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ABOUT COMPACTNESS TYPE OF UNIFORM SPACES

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Abstract. This paper introduces uniform analogues of compact, finally compact and countable compact spaces and investigates their connection with other uniform properties of compactness type.

1. Introduction and Preliminaries

It is well known that compact, finally compact and countable compact spaces play an important role in set-theoretical topology. Finding and studying uniform analogues of these classes of compactness types is an important and interesting problem in uniform topology.

Throughout this paper, all uniform spaces are assumed to be Hausdorff, topological space are Tychonoff and mappings are uniformly continuous.

A topological space X is called finally compact if every open cover has a countable open refinement [1]; a topological space X is called countable compact, if every countable open cover has a finite open refinement [1]; a uniform space (X, U) is called precompact if it has a base consisting of finite covers [2,4,17]; a uniform space (X, U) is called strongly uniformly paracompact if every finitely additive open cover has a star finite uniform refinement [5]; the filter F is called Cauchy filter if $\alpha \bigcap F \neq \varnothing$ for any $\alpha \in U$ [1,17]; a uniformly continuous mapping $f : (X, U) \to (Y, V)$ of uniform space (X, U) onto a uniform space (Y, V) is called a precompact, if for each $\alpha \in U$ there exist a uniform cover $\beta \in V$ and finite uniform cover $\gamma \in U$, such that $f^{-1}\beta \wedge \gamma \succ \alpha$ [1]; a continuous mapping $f : X \to Y$ of topological space X to a topological space Y

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is called ω-mapping, if for each point $y \in Y$ there exist neighborhood O_y and $W \in \omega$, such that $f^{-1}O_y \subset W$ [5].

For a cover α of a set X and $x \in X$, $M \subset X$, we have: $St(\alpha, x) = \{A \in \alpha : A \ni x\}$ $\alpha(x) = \bigcup St(\alpha, x)$, $St(\alpha, M) = \{A \in \alpha : A \bigcap M \neq \varnothing\}$, $\alpha(M) = \bigcup St(\alpha, M)$. For covers α and β of the set X, the symbol $\alpha \succ \beta$ means that the cover α is a refinement of the cover β, i.e., for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$ and, for covers α and β of a set X, we have: $\alpha \wedge \beta = \{A \bigcap B : A \in \alpha, B \in \beta\}$. For a cover α of a set X let $\alpha^{\angle} = \{\bigcup \alpha_0 : \alpha_0 \subset \alpha$ - is finite$\}$. The cover α is finitely additive if $\alpha^{\angle} = \alpha$. For the uniformity U by τ_U we denote the topology generated by the uniformity and symbol U_X means the universal uniformity.

2. Compactness Type of Uniform Spaces

This paper introduces and studies u-uniformly compact, u-uniformly finally compact and u-uniformly countable compact spaces.

The following lemma will serve as the basis for our definitions.

Lemma 2.1. *A topological space (X, τ) is compact (countable compact, finally compact) if and only if every finitely additive arbitrary (countable, arbitrary) open cover has an open finite (finite, countable) refinement.*

Proof. Necessity. Let α be an arbitrary (countable) open cover of (X, τ). Then, for the finitely additive (countable) open cover $\alpha^{\angle}$ there exists a finite (finite, countable) open cover β such that $\beta \succ \alpha^{\angle}$. For each $B \in \beta$ choose $A_B \in \alpha^{\angle}$ such that $B \subset A_B$, $A_B = \bigcup_{i=1}^{n} A_i$, $A_i \in \alpha$, $i = 1, 2, \ldots, n$. Let $\alpha_0 = \bigcup\{\alpha_B : B \in \beta\}$, $\alpha_B = \{B \bigcap A_i : i = 1, 2, \ldots, n\}$. Then α_0 is finite (finite, countable) open cover of the topological space (X, τ) such that $\alpha_0 \succ \alpha$. Consequently, the topological space (X, τ) is compact (countable compact, finally compact).

