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ON WEIGHTED GENERALIZATION OF TRAPEZOID TYPE INEQUALITIES FOR FUNCTIONS OF TWO VARIABLES WITH BOUNDED VARIATION

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Abstract. In this paper, we obtain generalized weighted trapezoid inequalities for functions of two independent variables with bounded variation. We also give some applications for qubature formulas.

1. Introduction

In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives [20].

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whoose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty:=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

This inequality is well known in the literature as the *Ostrowski inequality.*

In 2001, Dragomir [15] obtained following Ostrowski type inequality for functions of bounded variation.

Key words and phrases. Function of bounded variation, Ostrowski type inequalities, Riemann-Stieltjes integral.

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Theorem 1.2. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)dt-(b-a)f(x)\right|\le\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f)$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

In [21], Tseng et al. gave the following generalization of weighted trapezoidal inequality for functions of bounded variation.

Theorem 1.3. *Let $w:[a,b]\to[0,\infty)$ nonnegative and continuous and $h:[a,b]\to\mathbb{R}$ be differentiable such that $h'(t)=w(t)$ on $[a,b]$. If $f:[a,b]\to\mathbb{R}$ be mapping of bounded variation on $[a,b]$, then*

$$\left|\int_a^b w(t)f(t)dt-\left[(x-h(a))f(a)+(h(b)-x)f(b)\right]\right| \tag{1.2}$$

$$\le\left[\frac{1}{2}\int_a^b w(t)dt+\left|x-\frac{h(a)+h(b)}{2}\right|\right]\bigvee_a^b(f),$$

for all $x\in[h(a),h(b)]$. The constant $\frac{1}{2}$ is the best possible.

2. Preliminaries and Lemmas

In 1910, Fréchet [17] has given the following characterization for the double Riemann-Stieltjes integral. Assume that $f(x,y)$ and $g(x,y)$ are defined over the rectangle $Q=[a,b]\times[c,d]$; let R be the divided into rectangular subdivisions, or cells, by the net of straight lines $x=x_i$, $y=y_i$

$$a=x_0<x_1<...<x_n=b \text{ and } c=y_0<y_1<...<y_m=d,$$

let ξ_i,η_j be any numbers satisfying $\xi_i\in[x_{i-1},x_i]$, $\eta_j\in[y_{j-1},y_j]$, $i=1,2,...,n$, $j=1,2,...,m$ and for all i,j let

$$\Delta_{11}g(x_i,y_j)=g(x_{i-1},y_{j-1})-g(x_{i-1},y_j)-g(x_i,y_{j-1})+g(x_i,y_j).$$

Then if the sum

$$S=\sum_{i=1}^{n}\sum_{j=1}^{m}f(\xi_i,\eta_j)\Delta_{11}g(x_i,y_j)$$

tends to a finite limit as the norm of the subdivisions approaches zero, the integral of f with respect to g is said to exist. We call this limit the restricted integral, and designate it by the symbol

$$\int_a^b\int_c^d f(x,y)d_yd_xg(x,y). \tag{2.1}$$

If in the above formulation S is replaced by the sum

$$S^{*}=\sum_{i=1}^{n}\sum_{j=1}^{m}f\left(\xi_{ij},\eta_{ij}\right)\Delta_{11}g(x_i,y_j),$$

where ξ_{ij},η_{ij} are numbers satisfying $\xi_{ij}\in[x_{i-1},x_i]$, $\eta_{ij}\in[y_{j-1},y_j]$ we call the limit, when it exist, the unrestricted integral, and designate it by the symbol

$$(*)\int_a^b\int_c^d f(x,y)d_yd_xg(x,y). \tag{2.2}$$

Clearly, the existence of (2.2) implies both the existence of (2.1) and its equality (2.2). On the other hand, Clarkson [13] has shown that the existence of (2.1) does not imply the existence of (2.2).

In [12], Clarkson and Adams gave the following definitions of bounded variation for functions of two variables.

