Kragujevac Journal of Mathematics
Volume 50(9) (2026), Pages 1445–1465.

THE SOLVABILITY OF p-KIRCHHOFF TYPE PROBLEMS WITH CRITICAL EXPONENT

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Abstract. The article aims to study new and current problems in the theory of nonclassical partial differential equations and their applications, proving the existence and nonexistence of solutions to p-Kirchhoff type problems with critical exponent of Sobolev in $\mathbb{R}^n$, which are of great interest in the study of mathematical physics equations. We show the existence of a local minimizer with negative/positive energy by using variational methods. More precisely, we considered a minimization of E_λ constrained in a neighborhood of zero using the Ekeland variational principle, then we found the first critical point of E_λ which achieves the local minimum of E_λ whose level is negative; next around the zero point, using the mountain pass theorem, we also obtained a critical point whose level is positive. In addition, we studied the case of $\lambda = 0$, where there is no non-trivial solution using the contradiction principle. We also established infinite solutions and discuss the different cases.

1. Introduction and Position of Problem

In $\mathbb{R}^n$, we are concerned with the following problem

$$-\left(a\left(\int_{\mathbb{R}^n} |\nabla u|^p \, dx\right)^{\theta-1} + b\right) \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = |u|^{p^*-2}u + \lambda f(x), \tag{1.1}$$

where $1 < p < n$, $0 \leq a, b$, $0 < a + b$, $\theta > 1$, λ is a parameter, $p^* = pn/(n-p)$ is the critical Sobolev exponent and $f \in W^* \setminus \{0\}$. Here, W^* is the dual space of $W^{1,p}(\mathbb{R}^n)$.

Key words and phrases. Variational methods, Kirchhoff equations, critical Sobolev exponent, Lyapunov dunctions, nonlinear equations.

2020 *Mathematics Subject Classification.* Primary: 35J20. Secondary: 35IJ60, 47J30.

DOI

Received: April 25, 2024.

Accepted: January 27, 2025.

Since equation (1.1) contains an integral over $\mathbb{R}^n$, it is no longer a point-wise identity. Therefore, it is often called a nonlocal problem. It is also called non-degenerate if $0 < b$ and $a \geq 0$, while it is degenerate if $b = 0$ and $0 < a$.

The non-local elliptic problem (1.1) is related to the original Kirchhoff equation in [1] which was first introduced by Kirchhoff as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the strings produced by transverse vibrations.

Problems involving non-local operators have been widely studied due to their numerous and relevant applications in various fields of science. In particular, Kirchhoff-type problems proved to be valuable tools for modeling several physical and biological phenomena, and many works have been made to ensure the existence of solutions for such problems. We quote in particular the article of Lions [2]. Since this famous paper, very fruitful developments are given rise to many works in this promising direction, and in most of them, with most relying on topological methods. However, only a few improvements have been made concerning the multiplicity of solutions. In this regard, the variational approach was sought instead of topological methods to solve these kinds of problems and also to prove the existence of multiple solutions (see [3, 4]).

In the last few years, great attention has been paid to the study of Kirchhoff problems involving critical nonlinearities. This problem creates many difficulties in applying variational methods. It is worth mentioning that the semilinear Laplace equation of elliptic type involving the the critical Sobolev exponent was investigated in the crucial paper of Brézis and Nirenberg [5]. After that, many researchers have dedicated themselves to the study of several kinds of elliptic equations with a critical exponent in a bounded domain or in the whole space. For $p = 2$ and $a = 0$, Tarantello [6] treated the problem (1.1) in a bounded domain of $\mathbb{R}^n$ and proved the existence of at least two solutions using Nehari manifold methods. The first work on the Kirchhoff-type problem with the critical Sobolev exponent was by Alves, Corrêa and Figueiredo in [7]. Naimen in [8] showed a Brézis-Nirenberg type result for the Kirchhoff problem in a bounded domain. In [9], He et al. considered the following problem

$$
\begin{aligned}
&-\left(a \int_{\mathbb{R}^n} |\nabla u|^p \, dx + b\right) \operatorname{div} \left(|\nabla u|^{p-2} \nabla u\right) = f(u) + h \text{ in } \Omega, \\
&u = 0 \text{ on } \partial\Omega,
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain, $0 \leq h \in L^2(\Omega)$ and $f \in C(\mathbb{R}, \mathbb{R})$. They obtained the existence of at least one or two positive solutions using the monotonicity technique and the non-existence criterion by using the corresponding Pohožaev identity. Also, they showed nonexistence properties for the 3-sublinear case and the critical case. Under general assumptions on the nonlinearity, they also established the existence result for the whole space $\mathbb{R}^3$ by using the properties of the Pohožaev identity and some delicate analysis.

In [10], Miyagaki et al. obtained the existence of infinitely many solutions for the following problem

$$-\left(a \int_{\mathbb{R}^n} |\nabla u|^p \, dx + b\right) \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = \lambda g(x,u) + \mu |u|^{p^*-2} u, \quad \text{in } \mathbb{R}^n,$$

where $\lambda > 0$, $0 < \mu < \mu_*$, with μ_* a positive constant, $n \geq 2p$ and $g(x,u)$ satisfies certain subcritical growth conditions.

Ke et al. in [11] considered the problem (1.1) with $\theta = 2$ and $f \in L^{\frac{p^*}{p^*-1}}(\mathbb{R}^n)$. They obtained the existence of infinitely many solutions for problem (1.1) and the multiplicity of solutions for the non-degenerate case $(0 < b)$ with $p^* = 2p$ and $0 < a < S^{-2}$ and the degenerate case $(b = 0)$ with $p^* > 2p$ and $0 < a$.

Recently, Benaissa et al. in [12] discussed the problem $(\mathcal{P}_\lambda)$ with $\theta = 2$. When $p^* > 2p$, the authors showed the existence of $\lambda_* > 0$ such that for $0 < \lambda < \lambda_*$ (1.1) has at least two solutions in a particular dimension $n = 3p/2$.

The main results in the present paper can be considered as an extension of the work of [11] and [12] for a more general non-local problem (degenerate or non-degenerate problem) with a large range of n.

To our knowledge, many of the results are new for $p > 1$ and even in the case $\theta = 2$. Our results and setting are more general and delicate; it is not difficult to obtain the second solution in the degenerate case where $\theta < \frac{p^*}{p}$.

In the case where $a = \lambda = 0$, $b = 1$ and $1 < p < n$, our main problem can be reduced to the following problem

$$-\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = |u|^{p^*-2} u, \quad \text{in } \mathbb{R}^n. \tag{1.2}$$

Sciunzi in [13] provided that if u is a positive solution of (1.2) then $u(x) = v_{\varepsilon,x_0}(x)$ where

$$v_{\varepsilon,x_0}(x) := \left[\frac{\varepsilon^{\frac{1}{p-1}} n^{\frac{1}{p}} \left(\frac{n-p}{p-1}\right)^{\frac{p-1}{p}}}{\varepsilon^{\frac{p}{p-1}} + |x - x_0|^{\frac{p}{p-1}}}\right]^{\frac{n-p}{p}}, \quad \varepsilon > 0,\ x_0 \in \mathbb{R}^n. \tag{1.3}$$

Consequently,

$$S := \inf_{u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\}} \frac{\|u\|^p}{\left(\int_{\mathbb{R}^n} |u|^{p^*} \, dx\right)^{p/p^*}}$$

satisfies

$$\|v_{\varepsilon,x_0}\|^p = \int_{\mathbb{R}^n} |v_{\varepsilon,x_0}|^{p^*} \, dx = S^{\frac{p^*}{p^*-p}}. \tag{1.4}$$

For more details, see [14].

This paper is composed of four sections in addition to the introduction and the conclusion with discussion. In Section 2, we give some abstract conditions when the functional E_λ satisfies the Palais Smale condition, so to overcome the lack of compactness, we need to determine a good level of the Palais Smale condition. We state and prove our first main contribution results regarding the existence of local minimizer with negative/positive energy (Theorem 3.1 and Theorem 3.2) in Section 3, by using variational methods. More precisely, we consider a minimization of E_λ constrained in a neighborhood of zero using the Ekeland variational principle (see [15]). Then we can find the first critical point of E_λ that reaches the local minimum of E_λ whose level is negative, next around the zero point, using the mountain pass theorem (see [16]). We also obtain a critical point whose level is positive. The proof of the second results (Theorem 4.1 and Theorem 4.2) is given and proved in Section 4, where we study the case of $\lambda = 0$.