The sufficiency is obvious. □

Definition 2.1. A uniform space (X, U) is said to be u-uniformly compact if every finitely additive open cover has a finite uniform refinement.

Definition 2.2. A uniform space (X, U) is said to be u-uniformly countable compact if every finitely additive countable open cover has a finite uniform refinement.

Definition 2.3. A uniform space (X, U) is said to be u-uniformly finally compact if every finitely additive open cover has a countable uniform refinement.

Proposition 2.1. *If a uniform space (X, U) is u-uniformly compact (u-uniformly countable compact, u-uniformly finally compact), then topological space (X, τ_U) is compact (countable compact, finally compact). Conversely, if a topological space (X, τ) is compact (countable compact, finally compact), then the space (X, U_X) is compact (countable compact, finally compact), where U_X is the universal uniformity on topological space (X, τ) such that $\tau_{U_X} = \tau$.*

Proof. Let α be an arbitrary (countable) open cover of the space (X, τ_U). Then for a finitely additive (countable) open cover $\alpha^{\angle}$ of (X, U) there exist finite (finite, countable) uniform cover $\beta \in U$ such that $\beta \succ \alpha^{\angle}$. It is known that the interior $\langle \beta \rangle$ of uniform cover β is a uniform cover. Denote $\gamma = \langle \beta \rangle$. Then γ is a finite (finite, countable) open uniform cover of the uniform space (X, U). For each $\Gamma \in \gamma$ we choose $A_\Gamma \in \alpha^{\angle}$ such that $\Gamma \subset A_\Gamma$, where $A_\Gamma = \bigcup_{i=1}^{n} A_i$, $A_i \in \alpha$, $i = 1, 2, \dots, n$. Let's put $\alpha_0 = \bigcup\{\alpha_\Gamma : \Gamma \in \gamma\}$, $\alpha_\Gamma = \{\Gamma \bigcap A_i : i = 1, 2, \dots, n\}$. Then α_0 is finite (finite, countable) open cover of the space (X, τ_U) and $\alpha_0 \succ \alpha$. So (X, τ_U) is compact (countable compact, finally compact).

Conversely, let (X, τ) be compact (countable compact, finally compact). Then the set of all open covers form the base of the universal uniformity U_X of the topological space (X, τ). It is evident that the uniform space (X, U_X) is u-uniformly compact (u-uniformly countable compact, u-uniformly finally compact). □

The following theorem gives a characterization of u-uniformly compactness (u-uniformly finally compactness) in the spirit of Tamano.

Theorem 2.1. *Let (X, U) be a uniform space and bX be a certain compact Hausdorff extensions of the space (X, τ_U). A uniform space (X, U) is u-uniformly compact (u-uniformly finally compact) if and only if for compact $K \subset bX \backslash X$ there exist a finite (countable) uniformly cover $\alpha \in U$, such that $[A]_{bX} \bigcap K = \emptyset$ for any $A \in \alpha$.*

Proof. Necessity. Let (X, U) be a u-uniformly compact (u-uniformly finally compact) space and $K \subset bX \backslash X$ be an arbitrary compact. For each point $x \in X$ there is an open neighborhood O_x of the compact Hausdorff extensions bX of the space (X, τ_U), such that $[O_x]_{bX} \bigcap K = \emptyset$. Denote $\gamma = \{O_x \bigcap X : x \in X\}$. Obviously γ is open cover of the space (X, U). Let $\beta \in U$ be a finite (countable) uniform cover of the space (X, U) such that $\beta \succ \gamma^{\angle}$, then $[B]_{bX} \subset [\bigcup_{i=1}^{n} (O_{x_i} \bigcap X)]_{bX} \subset \bigcup_{i=1}^{n} [O_{x_i}]_{bX}$. Since $[O_{x_i}]_{bX} \bigcap K = \emptyset$, $i = 1, 2, \dots, n$, then $[B]_{bX} \bigcap K = \emptyset$, $B \in \beta$.