2.1. **Definitions.** The function $f(x,y)$ is assumed to be defined in rectangle $R(a\le x\le b,\ c\le y\le d)$. By the term *net* we shall, unless otherwise specified mean a set of parallels to the axes:

$$\begin{aligned}x=&x_i,\ i=0,1,2,...,m,\ a=x_0<x_1<\cdots<x_m=b,\\ y=&y_j,\ j=0,1,2,...,n,\ c=y_0<y_1<\cdots<y_n=d.\end{aligned}$$

Each of the smaller rectangles into which R is devided by a net will be called a *cell*. We employ the notation

$$\Delta_{11}f(x_i,y_j)=f(x_{i+1},y_{j+1})-f(x_{i+1},y_j)-f(x_i,y_{j+1})+f(x_i,y_j),$$

$$\Delta f(x_i,y_j)=f(x_{i+1},y_{j+1})-f(x_i,y_j).$$

The total variation function, $\phi(\overline{x})$ $[\psi(\overline{y})]$, is defined as the total variation of $f(\overline{x},y)$ $[f(x,\overline{y})]$ considered as a function of y $[x]$ alone in interval (c,d) $[(a,b)]$,or as $+\infty$ if $f(\overline{x},y)$ $[f(x,\overline{y})]$ is of unbounded variation.

Definition 2.1 (Vitali-Lebesque-Fréchet-de la Vallée Poussin)**.** The function $f(x,y)$ is said tobe of bounded variation if the sum

$$\sum_{i=0,j=0}^{m-1,n-1}\left|\Delta_{11}f(x_i,y_j)\right|$$

is bounded for all nets.

Definition 2.2 (Fréchet)**.** The function $f(x,y)$ is said tobe of bounded variation if the sum

$$\sum_{i=0,j=0}^{m-1,n-1}\epsilon_i\overline{\epsilon_j}\left|\Delta_{11}f(x_i,y_j)\right|$$

is bounded for all nets and all possible choices of $\epsilon_i=\pm1$ and $\overline{\epsilon_j}=\pm1$.

Definition 2.3 (Hardy-Krause)**.** The function $f(x,y)$ is said tobe of bounded variation if it satisfies the condition of Definition 2.1 and if in addition $f(\overline{x},y)$ is of bounded variation in y (i.e., $\phi(\overline{x})$ is finite) for at least one $\overline{x}$ and $f(x,\overline{y})$ is of bounded variation in y (i.e., $\psi(\overline{y})$ is finite) for at least one $\overline{y}$.

Definition 2.4 (Arzelà)**.** Let (x_i,y_i), $i=0,1,2,\ldots,m$ be any set of points satisfiying the conditions

$$\begin{aligned} a =& x_0 < x_1 < \cdots < x_m = b, \\ c =& y_0 < y_1 < \cdots < y_m = d. \end{aligned}$$

Then $f(x,y)$ is said tobe of bounded variation if the sum

$$\sum_{i=1}^{m} |\Delta f(x_i,y_i)|$$

is bounded for all such sets of points.

Therefore, one can define the consept of total variation of a function of variables, as follows.

Let f be of bounded variation on $Q=[a,b]\times[c,d]$ and let $\sum(P)$ denote the sum $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}|\Delta_{11}f(x_i,y_j)|$ corresponding to the partition P of Q. The number

$$\bigvee_Q(f) := \bigvee_c^d\bigvee_a^b(f) := \sup\left\{\sum(P) : P\in P(Q)\right\}$$

is called the total variation of f on Q.

The following lemmas will be used in our main result.

Lemma 2.1 (Integrating by parts)**.** [19, Lemma 2] *If $f(t,s)$ is continuous on renctangle $Q=[a,b]\times[c,d]$ and $\alpha(t,s)\in BV_H(Q)$, then $\alpha(t,s)$ is integrable with respect to $f(t,s)$ over Q in the Riemann-Stieltjes sense and*

$$\begin{aligned} \int_a^b\int_c^d f(t,s)d_td_s\alpha(t,s) =& \int_a^b\int_c^d \alpha(t,s)d_td_sf(t,s) \\ &- \int_a^b \alpha(t,d)d_tf(t,d) + \int_a^b \alpha(t,c)d_tf(t,c) \\ &- \int_c^d \alpha(b,s)d_sf(b,s) + \int_c^d \alpha(a,s)d_sf(a,s) \\ &+ f(b,d)\alpha(b,d) - f(b,c)\alpha(b,c) - f(a,d)\alpha(a,d) \\ &+ f(a,c)\alpha(a,c). \end{aligned}$$

Lemma 2.2. [18] *Assume that $\Omega \in RS(g)$ on Q and g is of bounded variation on Q, then*

$$\left|\int_c^d \int_a^b \Omega(x,y) d_x d_y g(x,y)\right| \leq \sup_{(x,y)\in Q} |\Omega(x,y)| \bigvee_Q (g)\,. \tag{2.3}$$

In [18], authors obtained the following Ostrowski type inequality for functions of two variables with bounded variation.