2. Preliminaries and Tools

In this section, we state several preliminary results needed.

First, we make use of the following assumptions:

$(\mathcal{H}_0)$ $1 < \theta < \frac{p^*}{p}$, $0 < a$ and $0 < b$;

$(\mathcal{H}_1)$ $1 < \theta < \frac{p^*}{p}$, $a \geq 0$, $0 \leq b$ and $a + 0 < b$;

$(\mathcal{H}_2)$ $\theta = \frac{p^*}{p}$, $a \geq 0$ and $0 < b$;

$(\mathcal{H}_3)$ $\theta = \frac{p^*}{p}$, $0 \leq a < S^{-\theta}$ and $0 < b$;

$(\mathcal{H}_4)$ $\theta = \frac{p^*}{p}$, $a > S^{-\theta}$ and $b = 0$;

$(\mathcal{H}_5)$ $\theta = \frac{p^*}{p}$, $a \geq S^{-\theta}$ and $0 < b$;

$(\mathcal{H}_6)$ $\theta = \frac{p^*}{p}$, $0 < a$ and $b = 0$;

$(\mathcal{H}_7)$ $\theta > \frac{p^*}{p}$, $0 < a$ and $b = b^*$, where

$$b^* = \frac{\theta p - p^*}{(\theta - 1)\, p} \left(\frac{(\theta - 1)\, p}{p^* - p} a \right)^{-\frac{p^* - p}{\theta p - p^*}} S^{-\frac{(\theta-1)p^*}{\theta p - p^*}};$$

$(\mathcal{H}_8)$ $\frac{p^*}{p} < \theta$, $0 < a$ and $b > b^*$;

$(\mathcal{H}_9)$ $\theta > \frac{p^*}{p}$, $0 < a$ and $b < b^*$.

The Sobolev space $W^{1,p}(\mathbb{R}^n)$ is the space of measurable functions $u : \mathbb{R}^n \to \mathbb{R}^n$ such that u and the distributional gradient $\nabla u = (\partial_1 u, \ldots, \partial_n u)$ are in Ł$^p(\mathbb{R}^n)$.

Definition 2.1. Let $c \in \mathbb{R}$. A sequence $(u_n) \subset W^{1,p}(\mathbb{R}^n)$ is a Palais Smale sequence at level c, so called $(PS)_c$ sequence, if

$$E_\lambda(u_n) \to c \quad \text{and} \quad E'_\lambda(u_n) \to 0.$$

Here, E_λ verifies the $(PS)_c$ condition at level c if any $(PS)_c$ sequence has a convergent subsequence in $W^{1,p}(\mathbb{R}^n)$.

Lemma 2.1. *Suppose that $f \in W^* \setminus \{0\}$ and assume that $(\mathcal{H}_1)$ or $(\mathcal{H}_2)$ holds. Let $c \in \mathbb{R}$ and $(u_n) \subset W^{1,p}(\mathbb{R}^n)$ be a $(PS)_c$ sequence for E_λ. Then,*

$$u_n \rightharpoonup u, \quad \text{in } W^{1,p}(\mathbb{R}^n),$$

for some $u \in W^{1,p}(\mathbb{R}^n)$ with $E'_\lambda(u) = 0$.

Proof. We have

$$E_\lambda(u_n) \to c \quad \text{and} \quad E'_\lambda(u_n) \to 0.$$

Then,

$$c + o_n(1) = E_\lambda(u_n) \quad \text{and} \quad o_n(1)\|v\| = \langle E'_\lambda(u_n), v\rangle,$$

for all $v \in W^{1,p}(\mathbb{R}^n)$, where $o_n(1)$ denotes any quantity that tends to zero as $n \to +\infty$.

Since $n \to +\infty$, we have

$$\begin{aligned} c + o_n(1) - \frac{1}{p^*} o_n(1)\|u_n\| &= E_\lambda(u_n) - \frac{1}{p^*}\langle E'_\lambda(u_n), u_n\rangle \\ &= a\frac{p^* - \theta p}{\theta p p^*}\|u_n\|^{\theta p} + b\frac{p^* - p}{pp^*}\|u_n\|^p - \lambda\frac{p^* - 1}{p^*}\int_{\mathbb{R}^n} f(x)\, u_n dx \\ &\geq a\frac{p^* - \theta p}{\theta p p^*}\|u_n\|^{\theta p} + b\frac{p^* - p}{pp^*}\|u_n\|^p - \lambda\frac{p^* - 1}{p^*}\|f\|_{W^*}\|u_n\|, \end{aligned}$$

that is, (u_n) is bounded in $W^{1,p}(\mathbb{R}^n)$ if $(\mathcal{H}_1)$ or $(\mathcal{H}_2)$ holds. Up to a subsequence if necessary, there exists a function $u \in W^{1,p}(\mathbb{R}^n)$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^n)$ and in $L^{p^*}(\mathbb{R}^n)$, $u_n \to u$ a.e. in $\mathbb{R}^n$, and$\int_{\mathbb{R}^n} f(x)\, u_n dx \to \int_{\mathbb{R}^n} f(x)\, u dx$. Then,

$$\langle E'_\lambda(u_n), v\rangle = 0, \quad \text{for all } v \in C_0^\infty(\mathbb{R}^n).$$

Thus, $E'_\lambda(u) = 0$. This completes the proof. □

In order to introduce the local Palais Smale condition, we must state the following lemma, which can be considered a key step in obtaining a solution with positive energy (a mountain pass-type solution).

Lemma 2.2. *Let $\theta > 1$, $0 \leq a, b$, $0 < a + b$, $\sigma \geq 0$ and $\tilde{x} = \left(\frac{\sigma}{a}S^{-\theta}\right)^{\frac{1}{1-\sigma}}$ for $\sigma \neq 1$. For $x \geq 0$, let us consider the function $\Psi : \mathbb{R}^+ \to \mathbb{R}^*$, where*

$$\Psi(x) = S^{-1}x^\sigma - aS^{\theta-1}x - b.$$

Then, the following hold.

(1) *If $\sigma = 1$, $S^{-\theta} > a \geq 0$ and $0 < b$, then the equation $\Psi(x) = 0$ has a unique positive solution such that*

$$x_1 = \frac{b}{(S^{-\theta} - a)\, S^{\theta-1}},$$

and $\Psi(x) \geq 0$ for all $x \geq x_1$.

(2) *If $\sigma > 1$, the equation $\Psi(x) = 0$ has a unique positive solution $x_2 > \tilde{x}$ and $\Psi(x) \geq 0$ for all $x \geq x_2$.*

(3) *If $\sigma < 1$, then we have the following cases.*

i) If $\Psi(\tilde{x}) < 0$, then we have $\Psi(x) \neq 0$ for all $x \geq 0$.

ii) If $\Psi(\tilde{x}) = 0$, then we have

$$b = S^{-1}(1 - \sigma)\left(\frac{\sigma}{a}S^{-\theta}\right)^{\frac{\sigma}{1-\sigma}},$$

and $\Psi(x) \neq 0$ for all $x \neq \tilde{x}$.

iii) If $\Psi(\tilde{x}) > 0$, then we have

$$b < S^{-1}(1-\sigma)\left(\frac{\sigma}{a}S^{-\theta}\right)^{\frac{\sigma}{1-\sigma}},$$

and $\Psi(x) = 0$ for two different points.

Proof. (1) For $\sigma = 1$, $S^{-\theta} > a \geq 0$ and $0 < b$, we have

$$\Psi(x) = S^{\theta-1}\left(S^{-\theta} - a\right)x - b.$$

That is, equation $\Psi(x) = 0$ has a unique positive solution

$$x_1 = \frac{b}{(S^{-\theta} - a)\, S^{\theta-1}},$$

and $\Psi(x) \geq 0$ for all $x \geq x_1$.

(2) For $\sigma > 1$, we have $\Psi'(x) = \sigma S^{-1}x^{\sigma-1} - aS^{\theta-1}$ and

$$\Psi''(x) = \sigma(\sigma-1)\, S^{-1}x^{\sigma-2} > 0, \quad \text{for all } x > 0.$$

Then, $\Psi'(\tilde{x}) = 0$, $\Psi'(x) < 0$ for $y < \tilde{x}$ and $\Psi'(y) > 0$ for $x > \tilde{x}$. Hence, Ψ is a concave function and

$$\min_{x\geq 0}\Psi(x) = \Psi(\tilde{x}) = -(\sigma-1)\, S^{-1}\left(\frac{a}{\sigma}S^{\theta}\right)^{\frac{\sigma}{\sigma-1}} < 0. \tag{2.1}$$

Moreover, we have $\Psi(\tilde{x}) < 0$ and $\lim_{x\to+\infty}\Psi(x) = +\infty$. Thus, from (2.1) and the concavity of Ψ, we can conclude that the equation $\Psi(x) = 0$ has a unique positive solution $x_2 > \tilde{x}$ and $\Psi(x) \geq 0$ for all $x \geq x_2$.