Sufficiency. Let α be an arbitrary finitely additive open cover of the space (X, U). Then there exists an open system β of the compact Hausdorff extensions bX, such that $\beta \wedge \{X\} = \alpha$. Denote $K = bX \backslash \bigcup \beta$. It is easy to see that K is compact. Accordingly to the condition of the theorem, there exists a finite (countable) cover $\gamma \in U$, such that $[\Gamma]_{bX} \bigcap K = \emptyset$, $\Gamma \in \gamma$. Since $[\Gamma]_{bX}$ is compact in bX, there exist $B_1, B_2, \dots, B_n \in \beta$, such that $[\Gamma]_{bX} \subset \bigcup_{i=1}^{n} B_i$. Then $\Gamma \subset \bigcup_{i=1}^{n} A_i$, $\bigcup_{i=1}^{n} A_i \in \alpha$. Thus, the space (X, U) is u-uniformly compact (u-uniformly finally compact). □

Proposition 2.2. *Any compact uniform space (X, U) is u-uniformly compact.*

Proof. Let α be an arbitrary (countable) open cover of (X, U). As is known, every open cover of a compact is a uniform cover. It follows that $\alpha \in U$. Since the space (X, U) is precompact, there exists a finite uniform cover $\beta \in U$ such that $\beta \succ \alpha$. □

Proposition 2.3. *Any u-uniformly compact uniform space (X, U) is u-uniformly finally compact (u-uniformly countable compact).*

Proof. Let α be an arbitrary (countable) open cover of (X, U). Since the space (X, U) is u-uniformly compact, there exists a finite uniform cover $\beta \in U$ such that $\beta \succ \alpha$. Thus, (X, U) is u-uniformly finally compact (u-uniformly countable compact). □

Proposition 2.4. *Any u-uniformly finally compact uniform space (X, U) is complete.*

Proof. Let (X, U) be a u-uniformly compact space and F be an arbitrary Cauchy filter. Suppose that a Cauchy filter F does not converge at any point of the space (X, U). Then, each point $x \in X$ has a open neighborhood O_x and $B \in F$ such that $O_x \bigcap B = \emptyset$. Denote $\alpha = \{O_x : x \in X\}$. Since the space (X, U) is u-uniformly finally compact, then $\alpha^{\angle} \in U$, therefore $\alpha^{\angle} \bigcap F \neq \emptyset$, i.e., there are $A^{\angle} \in \alpha^{\angle}$ such that $A^{\angle} \in F$, where $A^{\angle} = \bigcup_{i=1}^{n} O_{x_i}$, $O_{x_i} \in \alpha$. Hence, $(\bigcap_{i=1}^{n} B_{x_i}) \bigcap (\bigcup_{i=1}^{n} O_{x_i}) \neq \emptyset$. Then, there exists $i_0 \leq n$, such that $B_{x_{i_0}} \bigcap O_{x_{i_0}} \neq \emptyset$. A contradiction. Thus, (X, U) is complete. □

Corollary 2.1. *Any u-uniformly compact uniform space (X, U) is complete.*

Proposition 2.5. *For a uniform space (X, U_X) the following conditions are equivalent:*

1. *(X, U_X) is u-uniformly compact;*
2. *(X, U_X) is compact.*

Proof. 1. ⇒ 2. The completeness of (X, U_X) implies in Proposition 2.4. Let α be an arbitrary uniform cover of the space (X, U_X). Then, there exists a finite uniform cover $\beta \in U$, such that $\beta \succ \alpha^{\angle}$. For each $B_i \in \beta$ we choose $A_{B_i} \in \alpha^{\angle}$ such that $B_i \subset A_{B_i}$, where $A_{B_i} = \bigcup_{j=1}^{k} A_j$, $A_j \in \alpha$, $j = 1, 2, \ldots, k$. Denote $\alpha_0 = \bigcup \{\alpha_{B_i} : i = 1, 2, \ldots, n\}$, $\alpha_{B_i} = \{B_i \bigcap A_j : j = 1, 2, \ldots, k\}$. Then, α_0 is a finite open cover of the space (X, U_X) and $\alpha_0 \succ \alpha$. Since U_X is universal uniformity, then $\alpha_0 \in U_X$. Thus, (X, U_X) is compact.