Theorem 2.1. *If the function $f : Q = [a,b] \times [c,d] \to R$ is of bounded variation on Q, then for all $(x,y) \in Q$ we have the inequality*

(2.4)

$$\left|f(x,y) - \frac{1}{b-a}\int_a^b f(t,y)dt - \frac{1}{d-c}\int_c^d f(x,s)ds + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\leq \left[\frac{1}{2} + \frac{\left|x - \frac{a+b}{2}\right|}{b-a}\right]\left[\frac{1}{2} + \frac{\left|y - \frac{c+d}{2}\right|}{d-c}\right]\bigvee_a^b\bigvee_c^d(f),$$

where $\bigvee_Q (f)$ denotes the total (double) variation of f on Q.

Moreover, authors gave the following trapeozoid inequality for mappings of two variables with bounded variation.

Theorem 2.2. *Let $f : Q \to\to \mathbb{R}$ be mapping of bounded variation on Q. Then for all $(x,y) \in Q$, we have inequality*

$$\begin{aligned} &\left|\frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{4} - \frac{1}{2\,(d-c)}\left[\int_c^d f(a,s)ds + \int_c^d f(b,s)ds\right]\right. \\ &\left. -\frac{1}{2\,(b-a)}\left[\int_a^b f(t,c)dt + \int_a^b f(t,d)dt\right] + \frac{1}{(b-a)\,(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right| \\ &\leq \frac{1}{4}\bigvee_a^b\bigvee_c^d(f). \end{aligned} \tag{2.5}$$

The constant $\frac{1}{4}$ is the best possible. For a simple proof of sharpness of constant see [6].

In [8], Budak and Sarikaya obtained the following generalized trapezoid inequality.

Theorem 2.3. *Let $f : Q \to \mathbb{R}$ be a mapping of bounded variation on Q. Then for all $(x,y) \in Q$, we have inequality*

$$\begin{aligned} &|(b-x)\,(d-y)\,f(b,d) + (b-x)\,(y-c)\,f(b,c) \\ &+ (x-a)\,(d-y)\,f(a,d) + (x-a)\,(y-c)\,f(a,c) \end{aligned} \tag{2.6}$$

$$
\begin{aligned}
& -(d-y)\int_a^b f(t,d)dt-(y-c)\int_a^b f(t,c)dt \\
& -(b-x)\int_c^d f(b,s)ds-(x-a)\int_c^d f(a,s)ds+\int_a^b\int_c^d f(t,s)dsdt\Bigg| \\
\leq\ & \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]\bigvee_a^b\bigvee_c^d(f),
\end{aligned}
$$

where $\bigvee_a^b\bigvee_c^d(f)$ *denotes he total variation of* f *on* Q.

For more information and recent developments on inequalities for mappings of bounded variation, please refer to [1–11,14–16,18,21–26].

The aim of this paper is to establish weighted generalization of trapezoid inequality for functions of two independent variables with bounded variation.

3. Main Results

Let $w_1:[a,b]\to[0,\infty)$ continuous and nonnegative on (a,b) and $h_1:[a,b]\to\mathbb{R}$ be differentiable such that $h_1'(t)=w_1(t)$ on $[a,b]$. Similarly, let $w_2:[c,d]\to[0,\infty)$ continuous and positive on (c,d) and $h_2:[c,d]\to\mathbb{R}$ be differentiable such that $h_2'(t)=w_2(t)$ on $[c,d]$.