(3) For $\sigma < 1$, we have $\Psi'(\tilde{x}) = 0$, Ψ is increasing for $0 < x < \tilde{x}$ and Ψ is decreasing when $x > \tilde{x}$. Moreover, from $\Psi(0) = -b < 0$, we obtain *i*), *ii*) and *iii*). □

3. Existence of Local Minimizer with Negative/Positive Energy

3.1. With negative energy.

Definition 3.1. We say that $u \in W^{1,p}(\mathbb{R}^n)\setminus\{0\}$ is a weak solution of equation (1.1) if

$$\left(a\,\|u\|^{(\theta-1)p} + b\right)\int_{\mathbb{R}^n}|\nabla u|^{p-2}\nabla v dx - \int_{\mathbb{R}^n}\left(|u|^{p^*-2}u - \lambda f(x)\right)v dx = 0,$$

for any $v \in W^{1,p}(\mathbb{R}^n)$.

Next, we define the energy functional.

$$E_\lambda(u) = \frac{a}{\theta p}\|u\|^{\theta p} + \frac{b}{p}\|u\|^{p} - \frac{1}{p^*}\int_{\mathbb{R}^n}|u|^{p^*}dx - \lambda\int_{\mathbb{R}^n}f(x)\, u dx,$$

associated to problem (1.1), for all $u \in W^{1,p}(\mathbb{R}^n)$.

Notice that the functional E_λ is well defined in $W^{1,p}(\mathbb{R}^n)$, and belongs to

$$C^1\left(W^{1,p}(\mathbb{R}^n), \mathbb{R}\right),$$

and a critical point of E_λ is a weak solution of the problem (1.1). When $\lambda > 0$, we have the following result.

Theorem 3.1. *Suppose that $f \in W^*\setminus\{0\}$ and assume $(\mathcal{H}_1)$ or $(\mathcal{H}_2)$ holds. Then, there exist constants $\lambda_- > 0$ such that for any $\lambda \in (0, \lambda_-)$ problem (1.1) has a solution u_- with negative energy.*

Remark 3.1. If $\theta > \frac{p^*}{p}$, $a \geq 0$, $0 \leq b$ and $0 < a + b$ or $\theta = \frac{p^*}{p}$, $a = S^{-\theta}$ and $0 < b$ or $\theta = \frac{p^*}{p}$, $a > S^{-\theta}$ and $0 \leq b$, then for any $\lambda > 0$, we can easily show the existence of one solution which is a ground state solution.

Using the Ekeland variational principle, we now provide the proof of our first main result stated in Theorem 3.1.

Proof of Theorem 3.1. Let $u \in W^{1,p}(\mathbb{R}^n)\setminus\{0\}$, $0 < b$, $a \geq 0$ and $\theta \leq \frac{p^*}{p}$. By applying Hölder's and Sobolev inequalities, we have

$$\begin{aligned} E_\lambda(u) &= \frac{a}{\theta p}\|u\|^{\theta p} + \frac{b}{p}\|u\|^p - \frac{1}{p^*}\int_{\mathbb{R}^n}|u|^{p^*}dx - \lambda\int_{\mathbb{R}^n} f(x)\, u dx \\ &\geq \frac{b}{p}\|u\|^p + \frac{a}{\theta p}\|u\|^{\theta p} - \frac{S^{-p^*/p}}{p^*}\|u\|^{p^*} - \lambda\|f\|_{W^*}\|u\|. \end{aligned}$$

Now, we divide the proof into two cases.

Firstly, assume $0 < b$ and $a \geq 0$. If $(\mathcal{H}_1)$ or $(\mathcal{H}_2)$ holds, we get

$$E_\lambda(u) \geq \frac{b}{p}\|u\|^p - \frac{S^{-p^*/p}}{p^*}\|u\|^{p^*} - \left(\frac{b}{2}\right)^{\frac{-1}{p}}\lambda\|f\|_{W^*}\left(\frac{b}{2}\right)^{\frac{1}{p}}\|u\|.$$

It follows from the inequality $Xx \leq \frac{X^q}{q} + \frac{x^{q'}}{q'}$ for any X, $x \geq 0$ and $q, q' > 0$, with $\frac{1}{q} + \frac{1}{q'} = 1$, that

$$\begin{aligned} E_\lambda(u) &\geq \frac{b}{p}\|u\|^p - \frac{S^{-p^*/p}}{p^*}\|u\|^{p^*} - \frac{p-1}{p}\left(\left(\frac{b}{2}\right)^{\frac{-1}{p}}\lambda\|f\|_{W^*}\right)^{\frac{p}{p-1}} - \frac{1}{p}\left(\left(\frac{b}{2}\right)^{\frac{1}{p}}\|u\|\right)^p \\ &\geq \frac{b}{2p}\|u\|^p - \frac{S^{-p^*/p}}{p^*}\|u\|^{p^*} - \frac{p-1}{p}\left(\left(\frac{b}{2}\right)^{\frac{-1}{p}}\lambda\|f\|_{W^*}\right)^{\frac{p}{p-1}}. \end{aligned}$$

For $\rho \geq 0$, let us consider the function $h_1 \in C(\mathbb{R}^+, \mathbb{R}^*)$, given by

$$h_1(\rho) = \frac{b}{2p}\rho^p - \frac{S^{-p^*/p}}{p^*}\rho^{p^*}.$$

The direct calculation shows that

$$\max_{\rho\geq 0} h_1(\rho) = h_1(\rho_1) = \frac{p^*-p}{pp^*}S^{\frac{p^*}{p^*-p}}\left(\frac{b}{2}\right)^{\frac{p^*}{p^*-p}}, \quad \text{with } \rho_1 = \left[\frac{b}{2}S^{p^*/p}\right]^{\frac{1}{p^*-p}},$$

and $h_1(\rho) \geq 0$ for all $\rho \in B_{\rho_1}(0)$. Consequently,

$$E_\lambda(u)|_{B_{\rho_1(0)}} \geq -\frac{p-1}{p}\left(\left(\frac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_{W^*}\right)^{\frac{p}{p-1}}. \tag{3.1}$$

Moreover, for $\|u\| = \rho_1$ we have

$$\begin{aligned} E_\lambda(u) &\geq h_1(\rho_1) - \frac{p-1}{p}\left(\left(\frac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_{W^*}\right)^{\frac{p}{p-1}} \\ &\geq \frac{1}{p}h_1(\rho_1) + \frac{p-1}{p}h_1(\rho_1) - \frac{p-1}{p}\left(\left(\frac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_{W^*}\right)^{\frac{p}{p-1}} \geq \frac{1}{p}h_1(\rho_1) \\ &= \delta_1, \end{aligned}$$

for all $\lambda \in (0, \lambda_1)$, with

$$\lambda_1 = \left(\frac{p^*-p}{pp^*}S^{\frac{p^*}{p^*-p}}\right)^{\frac{p-1}{p}} \|f\|_{W^*}^{-1} \left(\frac{b}{2}\right)^{\frac{p^*-1}{p^*-p}}.$$

We turn to the case where $0 < a$ and $0 \leq b$. If $(\mathcal{H}_1)$ holds, we obtain

$$\begin{aligned} E_\lambda(u) \geq& \frac{a}{\theta p}\|u\|^{\theta p} - \frac{S^{-p^*/p}}{p^*}\|u\|^{p^*} - \left(\left(\frac{a}{2}\right)^{\frac{-1}{\theta p}} \lambda \|f\|_{W^*}\right)\left(\left(\frac{a}{2}\right)^{\frac{1}{\theta p}} \|u\|\right) \\ \geq& \frac{a}{\theta p}\|u\|^{\theta p} - \frac{S^{-p^*/p}}{p^*}\|u\|^{p^*} - \frac{\theta p-1}{\theta p}\left(\left(\frac{a}{2}\right)^{\frac{-1}{\theta p}} \lambda \|f\|_{W^*}\right)^{\frac{\theta p}{\theta p-1}} - \frac{1}{\theta p}\left(\left(\frac{a}{2}\right)^{\frac{1}{\theta p}} \|u\|\right)^{\theta p} \\ \geq& \frac{a}{2\theta p}\|u\|^{\theta p} - \frac{S^{-p^*/p}}{p^*}\rho^{p^*} - \frac{\theta p-1}{\theta p}\left(\left(\frac{a}{2}\right)^{\frac{-1}{\theta p}} \lambda \|f\|_{W^*}\right)^{\frac{\theta p}{\theta p-1}}. \end{aligned}$$