2. ⇒ 1. Is implied in Proposition 2.2. □

In the class of precompact uniform spaces compactness, u-uniform compactness and u-uniform finally compactness are equivalent.

Proposition 2.6. *Any u-uniformly compact space (X, U) is strongly uniformly paracompact.*

Proof. Let α be an arbitrary finitely additive open cover of the space (X, U). Then, there exists a finite uniform cover β of the space (X, U) such that the cover β is refined in a cover α. It is easily to see that the cover β is star finite uniform cover of the space (X, U). Thus, the uniform space (X, U) is strongly uniformly paracompact. □

If $(X, \tau)$is a strongly paracompact but not compact space, then a uniform space (X, U_X) according to Proposition 2.1. is strongly uniformly paracompact but not u-uniformly compact.

Example 2.1. Real line $\mathbb{R}$ with the natural uniformity is u-uniformly finally compact.

Proof. Let α be an arbitrary finitely additive cover of $\mathbb{R}$ and $\beta = \{(n-1, n+1) : n = 0, \pm 1, \pm 2, \ldots\}$. Then, $\beta \in U$ is countable uniform cover of $\mathbb{R}$. Since $[n-1, n+1]$ is compactum, then there exists a finite family $\{A_1, A_2, \ldots, A_n\}$, $A_i \in \alpha$, $i = 1, 2, \ldots, n$, such that $(n-1, n+1) \subset [n-1, n+1] \subset \bigcup_{i=1}^{n} A_i$, i.e., $\beta \succ \alpha^{\angle} = \alpha$. Hence, $\mathbb{R}$ is u-uniformly finally compact, but not u-uniformly compact. $\square$

Let (X, τ) be a countable compact, but not compact space. Then a uniform space (X, U_X) according to Proposition 2.1. is a u-uniformly countable compact but not u-uniformly compact.

u-uniformly compactness (u-uniformly countable compactness, u-uniformly finally compactness) inherented by taking closed subspaces.

Theorem 2.2. *Any closed subspace M of u-uniformly compact (u-uniformly countable compact, u-uniformly finally compact) space (X, U) is u-uniformly compact (u-uniformly finally compact) space.*

Proof. Let γ_M be an arbitrary finitely additive (countable) open cover of the subspace (M, U_M). Then there exits a finitely additive (countable) open family γ of the space (X, U), such that $\gamma \wedge \{M\} = \gamma_M$. Put $\hat{\gamma} = \{\gamma, X \backslash M\}$. Obviously the cover $\hat{\gamma}$ is an finitely additive (countable) open cover of the space (X, U). Then, there exists a finite (finite, countable) uniform cover $\beta \in U$, such that $\beta \succ \hat{\gamma}$. Denote $\beta_M = \beta \wedge \{M\}$. Then, it is easy to see that β_M is finite (finite, countable) uniform cover of the subspace (M, U_M) and $\beta_M \succ \gamma_M$. Thus, (M, U_M) is u-uniformly compact (u-uniformly countable compact, u-uniformly finally compact). $\square$

Theorem 2.3. *Let $f : (X, U) \to (Y, V)$ be a ω-mapping between uniform spaces (X, U) and (Y, V). If (Y, V) is u-uniformly compact (u-uniformly finally compact, u-uniformly countable compact) space, then the uniform space (X, U) is also u-uniformly compact (u-uniformly finally compact, u-uniformly countable compact) space.*