Theorem 3.1. *If* $f:[a,b]\times[c,d]\to\mathbb{R}$ *is a mapping of bounded variation on* $[a,b]\times[c,d]$*, then we have the following inequality for all* $(x,y)\in[h_1(a),h_1(b)]\times[h_2(c),h_2(d)]$

$$
\begin{aligned}
& |(h_1(b)-x)(h_2(d)-y)f(b,d)+(h_1(b)-x)(y-h_2(c))f(b,c) \\
& +(x-h_1(a))(h_2(d)-y)f(a,d)+(x-h_1(a))(y-h_2(c))f(a,c) \\
& -(h_2(d)-y)\int_a^b w_1(t)f(t,d)dt-(y-h_2(c))\int_a^b w_1(t)f(t,c)dt \\
& -(h_1(b)-x)\int_c^d w_2(s)f(b,s)ds-(x-h_1(a))\int_c^d w_2(s)f(a,s)ds \\
& +\int_a^b\int_c^d w_1(t)w_2(s)f(t,s)dsdt\Bigg| \\
\leq & \left[\frac{1}{2}\int_a^b w_1(t)dt+\left|x-\frac{h_1(a)+h_1(b)}{2}\right|\right] \\
& \times\left[\frac{1}{2}\int_c^d w_2(t)dt+\left|y-\frac{h_2(c)+h_2(d)}{2}\right|\right]\bigvee_a^b\bigvee_c^d(f),
\end{aligned}
\tag{3.1}
$$

where $\bigvee_a^b \bigvee_c^d (f)$ *denotes the total variation of* f *on interval* $[a,b]\times[c,d]$.

Proof. For $(x,y)\in[h_1(a),h_1(b)]\times[h_2(c),h_2(d)]$, using integration by parts, we have

$$\begin{aligned}
&\int_a^b\int_c^d (x-h_1(t))\,(y-h_2(s))\,d_td_sf(t,s) \\
&=(h_1(b)-x)\,(h_2(d)-y)\,f(b,d)+(h_1(b)-x)\,(y-h_2(c))\,f(b,c) \\
&\quad+(x-h_1(a))\,(h_2(d)-y)\,f(a,d)+(x-h_1(a))\,(y-h_2(c))\,f(a,c) \\
&\quad-(h_2(d)-y)\int_a^b w_1(t)f(t,d)dt-(y-h_2(c))\int_a^b w_1(t)f(t,c)dt \\
&\quad-(h_1(b)-x)\int_c^d w_2(s)f(b,s)ds-(x-h_1(a))\int_c^d w_2(s)f(a,s)ds \\
&\quad+\int_a^b\int_c^d w_1(t)w_2(s)f(t,s)dsdt.
\end{aligned}\tag{3.2}$$

Taking modulus (3.2) and using Lemma 2.2, we get

$$\begin{aligned}
&\Big|(h_1(b)-x)\,(h_2(d)-y)\,f(b,d)+(h_1(b)-x)\,(y-h_2(c))\,f(b,c) \\
&\quad+(x-h_1(a))\,(h_2(d)-y)\,f(a,d)+(x-h_1(a))\,(y-h_2(c))\,f(a,c) \\
&\quad-(h_2(d)-y)\int_a^b w_1(t)f(t,d)dt-(y-h_2(c))\int_a^b w_1(t)f(t,c)dt \\
&\quad-(h_1(b)-x)\int_c^d w_2(s)f(b,s)ds-(x-h_1(a))\int_c^d w_2(s)f(a,s)ds \\
&\quad+\int_a^b\int_c^d w_1(t)w_2(s)f(t,s)dsdt\Big| \\
&=\left|\int_a^b\int_c^d (x-h_1(t))\,(y-h_2(s))\,d_td_sf(t,s)\right| \\
&\le\sup_{t\in[a,b]}|x-h_1(t)|\sup_{s\in[c,d]}|y-h_2(s)|\bigvee_a^b\bigvee_c^d(f).
\end{aligned}$$

Since $x-h_1(t)$ is decreasing on $[a,b]$, $h_1(a)\le x\le h_1(b)$, and $h_1'(t)=w_1(t)$ on $[a,b]$, we have

$$\begin{aligned}
\sup_{t\in[a,b]}|x-h_1(t)| &=\max\{x-h_1(a),h_1(b)-x\} \\
&=\frac{h_1(a)+h_1(b)}{2}+\left|x-\frac{h_1(a)+h_1(b)}{2}\right|
\end{aligned}\tag{3.3}$$

$$=\frac{1}{2}\int_a^b w_1(t)dt+\left|x-\frac{h_1(a)+h_1(b)}{2}\right|.$$

Similarly, we have

$$\sup_{s\in[c,d]}|y-h_2(s)|=\frac{1}{2}\int_c^d w_2(t)dt+\left|y-\frac{h_2(c)+h_2(d)}{2}\right|.$$

