

CHALLENGES WITHIN FRAMEWORK OF SQUARE MATRIX PENCIL

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ABSTRACT. Matrix pencils emerged in the literature as an important topic of studies in all fields of mathematics, especially in linear algebra, in control systems with linear descriptors. Our findings explore the potential level of the above notion to develop a new technique allowing to describe the spectral values of unbounded operator matrix pencil block 3×3 defined with diagonal domain, $\mathbf{A}_P(t) := t\mathbf{P} - \mathbf{A}$. Specifically, abstract settings are developed. Proper discussions are presented to attend crucial insights into the structure of the eigenvalues associated to our studied matrix pencil $\mathbf{A}_P(t)$ within the context of Fredholm theory. An elegant use of partial differential algebraic equations is stated as an example to motivate the validity of our main results. Our findings come to enrich the ongoing development and enhancement of matrix pencil theory. Particularly, our results provide comprehensive affirmative answers from the inquiries posed by F. Abdmouleh et al. in [1, 2].

1. INTRODUCTION

An important development in operator theory throughout the 20th century was the notion of operator pencil. Such an operator type was thought to be more significant than ordinary operators, demonstrating a noteworthy development in operator theory. For among intensely works done in this prospect, we refer to the sources [14–17, 19, 32, 35]. The birth of such a notion dates backs to 1951 from M. V. Keldysh work's [22, 23] and was later developed by A. S. Markus in [26] and among others. Since then, the ground breaking findings established by M. V. Keldysh and others. So,

Key words and phrases. Operators theory, eigenvalues, spectrum, partial differential-algebraic equations.

2020 *Mathematics Subject Classification.* Primary: 15A60, 34A09, 34D20, 65F15. Secondary: 47A08, 47A10, 47A53, 34K08.

DOI

Received: July 28, 2025.

Accepted: December 23, 2025.

the theory's evolution embracing such notion of operator emerged in mathematical fields marking a pivotal event in operator theory. This concept motivates many mathematicians to produce some rigorously groundwork related within operator pencil form. For a better understanding, in this context we refer the readers to the works of [4, 7, 8, 14, 18, 25, 26, 30].

Subsequently, the above notion of operator has been enlarged and extensively developed through various studies within the matrix framework in a variety of scientific disciplines with significant results [1, 7, 8, 34]. More precisely, the invention done by M. Faierman et al. in [18] enlarged the spectral analysis of linear operator pencil to the framework of matrix pencil block 2×2 . Meanwhile, since such event, the analysis problems around the properties of block $n \times n$ of matrix pencil has developed rapidly linking to some classes of Fredholm perturbation those are indeed insightful to bring quite information in the study of specific spectral values within the domain of operator matrices. Among the works done in this direction for block 2×2 , we quote, for example, [1, 3, 4, 7, 8, 12, 19, 20, 25, 34] also for particular case of block 3×3 was initially explored in the recent relevant papers of I. Walha et al. [2, 5, 6, 9, 10, 27–29].

While this line of research is indeed meaningful spectral properties closely linked to the description of the eigenvalues of operator matrix pencil and the concepts of Fredholm perturbation theory but it limits the scope for particular cases of matrices pencils, thereby for matrices blocks $n \times n$ with $n > 2$. Thus motivates to reach such area within the structure of block 3×3 matrix pencil form.

Operator matrices pencils are quite beneficial in a lot of analytical scenarios. Specifically, they are extensively used in research of control systems with linear descriptors. The widespread problems of eigenvalues of matrix pencils have gained interest for decades mathematicians and engineers. However, in many analytical situations, the operator matrices pencils come in quite handy [4, 7, 8, 11, 16, 17, 19, 20]. It is especially useful for analyzing the generalized eigenvalues and eigenvalues of matrices pencils, which have attracted the attention of researchers for many years. So, as a brief overview of works avoid such analytical studies, we list the following references [4, 7, 8, 14, 16, 17, 19, 20, 32, 35].

More specifically, as a desire to identify the issue of finding the eigenvalues of operators matrices is quite pertinent in the operator theory as well as in linear systems. Nevertheless, our challenge comes back from the hard problem arising in the computation of the eigenvalues of some linear systems within the context of matrix pencil block 3×3 . Thus, keeping this analysis within our operator matrix pencil structure seemed intriguing to us.

Therefore, our challenge in this work keeps us interested in this context as well as we investigate its potential to produce improved structural insights and efficient methods for more effectively computing eigenvalues in linear systems.