Proof. Let ω be an arbitrary finitely additive (countable) open cover of the space (X, U) and $f : (X, U) \to (Y, V)$ be a ω-mapping of the uniform space (X, U) onto u-uniformly compact (u-uniformly countable compact, u-uniformly finally compact) space (Y, V). Then, for each point $y \in Y$ there exists a neighborhood O_y and $W \in \omega$, such that $f^{-1} O_y \subset W$. Denote $\beta = \{O_y : y \in Y\}$, $\beta^{\angle} = \{\bigcup \beta_0 : \beta_0 \subset \beta \text{ is finite}\}$. Since the space (Y, V) is u-uniformly compact (u-uniformly countable compact, u-uniformly finally compact), there exists a finite (finite, countable) uniform cover $\gamma \in V$, such that $\gamma \succ \beta^{\angle}$. Then finite (finite, countable) uniform cover $f^{-1}\gamma$ is refined of cover ω. Thus, (X, U) is u-uniformly compact (u-uniformly countable compact, u-uniformly finally compact). $\square$

Theorem 2.4. *Let $f : (X, U) \to (Y, V)$ be a perfect mapping between uniform spaces (X, U) and (Y, V). If (Y, V) is u-uniformly compact (u-uniformly finally compact, u-uniformly countable compact) space then the uniform space (X, U) is also u-uniformly compact (u-uniformly finally compact, u-uniformly countable compact) space.*

Proof. Let α be an arbitrary finite additive open cover of uniform space (X, U). It is clear that the cover $\{f^{-1}y : y \in Y\}$ is refined of the cover α. Then, $\beta = f^{\#}\alpha = \{f^{\#}A : A \in \alpha\}$, where $f^{\#}A = Y \backslash f(X \backslash A)$ is an open cover of the space (Y, V). Since the space (Y, V) is u-uniformly compact (u-uniformly finally compact, u-uniformly countable compact) space, there exists a finite (countable) cover $\gamma \in V$ such that $\gamma \succ \beta^{\angle}$. It is easy to see that $f^{-1}\gamma \succ \alpha$ and $f^{-1}\gamma$ is finite (countable) cover of the space (X, U). Consequently, the space (X, U) is u-uniformly compact (u-uniformly finally compact, u-uniformly countable compact). □

Corollary 2.2. *Let $f : (X, U) \to (Y, V)$ be a uniformly perfect mapping between uniform spaces. If (Y, V) is u-uniformly compact (u-uniformly finally compact, u-uniformly countable compact) space, then the uniform space (X, U) is also u-uniformly compact (u-uniformly finally compact, u-uniformly countable compact) space.*

Proposition 2.7. *Product $(X, U) \times (Y, V)$ of a u-uniformly compact (u-uniformly countable compact, u-uniformly finally compact) space (X, U) by a compact space (Y, V) is u-uniformly compact (u-uniformly countable compact, u-uniformly finally compact).*

Proof. Let (X, U) be a u-uniformly compact (u-uniformly countable compact, u-uniformly finally compact) and let (Y, V) be compact. It is known that projection $\pi_X : (X, U) \times (Y, V) \to (X, U)$ is uniformly perfect mapping of the product $(X, U) \times (Y, V)$ onto u-uniformly compact (u-uniformly countable compact, u-uniformly finally compact) space (Y, V). Then, projection π_X is ω-mapping for any finitely additive (countable) an open cover ω. Then, according to Theorem 2.3., the product $(X, U) \times (Y, V)$ is u-uniformly compact (u-uniformly countable compact, u-uniformly finally compact). □

Theorem 2.5. *A uniform space (X, U) is u-uniformly compact if and only if (X, U) is u-uniformly finally compact and u-uniformly countable compact.*

Proof. Necessity follows from Proposition 2.3.

Sufficiency. Let α be an arbitrary finally additive open cover of the space (X, U). Since the space (X, U) is u-uniformly finally compact space, there exists a countable cover $\beta \in U$ such that $\beta \succ \alpha$. Since the space (X, U) is u-uniformly countable compact there exists a finite uniform cover $\gamma \in U$, such that $\gamma \succ \beta^{\angle}$. Then, $\gamma \succ \beta^{\angle} \succ \alpha^{\angle} = \alpha$, this means that $\gamma \succ \alpha$. Thus, (X, U) is u-uniformly compact. □

Corollary 2.3. *A Tychonoff space (X, τ) is compact if and only if (X, τ) is finally compact and countable compact.*

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