This completes the proof. □

Remark 3.1. If we choose $w_1(t)\equiv 1$, $h_1(t)=t$ on $[a,b]$ and $w_2(s)=1$, $h_2(s)=s$ on $[c,d]$ in Theorem 3.1, then the inequality (3.1) reduces the inequality (2.6).

Corollary 3.1 (Weighted trapeozoid). *Under the assumption of Theorem* 3.1 *with* $x=\frac{h_1(a)+h_1(b)}{2}$ *and* $y=\frac{h_2(c)+h_2(d)}{2}$, *then we have the following weighted trapeozoid inequality*

$$\begin{aligned}(3.4)\qquad &\Bigg|\left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right)\frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{4}\\&-\left(\int_c^d w_2(t)dt\right)\left[\int_a^b w_1(t)f(t,d)dt+\int_a^b w_1(t)f(t,c)dt\right]\\&-\left(\int_a^b w_1(t)dt\right)\left[\int_c^d w_2(s)f(b,s)ds+\int_c^d w_2(s)f(a,s)ds\right]\\&+\int_a^b\int_c^d w_1(t)w_2(s)f(t,s)dsdt\Bigg|\\ \leq&\frac{1}{4}\left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right)\bigvee_a^b\bigvee_c^d(f).\end{aligned}$$

The constant $\frac{1}{4}$ *is the best possible.*

Remark 3.2. If we choose $w_1(t)\equiv 1$, $h_1(t)=t$ on $[a,b]$ and $w_2(s)=1$, $h_2(s)=s$ on $[c,d]$ in Corollary 3.1, then the inequality (3.4) reduces the inequality (2.5).

Corollary 3.2 (Weighted left rectangle inequality). *Under the assumption of Theorem* 3.1 *with* $x=h_1(b)$ *and* $y=h_2(d)$, *then we have the following weighted left rectangle inequality*

$$\begin{aligned}(3.5)\qquad &\Bigg|\left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right)f(a,c)-\left(\int_c^d w_2(t)dt\right)\int_a^b w_1(t)f(t,c)dt\\&-\left(\int_a^b w_1(t)dt\right)\int_c^d w_2(s)f(a,s)ds+\int_a^b\int_c^d w_1(t)w_2(s)f(t,s)dsdt\Bigg|\end{aligned}$$

$$\leq \left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right)\bigvee_a^b\bigvee_c^d(f).$$

Remark 3.3. If we choose $w_1(t) \equiv 1$, $h_1(t) = t$ on $[a,b]$ and $w_2(s) = 1$, $h_2(s) = s$ on $[c,d]$ in Corollary 3.2, then Corollary 3.2 reduces the Remark 1.a in [5].

Corollary 3.3 (Weighted right rectangle inequality)**.** *Under the assumption of Theorem* 3.1 *with* $x = h_1(a)$ *and* $y = h_2(c)$*, then we have the following weighted right rectangle inequality*

$$\Bigg|\left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right)f(b,d) - \left(\int_c^d w_2(t)dt\right)\int_a^b w_1(t)f(t,d)dt$$
$$-\left(\int_a^b w_1(t)dt\right)\int_c^d w_2(s)f(b,s)ds + \int_a^b\int_c^d w_1(t)w_2(s)f(t,s)dsdt\Bigg|$$
$$\leq \left(\int_a^b w_1(t)dt\right)\left(\int_c^d w_2(t)dt\right)\bigvee_a^b\bigvee_c^d(f).$$

Remark 3.4. If we choose $w_1(t) \equiv 1$, $h_1(t) = t$ on $[a,b]$ and $w_2(s) = 1$, $h_2(s) = s$ on $[c,d]$ in Corollary 3.3, then Corollary 3.3 reduces the Remark 1.b in [5].