To be precise, the key ideas of our findings lies to develop an abstract setting to provide a simple and fine description of the eigenvalues of an unbounded 3×3 block of matrix pencil acting in Banach spaces in terms of their diagonal entries without

imposing any blocks on the entries. Particularly, our motivation aims to draft an interesting use of Fredholm theory in the analysis problem of the left-right Fredholm, left-right Weyl, Fredholm and Weyl of a matrix pencil spectra. Among other things, we explicit an elegant use of the concept within Fredholm theory perturbation in order to provide some relationship between some types of essential spectra of our matrix pencil framework and those of its diagonal entries. Moreover, we derive a key formula of the resolvent form of their studied matrix form. Also, we establish sufficient conditions linking to some classes of Fredholm perturbation, those are used to bring quite information in the study of some relative essential spectra between an unbounded and diagonal blocks 3×3 of operator matrices pencils forms. However, the strategy of our approach avoids the need for partitioning on the entries of our studied matrix pencil, but allowing for a more pertinent analysis only on the entire components pencils entries.

Additionally, our invested results of this study have significant implications, as they not only refine some existing results in the setting of 3×3 block of matrix pencils developed in the works [2, 5, 6, 9, 10, 12, 27–29, 34], but also introduce new sufficient conditions and argumentations for understanding of the left-right Fredholm and Weyl spectra in the setting of unbounded block 3×3 of operator matrix pencil defined with diagonal domain case. These new proposed methodologies in this study insight a precise interaction between the structure of some relative essential spectra of our considered matrix and those of its diagonal entries, which make our results original and strengthen the theoretical foundation of operator matrix pencils theory that is not yet undertaken in such situation. Consequently, we are interested in using this technique to get a weaker criterion that will enable us to identify an interaction between some relative essential spectra of our studied matrix and their relative essential spectra of their diagonal components.

On a different note, partial differential-algebraic equations (DAEs) are introduced with relevant physical and theoretical significance. Whereas, which one hope to enrich the analysis problem within the framework of matrix pencil. Mainly, those are allowing us to understand some spectral properties of a linear. Based up, we are interested to explore a suitable idea making sense to attend the validity of our findings within the domain of partial differential-algebraic equations on $\mathcal{U} \times \mathcal{U} \times \mathcal{U}$, (where the space \mathcal{U} is nothing else $L_p((-a, a) \times (-1, 1), dx d\xi)$, $a > 0$, $p > 1$), as related to the theory of transport equations. Specifically, our physical example states in Section 4 not only demonstrates the importance and applicability of our theoretical framework within the form of unbounded 3×3 block of an operator matrix pencil nonetheless, but could prove their usefulness in the explicit computation of the eigenvalues problem of linear systems. Our contribution ultimately makes use of an improvement in the spectral values in the theory of operator matrices pencils and provides new directions for research and broadens the scope of the analysis problems in the realms of spectral theory.

Our paper is organized as follows. In Section 2, we review some initial findings and terminology that were utilized to make clear understanding for the reader's and how we would progress in the subsequent analysis. In Section 3, yielding an efficient tool to provide a new technique and argumentation in proving some spectral properties of square operator matrix pencil block 3×3 without resorting to any partitioning of their components entries. Specifically, a rigorous discussion and investment are made on the interplay between those diagonal operators pencil entries and several fundamental spectra of our studied model of operator matrix block 3×3 , demonstrating their robustness in the analysis problem of spectral values of our matrix framework. Our findings provide a comprehensive understanding of our primary contributions at the end of our paper to an example of partial differential-algebraic equations.

2. SECTIONAL UNITS

The purpose of this part is to provide some fundamental concepts and auxiliary notations that will be essential in the sequel. Firstly, we will indicate in this paper by:

- $\mathcal{L}(\mathbf{U}_1, \mathbf{U}_2)$: the set of all bounded linear operators acting between Banach spaces \mathbf{U}_1 and \mathbf{U}_2 ,
- $\mathcal{C}(\mathbf{U}_1, \mathbf{U}_2)$: the set of all closed densely defined linear operators acting between Banach spaces \mathbf{U}_1 and \mathbf{U}_2 ,
- $\mathcal{K}(\mathbf{U}_1, \mathbf{U}_2)$: the set of all compact linear operators from \mathbf{U}_1 into \mathbf{U}_2 .

Subsequently, we will be provided the set of common symbols and notations for $A \in \mathcal{C}(\mathbf{U}_1, \mathbf{U}_2)$ as:

- $\mathfrak{D}(A)$: denotes the domain of A ,
- $\mathcal{R}(A) = \{Ax : x \in \mathbf{U}_1\}$: denotes the range of A ,
- $\mathcal{N}(A) = \{x \in \mathbf{U}_1 : Ax = 0\}$: denotes its kernel,
- $\alpha(A) = \dim(\mathcal{N}(A))$: denotes the nullity of A ,
- $\beta(A) = \text{codim}(\mathcal{R}(A))$: denotes the deficiency of A ,
- $\sigma(A)$: the spectrum set of A .