Now, we define the function $h_2 \in (\mathbb{R}^+, \mathbb{R}^*)$ as follows

$$h_2(\rho) = \frac{a}{2\theta p}\rho^{\theta p} - \frac{S^{-p^*/p}}{p^*}\rho^{p^*}.$$

Then,

$$\max_{\rho \geq 0} h_2(\rho) = h_2(\rho_2) = \left(\frac{1}{\theta p} - \frac{1}{p^*}\right) S^{-p^*/p}\left[\frac{a}{2}S^{p^*/p}\right]^{\frac{p^*}{p^*-\theta p}}, \quad \text{with } \rho_2 = \left[\frac{a}{2}S^{p^*/p}\right]^{\frac{1}{p^*-\theta p}},$$

and $h_2(\rho) \geq 0$ for all $\rho \in B_{\rho_2}(0)$. Consequently,

$$E_\lambda(u)|_{B_{\rho_2}(0)} \geq -\frac{\theta p-1}{\theta p}\left(\left(\frac{a}{2}\right)^{\frac{-1}{\theta p}} \lambda \|f\|_{W^*}\right)^{\frac{\theta p}{\theta p-1}}.$$

Moreover, for $\|u\| = \rho_2$ we have

$$\begin{aligned} E_\lambda(u) &\geq h_2(\rho_2) - \frac{\theta p - 1}{\theta p}\left(\left(\frac{a}{2}\right)^{\frac{-1}{\theta p}} \lambda \|f\|_{W^*}\right)^{\frac{\theta p}{\theta p - 1}} \\ &\geq \frac{\theta p - 1}{\theta p} h_2(\rho_2) + \frac{1}{\theta p} h_2(\rho_2) - \frac{\theta p - 1}{\theta p}\left(\left(\frac{a}{2}\right)^{\frac{-1}{\theta p}} \lambda \|f\|_{W^*}\right)^{\frac{\theta p}{\theta p - 1}} \geq \frac{1}{\theta p} h_2(\rho_2) \\ &= \delta_2, \end{aligned}$$

for all $\lambda \in (0, \lambda_2)$, with

$$\lambda_2 = \left(\frac{p^* - \theta p}{\theta p p^*} S^{\frac{\theta p^*}{p^* - \theta p}}\right)^{\frac{\theta p - 1}{\theta p}} \left(\frac{a}{2}\right)^{\frac{p^* - 1}{p^* - \theta p}} \|f\|_{W^*}^{-1}.$$

We choose δ_*, ρ_* and λ_- such that

$$(\delta_*, \ \rho_*, \lambda_-) = \begin{cases} (\delta_1, \rho_1, \lambda_1), & \text{if } (\mathcal{H}_2) \text{ or } (\mathcal{H}_1) \text{ satisfies with } 0 < b, \\ (\delta_2, \rho_2, \lambda_2), & \text{if } (\mathcal{H}_1) \text{ satisfies with } 0 < a. \end{cases} \tag{3.2}$$

Then, for all $\lambda \in (0, \lambda_-)$, we have

$$E_\lambda(u)|_{\partial B_{\rho_*}(0)} \geq \delta_* \tag{3.3}$$

and

$$E_\lambda(u)|_{B_{\rho_*}(0)} \geq -C_\lambda, \tag{3.4}$$

with

(3.5)

$$C_\lambda := \begin{cases} \frac{p-1}{p}\left(\left(\frac{b}{2}\right)^{\frac{-1}{p}} \lambda \|f\|_{W^*}\right)^{\frac{p}{p-1}}, & \text{if } (\mathcal{H}_2) \text{ or } (\mathcal{H}_1) \text{ is satisfied with } 0 < b, \\ \frac{\theta p - 1}{\theta p}\left(\left(\frac{a}{2}\right)^{\frac{-1}{\theta p}} \lambda \|f\|_{W^*}\right)^{\frac{\theta p}{\theta p - 1}}, & \text{if } (\mathcal{H}_1) \text{ is satisfied with } 0 < a. \end{cases}$$

Now, we define

$$c_- = \inf\left\{E_\lambda(u), u \in \overline{B}_{\rho_*}(0)\right\}. \tag{3.6}$$

As $f \in W^* \backslash \{0\}$ we can choose $\varphi \in W^{1,p}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} f(x)\varphi dx > 0$. Then, for a fixed $\lambda \in (0, \lambda_-)$, there exists $t_0 > 0$ such that $\|t_0\varphi\| < \rho_*$ and

$$c_- \leq E_\lambda(t_0\varphi) < 0, \quad \text{for } t \in (0, t_0).$$

Hence, $c_- < E_\lambda(0) = 0$. Using the Ekeland variational principle, for the complete metric space $\overline{B}_{\rho_*}(0)$ with respect to the norm of $W^{1,p}(\mathbb{R}^n)$, we obtain the result that there exists a Palais Smale sequence $u_n \in \overline{B}_{\rho_*}(0)$ at level c_-. By Lemma 2.1, there exists $u_- \in \overline{B}_{\rho_*}(0)$ such that $u_n \rightharpoonup u_-$ in $W^{1,p}(\mathbb{R}^n)$ and $E'_\lambda(u_-) = 0$.

Now, we shall show that $u_n \to u_-$ in $W^{1,p}$. Suppose otherwise. Then, $\|u_-\| < \liminf_{n\to+\infty} \|u_n\|$, which implies that

$$\begin{aligned}
c_- &\leq E_\lambda(u_-) \\
&= E_\lambda(u_-) - \frac{1}{p^*}\langle E'_\lambda(u_-), u_-\rangle \\
&= a\frac{p^*-\theta p}{\theta p p^*}\|u_-\|^{\theta p} + b\frac{p^*-p}{pp^*}\|u_-\|^p - \lambda\frac{p^*-1}{p^*}\int_{\mathbb{R}^n} f(x)\, u_- dx \\
&< \liminf_{n\to+\infty}\left[a\frac{p^*-\theta p}{\theta p p^*}\|u_n\|^{\theta p} + b\frac{p^*-p}{pp^*}\|u_n\|^p - \lambda\frac{p^*-1}{p^*}\int_{\mathbb{R}^n} f(x)\, u_n dx\right] \\
&= \liminf_{n\to+\infty}\left[E_\lambda(u_n) - \frac{1}{p^*}\langle E'_\lambda(u_n), u_n\rangle\right] \\
&= c_-.
\end{aligned}$$

This is a contradiction. We conclude that $u_n \to u_-$ strongly in $W^{1,p}(\mathbb{R}^n)$. Therefore, $E'_\lambda(u_-) = 0$ and $E_\lambda(u_-) = c_- < 0 = E_\lambda(0)$. Hence, u_- is a nonzero solution of (1.1) with negative energy. □

3.2. **Existence of a mountain pass type solution.** The next theorem guarantees a second solution of (1.1) of mountain pass type.

Theorem 3.2. *Suppose that $f \in W^*\backslash\{0\}$ such that $\int_{\mathbb{R}^n} f(x)\, v_{\varepsilon,x_0} dx \neq 0$. Assume that $(\mathcal{H}_1)$ or $(\mathcal{H}_3)$ holds. Then, there exists a constant $\lambda_+ \in (0, \lambda_-]$ such that for any $\lambda \in (0, \lambda_+)$ the problem (1.1) has a second solution u_+ with positive energy.*

Notice that the assumption $\int_{\mathbb{R}^n} f(x)\, v_{\varepsilon,x_0} dx \neq 0$ certainly holds if $f \in W^*\backslash\{0\}$ does not change its sign. Also, we have $f \in L^{\frac{p^*}{p^*-1}}(\mathbb{R}^n)$ since $f \in W^*\backslash\{0\}$ and $u_-, u_+ \geq 0$ if $f \geq 0$. Furthermore, in Remark 1.2 [11], the authors mentioned that it is not easy to obtain the second solution for the case $p < n$, $0 < a$ and $0 < b$. For the special dimension $n = 3p/2$, this case is studied in [12]. In our work, we have overcome these difficulties and we give the results for general $\theta \in \left(1, \frac{p^*}{p}\right]$. Moreover, for $\theta = 2$, the condition $\theta \leq \frac{p^*}{p}$ is equivalent to $n \in]p, 2p]$ and $n \in \{3, 4\}$ for $p = 2$. Our results imply that suitable real θ can release the restriction on the spatial dimension n, for example. If $\theta = 1 + \varepsilon$ with $\varepsilon > 0$ is small enough, we have $\theta \leq \frac{p^*}{p}$ for a large range of n.