4. Application to a Cubature Rule

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < \cdots < x_n = b$, and $J_m : c = y_0 < y_1 < \cdots < y_m = d$ with $l_1^i := x_{i+1} - x_i$ and $l_2^j := y_{j+1} - y_j$, and let

$$\upsilon(l_1) := \max\left\{l_1^i \mid i = 0,\ldots,n-1\right\},$$
$$\upsilon(l_2) := \max\left\{l_2^j \mid j = 0,\ldots,m-1\right\},$$

$$\upsilon(W_1) := \max\left\{W_1^i \mid i = 0,\ldots,n-1\right\},\ W_1^i := \int_{x_i}^{x_{i+1}} w_1(u)du = h_1(x_{i+1}) - h_1(x_i)$$

and

$$\upsilon(W_2) := \max\left\{W_2^j \mid j = 0,\ldots,m-1\right\},\ W_2^j := \int_{y_j}^{y_{j+1}} w_2(u)du = h_2(y_{j+1}) - h_2(y_j).$$

Let us have w_1, h_1, w_2 and h_2 defined as in Theorem 3.1 and let $\xi_i \in [h_1(x_i), h_1(x_{i+1})]$, $i = 0,\ldots,n-1$ and $\eta_j \in [h_2(y_j), h_2(y_{j+1})]$, $j = 0,\ldots,m-1$.

Define the sum

$$(4.1)\quad A(f,w_1,h_1,w_2,h_2,I_n,J_m,\xi,\eta)$$
$$:= \sum_{j=0}^{m-1}(h_2(y_{j+1}) - \eta_j)\int_a^b w_1(t)f(t,y_{j+1})dt + \sum_{j=0}^{m-1}(\eta_j - h_2(y_j))\int_a^b w_1(t)f(t,y_j)dt$$

$$
\begin{aligned}
&+\sum_{i=0}^{n-1}\left(h_1(x_{i+1})-\xi_i\right)\int_c^d w_2(s)f(x_{i+1},s)ds+\sum_{i=0}^{n-1}\left(\xi_i-h_1(x_i)\right)\int_c^d w_2(s)f(x_i,s)ds\\
&-\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(h_1(x_{i+1})-\xi_i\right)\left(h_2(y_{j+1})-\eta_j\right)f(x_{i+1},y_{j+1})\\
&-\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(h_1(x_{i+1})-\xi_i\right)\left(\eta_j-h_2(y_j)\right)f(x_{i+1},y_j)\\
&-\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(\xi_i-h_1(x_i)\right)\left(h_2(y_{j+1})-\eta_j\right)f(x_i,y_{j+1})\\
&-\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(\xi_i-h_1(x_i)\right)\left(\eta_j-h_2(y_j)\right)f(x_i,y_j).
\end{aligned}
$$

Theorem 4.1. *Let f defined as in Theorem 3.1 and let*

$$
\begin{aligned}
&\int_a^b\int_c^d w_1(t)w_2(s)f(t,s)dsdt\\
=&A(f,w_1,h_1,w_2,h_2,I_n,J_m,\xi,\eta)+R(f,w_1,h_1,w_2,h_2,I_n,J_m,\xi,\eta),
\end{aligned}
$$

where the remainder term $R(f,w_1,h_1,w_2,h_2,I_n,J_m,\xi,\eta)$ satisfies

$$
\begin{aligned}
&\left|R(f,w_1,h_1,w_2,h_2,I_n,J_m,\xi,\eta)\right| \\
\leq&\left[\frac{1}{2}\upsilon(W_1)+\max_{0\leq i\leq n}\left|\xi_i-\frac{h_1(x_i)+h_1(x_{i+1})}{2}\right|\right]\\
&\times\left[\frac{1}{2}\upsilon(W_2)+\max_{0\leq j\leq m}\left|\eta_j-\frac{h_2(y_j)+h_2(y_{j+1})}{2}\right|\right]\bigvee_a^b\bigvee_c^d(f)\\
\leq&\upsilon(W_1)\upsilon(W_2)\bigvee_a^b\bigvee_c^d(f).
\end{aligned}\tag{4.2}
$$