Definition 2.1 (Resolvent set, spectrum and resolvent of an operator pencil). Let $(P, A) \in \mathcal{L}(\mathbf{U}_1, \mathbf{U}_2) \times \mathcal{C}(\mathbf{U}_1, \mathbf{U}_2)$ such that $P \neq 0$ and $P \neq A$. Then, for $t \in \mathbb{C}$, we define by:

$$\varrho_P(A) := \left\{ t \in \mathbb{C} : (tP - A)^{-1} \in \mathcal{L}(\mathbf{U}_1, \mathbf{U}_2) \right\} : \text{the resolvent set of the operator pencil } tP - A,$$

$$\sigma_P(A) := \mathbb{C} \setminus \varrho_P(A) : \text{the spectrum of the operator pencil } tP - A,$$

$$\mathbf{R}_P(t, A) := (tP - A)^{-1} : \text{the resolvent of the operator pencil } tP - A.$$

To keep the final terminology interesting, we need to present a few important subsets of $\mathcal{C}(\mathbf{U}_1, \mathbf{U}_2)$:

$$\Phi_+(\mathbf{U}_1, \mathbf{U}_2) := \{A \in \mathcal{C}(\mathbf{U}_1, \mathbf{U}_2) : \alpha(A) < +\infty \text{ and } \mathcal{R}(A) \text{ is closed}\},$$

$$\begin{aligned} \Phi_-(\mathbf{U}_1, \mathbf{U}_2) &:= \{A \in \mathcal{C}(\mathbf{U}_1, \mathbf{U}_2) : \beta(A) < +\infty \text{ and } \mathcal{R}(A) \text{ is closed}\}, \\ \Phi_{\pm}(\mathbf{U}_1, \mathbf{U}_2) &:= \Phi_+(\mathbf{U}_1, \mathbf{U}_2) \cup \Phi_-(\mathbf{U}_1, \mathbf{U}_2), \\ \Phi_l(\mathbf{U}_1, \mathbf{U}_2) &:= \{A \in \Phi_+(\mathbf{U}_1, \mathbf{U}_2) : \mathcal{R}(A) \text{ is a complemented subspace of } \mathbf{U}_2\}, \\ \Phi_r(\mathbf{U}_1, \mathbf{U}_2) &:= \{A \in \Phi_-(\mathbf{U}_1, \mathbf{U}_2) : \mathcal{N}(A) \text{ is a complemented subspace of } \mathbf{U}_1\}, \\ \Phi(\mathbf{U}_1, \mathbf{U}_2) &:= \Phi_+(\mathbf{U}_1, \mathbf{U}_2) \cap \Phi_-(\mathbf{U}_1, \mathbf{U}_2). \end{aligned}$$

Since $A \in \Phi_{\pm}(\mathbf{U}_1, \mathbf{U}_2)$, then we define its index by:

$$i : \Phi_{\pm}(\mathbf{U}_1, \mathbf{U}_2) \rightarrow \mathbb{Z}, \quad A \mapsto i(A) := \alpha(A) - \beta(A).$$

The classes $\Phi_+(\mathbf{U}_1, \mathbf{U}_2)$, $\Phi_-(\mathbf{U}_1, \mathbf{U}_2)$, $\Phi_{\pm}(\mathbf{U}_1, \mathbf{U}_2)$, $\Phi_l(\mathbf{U}_1, \mathbf{U}_2)$, $\Phi_r(\mathbf{U}_1, \mathbf{U}_2)$ and $\Phi(\mathbf{X}, \mathbf{Y})$, consist of all upper semi-Fredholm, lower semi-Fredholm, Fredholm, left semi-Fredholm, right semi-Fredholm, and Fredholm operators, respectively.

The set of left Weyl, right Weyl, and Weyl operators are defined, respectively, as:

$$\begin{aligned} \mathcal{W}_l(\mathbf{U}_1, \mathbf{U}_2) &:= \{A \in \mathcal{C}(\mathbf{U}_1, \mathbf{U}_2) : A \in \Phi_l(\mathbf{U}_1, \mathbf{U}_2) \text{ and } i(A) \leq 0\}, \\ \mathcal{W}_r(\mathbf{U}_1, \mathbf{U}_2) &:= \{A \in \mathcal{C}(\mathbf{U}_1, \mathbf{U}_2) : A \in \Phi_r(\mathbf{U}_1, \mathbf{U}_2) \text{ and } i(A) \geq 0\}, \\ \mathcal{W}(\mathbf{U}_1, \mathbf{U}_2) &:= \mathcal{W}_l(\mathbf{U}_1, \mathbf{U}_2) \cap \mathcal{W}_r(\mathbf{U}_1, \mathbf{U}_2). \end{aligned}$$