First, to ensure the local compactness of the Palais Smale sequence for E_λ, we have to prove an important result.

For $i \in \{1,2\}$, let x_i be defined as in Lemma 2.2 and let

$$
\begin{aligned}
C_i =& a\left(\frac{1}{\theta p}-\frac{1}{p^*}\right)\left(Sx_i^{\frac{1}{\theta-1}}\right)^{\theta}+b\left(\frac{1}{p}-\frac{1}{p^*}\right)Sx_i^{\frac{1}{\theta-1}},\\
C^* =& \begin{cases} C_1, & \text{if } \theta=\frac{p^*}{p},\ S^{-\theta}>a\geq 0,\ 0<b,\\ C_2, & \text{if } \theta<\frac{p^*}{p},\ a\geq 0,\ 0\leq b,\ a+0<b.\end{cases}
\end{aligned}
\tag{3.7}
$$

Lemma 3.1. *Suppose that $f \in W^*\backslash\{0\}$ and $(\mathcal{H}_1)$ or $(\mathcal{H}_3)$ hold. Let $(u_n) \subset W^{1,p}(\mathbb{R}^n)$ be a Palais Smale sequence for E_λ for some $c \in \mathbb{R}$. Then, either $u_n \to u$ or $c \geq E_\lambda(u) + C^*$.*

Proof. From the proof of Lemma 2.1, we have that (u_n) is a bounded sequence in $W^{1,p}(\mathbb{R}^n)$ and $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^n)$ for some $u \in W^{1,p}(\mathbb{R}^n)$ with $E'_\lambda(u) = 0$. Furthermore, if we write $v_n = u_n - u$, we derive

$$
\begin{cases} v_n \rightharpoonup 0 \text{ in } W^{1,p}(\mathbb{R}^n) \text{ and in } L^{p^*}(\mathbb{R}^n),\\ v_n \to 0 \text{ a.e. in } \mathbb{R}^n,\\ \int_{\mathbb{R}^n} f(x)\, v_n dx \to 0.\end{cases}
\tag{3.8}
$$

On the one hand, by using Brézis-Lieb's lemma [17], one has

$$
\begin{cases} \|u\|^p = \|v_n\|^p + \|u\|^p + o_n(1),\\ \int_{\mathbb{R}^n} |u_n|^{p^*}dx = \int_{\mathbb{R}^n} |v_n|^{p^*}dx + \int_{\mathbb{R}^n} |u|^{p^*}dx + o_n(1).\end{cases}
\tag{3.9}
$$

Since $\langle E'_\lambda(u), u\rangle = 0$, we obtain by (3.9) that

$$
o_n(1) = \langle E'_\lambda(u_n), u_n\rangle = \|v_n\|^p - \int_{\mathbb{R}^n} |v_n|^{p^*}dx,
\tag{3.10}
$$

and

$$
\begin{aligned}
c + o_n(1) =& E_\lambda(u_n) - \frac{1}{p^*}\langle E'_\lambda(u_n), u_n\rangle\\
=& a\left(\frac{1}{\theta p}-\frac{1}{p^*}\right)\left(\|v_n\|^p+\|u\|^p\right)^\theta + b\left(\frac{1}{p}-\frac{1}{p^*}\right)\left(\|v_n\|^p+\|u\|^p\right)\\
&+\lambda\left(\frac{1}{p^*}-1\right)\int_{\mathbb{R}^n} f(x)\, v_n dx - \lambda\left(\frac{1}{p^*}-1\right)\int_{\mathbb{R}^n} f(x)\, u dx\\
\geq& a\left(\frac{1}{\theta p}-\frac{1}{p^*}\right)\|v_n\|^{\theta p} + b\left(\frac{1}{p}-\frac{1}{p^*}\right)\|v_n\|^p + E_\lambda(u) - \frac{1}{p^*}\langle E'_\lambda(u), u\rangle.
\end{aligned}
$$

Consequently,

$$
c + o_n(1) \geq E_\lambda(u) + \left(\frac{a}{\theta p}-\frac{a}{p^*}\right)\|v_n\|^{\theta p} + \left(\frac{b}{p}-\frac{b}{p^*}\right)\|v_n\|^p.
\tag{3.11}
$$

Assume that $\lim\limits_{n\to+\infty}\|v_n\| = l > 0$. Then, by (3.10) and the Sobolev inequality, we obtain

$$
l^p \geq S\left(bl^p + al^{\theta p}\right)^{\frac{p}{p^*}}.
$$

This implies that

$$S^{-\frac{p^*}{p}} l^{p^*-p} - a l^{p(\theta-1)} - b \geq 0. \tag{3.12}$$

Let $x = S^{-(\theta-1)} l^{p(\theta-1)}$ and $\sigma = \frac{p^*-p}{(\theta-1)p}$. Then, by (3.12), we get

$$S^{-1}x^{\sigma} - aS^{\theta-1}x - 0 \leq b.$$

It is clear that $\sigma \geq 1$ thanks to $\theta \leq \frac{p^*}{p}$. Then, by definition of Ψ, we get $\Psi(x) \geq 0$. We are now in a position to discuss two cases.

Case 1. $\theta = \frac{p^*}{p}$, $S^{-\theta} > a \geq 0$ and $0 < b$. According to Lemma 2.2, we have $\Psi(x) \geq 0$ if $x \geq x_1$, with

$$x_1 = \frac{b}{\left(S^{-\theta} - a\right) S^{\theta-1}},$$

which implies that $l^p \geq S x_1^{\frac{1}{\theta-1}}$.

Case 2. $\theta < \frac{p^*}{p}$, $a \geq 0$, $0 \leq b$ and $a + 0 < b$. In this case, it follows from Lemma 2.2 that $\Psi(x) \geq 0$ if $x \geq x_2$, with

$$x_2 > \left(\frac{a(\theta-1)p}{p^*-p} S^{\theta}\right)^{\frac{(\theta-1)p}{p^*-\theta p}},$$

which implies that $l^p \geq S x_2^{\frac{1}{\theta-1}}$. Then, by (3.11), one has

$$\begin{aligned}
c &\geq E_\lambda(u) + \left(\frac{a}{\theta p} - \frac{a}{p^*}\right) l^{\theta p} + \left(\frac{b}{p} - \frac{b}{p^*}\right) l^p \\
&\geq E_\lambda(u) + \begin{cases} b\frac{p^*-p}{pp^*} S x_1^{\frac{1}{\theta-1}}, & \text{if } \theta = \frac{p^*}{p}, S^{-\theta} > a \geq 0 \text{ and } 0 < b, \\ a\frac{p^*-\theta p}{\theta p p^*}\left(S x_2^{\frac{1}{\theta-1}}\right)^{\theta} + b\frac{p^*-p}{pp^*} S x_2^{\frac{1}{\theta-1}}, & \text{if } \theta < \frac{p^*}{p}, 0 \leq a, b \text{ and } a < b, \end{cases} \\
&= E_\lambda(u) + C^*.
\end{aligned}$$

The proof of Lemma 3.1 is complete. □

Next, we estimate the level energy.