Proof. Appliying Theorem 3.1 to the bidimentional interval $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, we have

$$
\begin{aligned}
&|\left(h_1(x_{i+1})-\xi_i\right)\left(h_2(y_{j+1})-\eta_j\right)f(x_{i+1},y_{j+1})\\
&+\left(h_1(x_{i+1})-\xi_i\right)\left(\eta_j-h_2(y_j)\right)f(x_{i+1},y_j)\\
&+\left(\xi_i-h_1(x_i)\right)\left(h_2(y_{j+1})-\eta_j\right)f(x_i,y_{j+1})\\
&+\left(\xi_i-h_1(x_i)\right)\left(\eta_j-h_2(y_j)\right)f(x_i,y_j)\\
&-\left(h_2(y_{j+1})-\eta_j\right)\int_{x_i}^{x_{i+1}}w_1(t)f(t,y_{j+1})dt-\left(\eta_j-h_2(y_j)\right)\int_{x_i}^{x_{i+1}}w_1(t)f(t,y_{j+1})dt
\end{aligned}\tag{4.3}
$$

$$
\begin{aligned}
&- (h_1(x_{i+1}) - \xi_i) \int\limits_{y_j}^{y_{j+1}} w_2(s) f(x_{i+1}, s) ds - (\xi_i - h_1(x_i)) \int\limits_{y_j}^{y_{j+1}} w_2(s) f(x_i, s) ds \\
&+ \left. \int\limits_{x_i}^{x_{i+1}} \int\limits_{y_j}^{y_{j+1}} f(t,s) ds dt \right| \\
\le &\left[\frac{1}{2} W_1^i + \left| \xi_i - \frac{h_1(x_i) + h_1(x_{i+1})}{2} \right| \right] \\
&\times \left[\frac{1}{2} W_2^j + \left| \eta_j - \frac{h_2(y_j) + h_2(y_{j+1})}{2} \right| \right] \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f),
\end{aligned}
$$

for any $\xi_i \in [h_1(x_i), h_1(x_{i+1})]$, $i = 0, \dots, n-1$ and $\eta_j \in [h_2(y_j), h_2(y_{j+1})]$, $j = 0, \dots, m-1$.

Summing the inequality (4.3) over i from 0 to $n-1$ and j from 0 to $m-1$ and using the generalized triangle inequality, we get

$$
\begin{aligned}
&|R(f, w_1, h_1, w_2, h_2, I_n, J_m, \xi, \eta)| \\
&\le \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[\frac{1}{2} W_1^i + \left| \xi_i - \frac{h_1(x_i) + h_1(x_{i+1})}{2} \right| \right] \\
&\quad \times \left[\frac{1}{2} W_2^j + \left| \eta_j - \frac{h_2(y_j) + h_2(y_{j+1})}{2} \right| \right] \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f) \\
&\le \max_{0 \le i \le n} \left[\frac{1}{2} W_1^i + \left| \xi_i - \frac{h_1(x_i) + h_1(x_{i+1})}{2} \right| \right] \\
&\quad \times \max_{0 \le j \le m} \left[\frac{1}{2} W_2^j + \left| \eta_j - \frac{h_2(y_j) + h_2(y_{j+1})}{2} \right| \right] \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f),
\end{aligned} \tag{4.4}
$$

which finishes the proof of the first inequality in (4.2).

In the last inequality in (4.4), we have

$$
\left| h_1(\xi_i) - \frac{h_1(x_i) + h_1(x_{i+1})}{2} \right| \le \frac{1}{2} W_1^i, \quad i = 0, 1, \dots, n-1,
$$

and so,

$$
\max_{i=0,\dots,n-1} \left| h_1(\xi_i) - \frac{h_1(x_i) + h_1(x_{i+1})}{2} \right| \le \frac{1}{2} \upsilon(W_1).
$$

Similarly, we get

$$
\max_{j=0,\dots,m-1} \left| h_2(\eta_j) - \frac{h_2(y_j) + h_2(y_{j+1})}{2} \right| \le \frac{1}{2} \upsilon(W_2).
$$

The proof of the theorem is completely completed. □

Remark 4.1. If we choose $w_1(t) \equiv 1$, $h_1(t) = t$ on $[a, b]$ and $w_2(s) \equiv 1$, $h_2(s) = s$ on $[c, d]$ in Theorem 4.1, then the inequalities (4.2) reduce to the inequalities (4.2) in [8].

