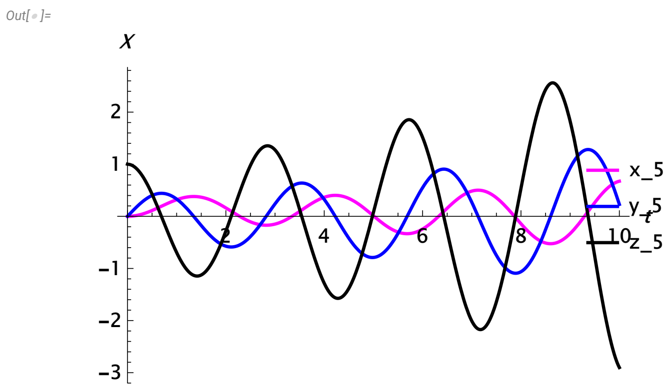

FIGURE 5. The plot of triple $(x_5(t), y_5(t), z_5(t))$ where $x_5(t)$ (in pink), $y_5(t)$ (in blue) and $z_5(t)$ (black) respectively of system (4.11) meeting the conditions of Theorem 4.1

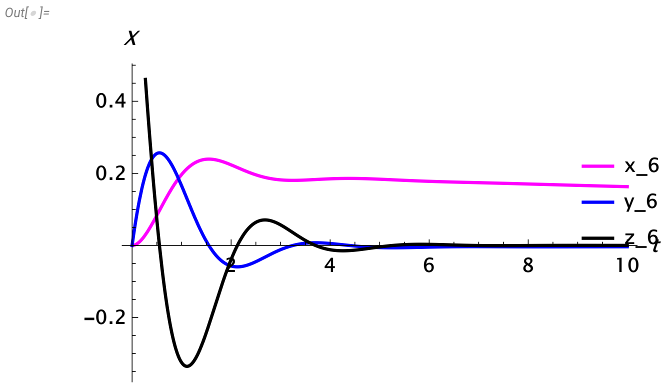

FIGURE 6. The plot of triple $(x_6(t), y_6(t), z_6(t))$ where $x_6(t)$ (in pink), $y_6(t)$ (in blue) and $z_6(t)$ (black) respectively of system (4.11) meeting the conditions of Theorem 4.1

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