Lemma 3.2. *Assume that all the conditions in Theorem* 3.2 *are fulfilled. Then, there exists* $z_\varepsilon \in W^{1,p}(\mathbb{R}^n)$ *and* $\lambda^* > 0$ *such that*

$$\sup_{t \geq 0} E_\lambda(t z_\varepsilon) < c_- + C^*, \quad \text{for all } \lambda \in (0, \lambda^*),$$

where c_- *and* C^* *are given in* (3.6) *and* (3.7), *respectively.*

Proof. Since $\int_{\mathbb{R}^n} f(x) v_{\varepsilon,x_0}(x)\,dx \neq 0$, there exists $z_\varepsilon = \pm v_{\varepsilon,x_0}$ that satisfies

$$\int_{\mathbb{R}^n} f(x) z_\varepsilon(x)\,dx > 0.$$

Given any $\lambda > 0$ and fixed $t > 0$, using (1.4), we have

$$\begin{aligned} E_\lambda(tz_\varepsilon) &= \frac{a}{\theta p} t^{\theta p} \|z_\varepsilon\|^{\theta p} + \frac{b}{p} t^p \|z_\varepsilon\|^p - \frac{t^{p^*}}{p^*} \int_{\mathbb{R}^n} |z_\varepsilon|^{p^*} dx - \lambda t \int_{\mathbb{R}^n} f(x) z_\varepsilon dx \\ &= \frac{a}{\theta p} t^{\theta p} S^{\frac{\theta p^*}{p^*-p}} + \frac{b}{p} t^p S^{\frac{p^*}{p^*-p}} - \frac{t^{p^*}}{p^*} S^{\frac{p^*}{p^*-p}} - \lambda t \int_{\mathbb{R}^n} f(x) z_\varepsilon dx. \end{aligned}$$

Define $g, h :]0, +\infty[\to \mathbb{R}$ by $g(t) = E_\lambda(tz_\varepsilon)$ and

$$h(t) = \frac{a}{\theta p} t^{\theta p} S^{\frac{\theta p^*}{p^*-p}} + \frac{b}{p} t^p S^{\frac{p^*}{p^*-p}} - \frac{t^{p^*}}{p^*} S^{\frac{p^*}{p^*-p}}.$$

Then,

$$h'(t) = -t^{p-1} S^{\frac{p^*}{p^*-p}} \left(t^{p^*-p} - a S^{\frac{(\theta-1)p^*}{p^*-p}} t^{(\theta-1)p} - b \right).$$

It follows from $h'(t) = 0$ that

$$aS^{\frac{(\theta-1)p^*}{p^*-p}} t^{(\theta-1)p} + b - t^{p^*-p} = 0. \tag{3.13}$$

So,

$$t^{p^*} = aS^{\frac{(\theta-1)p^*}{p^*-p}} t^{\theta p} + bt^p. \tag{3.14}$$

Let $x = S^{\frac{p(\theta-1)}{p^*-p}} t^{p(\theta-1)}$, $\sigma = \frac{p^*-p}{(\theta-1)p}$ and

$$x_* := \begin{cases} x_1, & \text{if } (\mathcal{H}_3) \text{ holds,} \\ x_2, & \text{if } (\mathcal{H}_1) \text{ holds.} \end{cases}$$

Then, by (3.13) and the definition of Ψ, we get

$$\Psi(x) = S^{-1} x^\sigma - aS^{\theta-1} x - b = 0. \tag{3.15}$$

By Lemma 2.2, we can conclude that $\Psi(x_*) = 0$, $\Psi(x) < 0$ for all $x \in]0, x_*[$ and $\Psi(x) > 0$ for all $x \in]x_*, +\infty[$. Therefore, $h'(t_*) = 0$, $h'(t) > 0$ for all $t \in]0, t_*[$ and $h'(t) < 0$ for all $t \in]t_*, +\infty[$, where

$$t_* := \begin{cases} S^{\frac{-1}{p^*-p}} x_1^{\frac{1}{(\theta-1)p}}, & \text{if } (\mathcal{H}_3) \text{ holds,} \\ S^{\frac{-1}{p^*-p}} x_2^{\frac{1}{(\theta-1)p}}, & \text{if } (\mathcal{H}_1) \text{ holds.} \end{cases}$$

Moreover, since $h(0) = 0$ and $\lim\limits_{t \to +\infty} h(t) = -\infty$ if $(\mathcal{H}_1)$ or $(\mathcal{H}_3)$ holds, then h attains its maximum at t_*.

So, from (3.14), we have

$$\begin{aligned}\max_{t\geq 0} h(t) &= h(t_*)\\ &= \frac{a}{\theta p} t_*^{\theta p} S^{\frac{\theta p^*}{p^*-p}} + \frac{b}{p} t_*^p S^{\frac{p^*}{p^*-p}} - \frac{t_*^{p^*}}{p^*} S^{\frac{p^*}{p^*-p}}\\ &= \frac{a}{\theta p} t_*^{\theta p} S^{\frac{\theta p^*}{p^*-p}} + \frac{b}{p} t_*^p S^{\frac{p^*}{p^*-p}} - \left(\frac{a}{p^*} t_*^{\theta p} S^{\frac{\theta p^*}{p^*-p}} + \frac{b}{p^*} t_*^p S^{\frac{p^*}{p^*-p}}\right)\\ &= a\left(\frac{1}{\theta p} - \frac{1}{p^*}\right) t_\varepsilon^{\theta p} S^{\frac{\theta p^*}{p^*-p}} + b\left(\frac{1}{p} - \frac{1}{p^*}\right) t_\varepsilon^p S^{\frac{p^*}{p^*-p}}\\ &= a\left(\frac{1}{\theta p} - \frac{1}{p^*}\right) S^\theta x_*^{\frac{\theta}{\theta-1}} + b\left(\frac{1}{p} - \frac{1}{p^*}\right) S x_*^{\frac{1}{\theta-1}}\\ &= C^*.\end{aligned}$$

By (3.4) and (3.6), we have $c_- \geq -C_\lambda$ for all $\lambda \in (0, \lambda_*)$. So, we can choose $\lambda_3 \leq \lambda_-$ such that for any $\lambda \in (0, \lambda_3)$ we have $C^* - c_- \geq C^* - C_\lambda > 0$. Hence, $C^* - c_- > 0$ for all $\lambda \in (0, \lambda_3)$.

Now, we consider the function $g(t) := E_\lambda(tz_\varepsilon)$, $t \geq 0$. Then, $g(t) = h(t) - \lambda t \int_{\mathbb{R}^n} f(x) z_\varepsilon dx$. So, for all $\lambda \in (0, \lambda_3)$, we have $g(0) = 0 < C^* - C_\lambda$. Hence, by the continuity of $g(t)$, there exists $t_1 > 0$ small enough such that $g(t) < C^* - C_\lambda$ for all $t \in (0,\ t_1)$.

We know also that $\lim_{t\to+\infty} g(t) = -\infty$ if $(\mathcal{H}_1)$ or $(\mathcal{H}_3)$ holds. Then, for $t_2 > 0$ sufficiently large one has$g(t) < C^* - C_\lambda$, for all $t \in (t_2, +\infty)$. On the other hand, as $\int_{\mathbb{R}^n} f(x) z_\varepsilon dx > 0$, we can deduce from the above estimate on $h(t)$ that for all $t \in [t_1, t_2]$

$$g(t) < C^* - \lambda t_1 \int_{\mathbb{R}^n} f(x) z_\varepsilon dx.$$

Set

$$\lambda_4 = \begin{cases} \left(\frac{p}{p-1} t_1 \int_{\mathbb{R}^n} f(x) z_\varepsilon dx\right)^{p-1} \frac{b}{2} \|f\|_{W^*}^{-p}, & \text{if } (\mathcal{H}_3) \text{ or } (\mathcal{H}_1) \text{ with } 0 < b \text{ holds},\\ \left(\frac{\theta p}{\theta p-1} t_1 \int_{\mathbb{R}^n} f(x) z_\varepsilon dx\right)^{\theta p-1} \frac{a}{2} \|f\|_{W^*}^{-\theta p}, & \text{if } (\mathcal{H}_1) \text{ with } 0 < a \text{ holds}.\end{cases}$$

Then, for any $\lambda \in (0, \lambda_4)$, one has

$$-\lambda t_1 \int_{\mathbb{R}^n} f(x) z_\varepsilon dx < -C_\lambda.$$

Taking $\lambda_+ = \min\{\lambda_-, \lambda_3, \lambda_4\}$, then we deduce that

$$\sup_{t\geq 0} E_\lambda(tz_\varepsilon) < C^* + c_-, \quad \text{for all } \lambda \in (0, \lambda_+).$$

This concludes the proof of Lemma 3.2. □

Now, we are ready to prove the existence of the mountain pass-type solution and give the proof of Theorems 3.2 with the help of Theorem 3.1.