Corollary 4.1. *Under the assumption of Theorem 4.1 with $\xi_i = h_1(x_{i+1})$ and $\eta_j = h_2(y_{j+1})$, we get*

$$\int_a^b \int_c^d w_1(t)w_2(s)f(t,s)dsdt = A_L(f,w_1,h_1,w_2,h_2,I_n,J_m)+R_L(f,w_1,h_1,w_2,h_2,I_n,J_m),$$

where $A_L(f,w_1,h_1,w_2,h_2,I_n,J_m)$ is built from the weighted left rectangle rule

$$\begin{aligned}&A_L(f,w_1,h_1,w_2,h_2,I_n,J_m)\\&=\sum_{j=0}^{m-1} W_2^i \int_a^b w_1(t)f(t,y_j)dt+\sum_{i=0}^{n-1} W_1^i \int_c^d w_2(s)f(x_i,s)ds\\&-\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_i,y_j)W_1^iW_2^j\end{aligned}$$

and remainder term $R_L(f,w_1,h_1,w_2,h_2,I_n,J_m)$ satisfies

$$|R_L(f,w_1,h_1,w_2,h_2,I_n,J_m)| \le \upsilon(W_1)\upsilon(W_2)\bigvee_a^b\bigvee_c^d(f).$$

Corollary 4.2. *Under the assumption of Theorem 4.1 with $\xi_i = h_1(x_i)$ and $\eta_j = h_2(y_j)$, we have*

$$\int_a^b \int_c^d w_1(t)w_2(s)f(t,s)dsdt = A_R(f,w_1,h_1,w_2,h_2,I_n,J_m)+R_R(f,w_1,h_1,w_2,h_2,I_n,J_m),$$

where $A_R(f,w_1,h_1,w_2,h_2,I_n,J_m)$ is constructed from the weighted right rectangle rule

$$\begin{aligned}&A_R(f,w_1,h_1,w_2,h_2,I_n,J_m)\\&=\sum_{j=0}^{m-1} W_2^i \int_a^b w_1(t)f(t,y_{j+1})dt+\sum_{i=0}^{n-1} W_1^i \int_c^d w_2(s)f(x_{i+1},s)ds\\&-\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_{i+1},y_{j+1})W_1^iW_2^j\end{aligned}$$

and remainder term satisfies

$$|R_L(f,w_1,h_1,w_2,h_2,I_n,J_m)| \le \upsilon(W_1)\upsilon(W_2)\bigvee_a^b\bigvee_c^d(f).$$

Corollary 4.3. *Under the assumption of Theorem 4.1 with $\xi_i = \frac{h_1(x_i)+h_1(x_{i+1})}{2}$ and $\eta_j = \frac{h_2(y_j)+h_2(y_{j+1})}{2}$ then*

$$\int_a^b \int_c^d w_1(t)w_2(s)f(t,s)dsdt = A_T(f,w_1,h_1,w_2,h_2,I_n,J_m)+R_T(f,w_1,h_1,w_2,h_2,I_n,J_m),$$

where $A_T(f, w_1, h_1, w_2, h_2, I_n, J_m)$ *is constructed from the weighted trapezoid rule*

$$\begin{aligned}
&A_T(f, w_1, h_1, w_2, h_2, I_n, J_m)\\
=&\frac{1}{2}\sum_{j=0}^{m-1} W_2^i \int_a^b w_1(t)\left[f(t, y_j) + f(t, y_{j+1})\right] dt\\
&+ \frac{1}{2}\sum_{i=0}^{n-1} W_1^i \int_c^d w_2(s)\left[f(x_i, s) + f(x_{i+1}, s)\right] ds\\
&- \frac{1}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f(x_{i+1}, y_{j+1}) + f(x_{i+1}, y_j) + f(x_i, y_{j+1}) + f(x_i, y_j)\right] W_1^i W_2^j
\end{aligned}$$

and remainder term satisfies

$$|R_T(f, w_1, h_1, w_2, h_2, I_n, J_m)| \leq \frac{1}{4}\upsilon(W_1)\upsilon(W_2)\bigvee_a^b\bigvee_c^d(f).$$

The constant $\frac{1}{4}$ *is the best possible.*

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