Proof of Theorem 3.2. Note that $E_\lambda(0) = 0$ and from (3.3), we have $E_\lambda(u)|_{\partial B_{\rho_*}(0)} \geq \delta_* > 0$ for all $\lambda \in (0, \lambda_-)$, where ρ_*, δ_* are defined in (3.2). We know also that $\lim_{t\to\infty} E_\lambda(tz_\varepsilon) = -\infty$ if $(\mathcal{H}_1)$ or $(\mathcal{H}_3)$ holds, then $E_\lambda(Tz_\varepsilon) < 0$ for T large enough. Hence, E_λ satisfies the geometry conditions of the mountain pass theorem [16]. Then, there exists a Palais Smale sequence (u_n) at level c_+, such that

$$E_\lambda(u_n) \to c_+ \quad \text{and} \quad E'_\lambda(u_n) \to 0, \qquad \text{as } n \to +\infty,$$

with

$$0 < c_+ = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} E_\lambda(\gamma(t)) \leq \sup_{t\geq 0} E_\lambda(tTz_\varepsilon) < C^* + c_-, \qquad \text{for all } \lambda \in (0, \lambda_+),$$

where

$$\Gamma = \left\{\gamma \in C\left([0,1], W^{1,p}(\mathbb{R}^n)\right), \gamma(0) = 0, \gamma(1) = Tz_\varepsilon\right\}.$$

Using Lemma 2.1, we find that (u_n) has a subsequence, still denoted by (u_n), such that $u_n \rightharpoonup u_+$ in $W^{1,p}(\mathbb{R}^n)$ as $n \to +\infty$. Hence, from Lemma 3.1 if $u_n \nrightarrow u_+$ in $W^{1,p}(\mathbb{R}^n)$ as $n \to +\infty$, it holds

$$c_+ \geq E_\lambda(u_+) + C^* \geq c_- + C^*,$$

which is in contradiction with Lemma 3.2. Hence, $E'_\lambda(u_+) = 0$ and

$$E_\lambda(u_+) = c_+ > 0.$$

So, since $c_+ > 0 = E_\lambda(0)$, we conclude that u_+ is a non-zero solution of (1.1) with positive energy. This completes the proof of Theorem 3.2. □

4. Results in the Case when $\lambda = 0$

Now, in the case when $\lambda = 0$, we have the following results.

4.1. Infinitely solutions.

Theorem 4.1. *Let $\lambda = 0$, $0 < a$, $0 \leq b$, $1 < p < n$ and $\theta > 1$. For v_{ε,x_0}, given by* (1.3)*, the following conclusions hold.*

(1) *If $\theta = \frac{p^*}{p}$, then under the hypothesis $(\mathcal{H}_3)$, problem* (1.1) *for $\lambda = 0$ has infinitely many nonnegative solutions and these solutions are*

$$\left(\frac{b}{1 - S^\theta a}\right)^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}, \quad \textit{for all } \varepsilon > 0,$$

and under the hypothesis $(\mathcal{H}_6)$, the problem (1.1) *for $\lambda = 0$ has infinitely many positive solutions $\delta^{\frac{1}{(\theta-1)p}} v_{\varepsilon,x_0}$ (for any $\delta > 0$) if and only if $a = S^{-\theta}$.*

(2) *If $\theta \neq \frac{p^*}{p}$, $b = 0$ and $0 < a$, then problem* (1.1) *has infinitely many nonnegative solutions and these solutions are*

$$\left(aS^{\frac{p^*(\theta-1)}{p^*-p}}\right)^{-\frac{1}{\theta p - p^*}} v_{\varepsilon,x_0}, \quad \textit{for all } \varepsilon > 0.$$

(3) *If* $(\mathcal{H}_0)$ *is satisfied, then there exists* $\delta_2 > S^{-1}\left(\frac{a(\theta-1)p}{p^*-p}S^{\theta}\right)^{\frac{p^*-p}{p^*-\theta p}}$ *such that* $\delta_2^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$ *are solutions of problem* (1.1)*, for all* $\varepsilon > 0$*.*

(4) *If* $(\mathcal{H}_7)$ *is satisfied, then problem* (1.1) *has infinitely many nonnegative solutions and these solutions are*

$$S^{-\frac{1}{p^*-p}}\left(\frac{p^*-p}{(\theta-1)\,pa}S^{-\theta}\right)^{\frac{1}{\theta p-p^*}} v_{\varepsilon,x_0}, \qquad \textit{for all } \varepsilon > 0.$$

(5) *If* $(\mathcal{H}_9)$ *is satisfied, then there exist*

$$\delta_3 \in \left(0, S^{-1}\left(\frac{p^*-p}{(\theta-1)\,pa}S^{-\theta}\right)^{\frac{p^*-p}{\theta p-p^*}}\right)$$

and

$$\delta_4 \in \left(S^{-1}\left(\frac{p^*-p}{(\theta-1)\,pa}S^{-\theta}\right)^{\frac{p^*-p}{\theta p-p^*}}, +\infty\right),$$

such that $\delta_3^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$ *and* $\delta_4^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$ *are solutions of problem* (1.1) *for all* $\varepsilon > 0$*.*

Remark 4.1. Ke et al. in [11] have obtained Theorem 4.1 for the case $\theta = 2$.

We give the proof of Theorem 4.1.

Proof of Theorem 4.1. For any $\delta > 0$, we define $V_{\varepsilon,\delta} = \delta^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$, where v_{ε,x_0} is given in (1.4). Since v_{ε,x_0} is a solution of problem (1.2), we have that $V_{\varepsilon,\delta}$ solves the following equation

$$-\delta \mathrm{div}\left(|\nabla V_{\varepsilon,\delta}|^{p-2}\nabla V_{\varepsilon,\delta}\right) = |V_{\varepsilon,\delta}|^{p^*-2}V_{\varepsilon,\delta}.$$

Moreover, according to (1.4), one has

$$\delta = a\left\|V_{\varepsilon,\delta}\right\|^{(\theta-1)p} + b = a\delta^{\frac{(\theta-1)p}{p^*-p}}\left\|v_{\varepsilon,x_0}\right\|^{(\theta-1)p} + b = aS^{\frac{p^*(\theta-1)}{p^*-p}}\delta^{\frac{(\theta-1)p}{p^*-p}} + b.$$

Therefore, the positive solution of the problem (1.1) corresponds to the solution of the following equation with respect to $\delta > 0$

$$\delta - aS^{\frac{p^*(\theta-1)}{p^*-p}}\delta^{\frac{(\theta-1)p}{p^*-p}} - b = 0. \tag{4.1}$$

(1) For $\theta = \frac{p^*}{p}$, equation (4.1) is equivalent to

$$\delta\left(1 - aS^{\theta}\right) - b = 0.$$

If $0 < b$ and $S^{-\theta} > a \geq 0$, we have that

$$\delta_0 = \frac{b}{1 - S^{\theta}a}$$

is a solution of equation (4.1). Hence, $V_{\varepsilon,\delta_0} = \delta_0^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$ satisfies the following equation in the weak sense

$$-\left(a \|u\|^{(\theta-1)p} + b\right) \operatorname{div} \left(|\nabla u|^{p-2} \nabla u\right) = |u|^{p^*-2} u.$$

If $b = 0$ and $0 < a$, then equation (4.1) is equivalent to

$$\delta \left(1 - aS^{\theta}\right) = 0,$$

then, for $\delta > 0$, we get $1 - aS^{\theta} = 0$. Thus, when $\theta = \frac{p^*}{p}$, problem (1.1) has infinitely many positive solutions $V_{\varepsilon,\delta} = \delta^{\frac{1}{(\theta-1)p}} v_{\varepsilon,x_0}$ if and only if $a = S^{-\theta}$.

(2) For $\theta \neq \frac{p^*}{p}$, $b = 0$ and $0 < a$ it is easy to see that

$$\delta_1 = \left(aS^{\frac{p^*(\theta-1)}{p^*-p}}\right)^{-\frac{p^*-p}{\theta p - p^*}}$$

is a solution of equation (4.1). Then, problem (1.1) for $\lambda = 0$ has infinity many positive solutions $V_{\varepsilon,\delta_1} = \delta_1^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$.

(3) Let $x = (S\delta)^{\frac{(\theta-1)p}{p^*-p}}$, by (4.1) and Lemma 2.2, we get

$$S^{-1} x^{\frac{p^*-p}{(\theta-1)p}} - aS^{\theta-1} x - b = 0. \tag{4.2}$$

So, $\Psi(x) = 0$ with $\sigma = \frac{p^*-p}{(\theta-1)p}$. Hence, for $\theta < \frac{p^*}{p}$, according to Lemma 2.2, we have that $\Psi(x) = 0$ has a unique positive solution

$$x_2 > \left(\frac{a(\theta-1)p}{p^*-p} S^{\theta}\right)^{\frac{(\theta-1)p}{p^*-\theta p}}.$$

Thus, problem (1.1) has infinity many positive solutions

$$V_{\varepsilon,\delta_2} = \delta_2^{\frac{1}{p^*-p}} v_{\varepsilon,x_0},$$

with

$$\delta_2 = S^{-1} x_2^{\frac{p^*-p}{(\theta-1)p}} > S^{-1} \left(\frac{a(\theta-1)p}{p^*-p} S^{\theta}\right)^{\frac{p^*-p}{p^*-\theta p}}.$$

For $\theta > \frac{p^*}{p}$, by using (4.2) and according to Lemma 2.2, we have the following.

If $b = b^*$, then the equation $\Psi(x) = 0$ has a unique positive solution

$$\tilde{x} = \left(\frac{p^*-p}{(\theta-1)pa} S^{-\theta}\right)^{\frac{(\theta-1)p}{\theta p - p^*}}.$$

Thus, problem (1.1) has infinity many positive solutions

$$V_{\varepsilon,\tilde{\delta}} = \tilde{\delta}^{\frac{1}{p^*-p}} v_{\varepsilon,x_0},$$

with

$$\tilde{\delta} = S^{-1} \tilde{x}^{\frac{p^*-p}{(\theta-1)p}}.$$

This proves (4).

If $b < b^*$, then Ψ has two different zero points x_3 and x_4 with $0 < x_3 < \tilde{x} < x_4$. Consequently, problem (1.1) has infinitely many positive solutions

$$V_{\varepsilon,\delta_3} = \delta_3^{\frac{1}{p^*-p}} v_{\varepsilon,x_0}$$

and

$$V_{\varepsilon,\delta_4} = \delta_4^{\frac{1}{p^*-p}} v_{\varepsilon,x_0},$$

with

$$\delta_3 = S^{-1} x_3^{\frac{p^*-p}{(\theta-1)p}} \in \left(0, S^{-1}\left(\frac{p^*-p}{(\theta-1)\,pa} S^{-\theta}\right)^{\frac{p^*-p}{\theta p - p^*}}\right)$$

and

$$\delta_4 = S^{-1} x_4^{\frac{p^*-p}{(\theta-1)p}} \in \left(S^{-1}\left(\frac{p^*-p}{(\theta-1)\,pa} S^{-\theta}\right)^{\frac{p^*-p}{\theta p - p^*}}, +\infty\right).$$

This proves (5). $\square$

4.2. **Non-existence result.**

Theorem 4.2. *Assume that one of the hypotheses* $(\mathcal{H}_i)$ *holds for* $i \in \{4, 5, 8\}$. *Then, problem* (1.1) *has no non-trivial solution for* $\lambda = 0$.

Remark 4.2. For $\theta = 2$, the authors in [11] proved the non-existence of solutions only in the case $p^* < 2p$, while the case $p^* = 2p$ is considered in our case. From this point of view, Theorem 4.2 could be viewed as some extension and completeness of the related results in [11].

We give the proof of Theorem 4.2.

Proof of Theorem 4.2. Suppose that $(\mathcal{H}_4)$ is satisfied and let $u \in W^{1,p}(\mathbb{R}^n)\backslash\{0\}$ be a solution of the problem (1.1). Then,

$$a \left\| u \right\|^{\theta p} = \int_{\mathbb{R}^n} |u|^{p^*} dx. \tag{4.3}$$

If $\theta = \frac{p^*}{p}$, $a > S^{-\theta}$ and $\int_{\mathbb{R}^n} |u|^{p^*} dx \leq S^{-\frac{p^*}{p}} \left\| u \right\|^{p^*}$, we have, by (4.3),

$$S^{-\theta} \left\| u \right\|^{\theta p} < a \left\| u \right\|^{\theta p} = \int_{\mathbb{R}^n} |u|^{p^*} dx \leq S^{-\theta} \left\| u \right\|^{\theta p},$$

which leds to a contradiction.

Suppose now that $(\mathcal{H}_5)$ is satisfied and let $u \in W^{1,p}(\mathbb{R}^n)\backslash\{0\}$ be a solution of (1.1). Then,

$$a \left\| u \right\|^{\theta p} + b \left\| u \right\|^{p} = \int_{\mathbb{R}^n} |u|^{p^*} dx.$$

If $\theta = \frac{p^*}{p}$, $a \geq S^{-\theta}$, $0 < b$ and

$$\int_{\mathbb{R}^n} |u|^{p^*} dx \leq S^{-\frac{p^*}{p}} \left\| u \right\|^{p^*},$$

we get

$$S^{-\theta} \|u\|^{\theta p} < a \|u\|^{\theta p} + b \|u\|^{p} = \int_{\mathbb{R}^n} |u|^{p^*} dx \leq S^{-\theta} \|u\|^{\theta p},$$

which is a contradiction.

In the same way as above, under the condition $(\mathcal{H}_8)$, suppose that $u \in W^{1,p}(\mathbb{R}^n)\backslash\{0\}$ is a solution of $(\mathcal{P}_0)$, that is,

$$a \|u\|^{\theta p} + b \|u\|^{p} = \int_{\mathbb{R}^n} |u|^{p^*} dx.$$

Then,

$$\begin{aligned}
\int_{\mathbb{R}^n} |u|^{p^*} dx \leq & S^{-\frac{p^*}{p}} \|u\|^{p^*} = \|u\|^{p^* - \frac{\theta p - p^*}{\theta - 1}} S^{-\frac{p^*}{p}} \|u\|^{\frac{\theta p - p^*}{\theta - 1}} \\
= & \left(\frac{(\theta-1)\,p}{p^*-p} a\right)^{\frac{p^*-p}{(\theta-1)p}} \|u\|^{\frac{\theta(p^*-p)}{\theta-1}} \left(\frac{(\theta-1)\,p}{p^*-p} a\right)^{-\frac{p^*-p}{(\theta-1)p}} S^{-\frac{p^*}{p}} \|u\|^{\frac{\theta p - p^*}{\theta-1}} \\
\leq & \frac{p^*-p}{(\theta-1)\,p} \left(\left(\frac{(\theta-1)\,p}{p^*-p} a\right)^{\frac{p^*-p}{(\theta-1)p}} \|u\|^{\frac{\theta(p^*-p)}{\theta-1}} \right)^{\frac{(\theta-1)p}{p^*-p}} \\
& + \frac{\theta p - p^*}{(\theta-1)\,p} \left(\left(\frac{(\theta-1)\,p}{p^*-p} a\right)^{-\frac{p^*-p}{(\theta-1)p}} S^{-\frac{p^*}{p}} \|u\|^{\frac{\theta p - p^*}{\theta-1}} \right)^{\frac{(\theta-1)p}{\theta p - p^*}} \\
\leq & a \|u\|^{\theta p} + \frac{\theta p - p^*}{(\theta-1)\,p} \left(\left(\frac{(\theta-1)\,p}{p^*-p} a\right)^{-\frac{p^*-p}{(\theta-1)p}} S^{-\frac{p^*}{p}} \right)^{\frac{(\theta-1)p}{\theta p - p^*}} \|u\|^{p} \\
= & a \|u\|^{\theta p} + \frac{\theta p - p^*}{(\theta-1)\,p} \left(\frac{(\theta-1)\,p}{p^*-p} a\right)^{-\frac{p^*-p}{\theta p - p^*}} S^{-\frac{(\theta-1)p^*}{\theta p - p^*}} \|u\|^{p} \\
< & a \|u\|^{\theta p} + b \|u\|^{p} \\
= & \int_{\mathbb{R}^n} |u|^{p^*} dx,
\end{aligned}$$

which leads to a contradiction (see [18–22]). This completes the proof. □

5. Conclusion

The objective of this article is to investigate contemporary challenges within the field of non-classical partial differential equations and to explore their practical applications. Specifically, it seeks to establish the presence or absence of solutions for p-Kirchhoff-type problems with critical Sobolev exponents in the Euclidean space $\mathbb{R}^n$. These investigations are pivotal for gaining a deeper understanding of mathematical physics equations and addressing pertinent issues in this domain. The proposed article is an expanded discussion of the question of existence and non-existence of solutions. We found several new results related to the discussion of the existence of a solution with negative and positive energy using the latest methods in this field to support the rapidly developing literature. After that, we discussed different cases depending on the change in parameters to show that solutions do not exist.

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