For $t \in \mathbb{C}$, let $A \in \mathcal{C}(\mathbf{U}_1, \mathbf{U}_2)$ and $P \in \mathcal{L}(\mathbf{U}_1, \mathbf{U}_2)$ a non null operator. The operator pencil $tP - A$ has corresponding essential spectra that are specified as follows:

$$\begin{aligned} \sigma_{P, \Phi_l}(A) &:= \{t \in \mathbb{C} : tP - A \notin \Phi_l(\mathbf{U}_1, \mathbf{U}_2)\} : \text{the } P\text{-left Fredholm spectrum,} \\ \sigma_{P, \Phi_r}(A) &:= \{t \in \mathbb{C} : tP - A \notin \Phi_r(\mathbf{U}_1, \mathbf{U}_2)\} : \text{the } P\text{-right Fredholm spectrum,} \\ \sigma_{P, \Phi}(A) &:= \{t \in \mathbb{C} : tP - A \notin \Phi(\mathbf{U}_1, \mathbf{U}_2)\} : \text{the } P\text{-Fredholm spectrum,} \\ \sigma_{P, \mathcal{W}_l}(A) &:= \{t \in \mathbb{C} : tP - A \notin \mathcal{W}_l(\mathbf{U}_1, \mathbf{U}_2)\} : \text{the } P\text{-left Weyl spectrum,} \\ \sigma_{P, \mathcal{W}_r}(A) &:= \{t \in \mathbb{C} : tP - A \notin \mathcal{W}_r(\mathbf{U}_1, \mathbf{U}_2)\} : \text{the } P\text{-right Weyl spectrum,} \\ \sigma_{P, \mathcal{W}}(A) &:= \{t \in \mathbb{C} : tP - A \notin \mathcal{W}(\mathbf{U}_1, \mathbf{U}_2)\} : \text{the } P\text{-Weyl spectrum.} \end{aligned}$$

The essential spectra before introduced can be ordered as:

$$(2.1) \quad \begin{array}{ccccccc} & & \sigma_{P, \Phi_+}(A) & \subset & \sigma_{P, \Phi_l}(A) & \subset & \sigma_{P, \mathcal{W}_l}(A) \\ & & \subset & & \subset & & \subset \\ \sigma_{P, \Phi_{\pm}}(A) & \subset & & & & \sigma_{P, \Phi}(A) & \subset & \sigma_{P, \mathcal{W}}(A) \\ & & \subset & & \subset & & \subset & \\ & & \sigma_{P, \Phi_-}(A) & \subset & \sigma_{P, \Phi_r}(A) & \subset & \sigma_{P, \mathcal{W}_r}(A) \end{array}$$

where

$$\begin{aligned} \sigma_{P, \Phi_*}(A) &:= \{t \in \mathbb{C} : tP - A \notin \Phi_*(\mathbf{U}_1, \mathbf{U}_2)\}, \quad \text{for } * = \{+, -\}, \\ \sigma_{P, \Phi_{\pm}}(A) &:= \sigma_{P, \Phi_+}(A) \cap \sigma_{P, \Phi_-}(A). \end{aligned}$$

3. BACKGROUND THEORY AND KEY FINDINGS

Our main contribution in this section aims to develop some spectral inclusions between an unbounded block 3×3 of operator matrix pencil and their diagonal entries. For such formulations, we start to introduced the following terminologies:

- a bounded operator matrix block 3×3 :

$$\mathbf{P} := \begin{pmatrix} P_1 & P_2 & P_3 \\ P_4 & P_5 & P_6 \\ P_7 & P_8 & P_9 \end{pmatrix},$$

- an unbounded 3×3 block of operator matrix \mathbf{A} as:

$$\mathbf{M} := \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix}, \quad \text{with diagonal domain } \mathfrak{D}(\mathbf{M}) := \mathfrak{D}(A) \times \mathfrak{D}(E) \times \mathfrak{D}(K),$$

with unbounded component entries having an appropriate domain and acting between their corresponding spaces $\mathbf{U}_k, k \in \{1, 2, 3\}$.

First, we are able to introduce the unbounded 3×3 block of operator matrix pencil with diagonal domain.

Definition 3.1 (Unbounded 3×3 block of operator matrix pencil with diagonal domain). In the product of Banach spaces $\prod_{k=1}^3 \mathbf{U}_k$, an unbounded 3×3 block of operator matrix pencil $\mathbf{A}_P(t) := t\mathbf{P} - \mathbf{A}$ with diagonal domain $\mathfrak{D}(\mathbf{A}_P(t))$ is defined as:

$$(3.1) \quad \mathbf{A}_P(t) := \begin{pmatrix} tP_1 - A & tP_2 - B & tP_3 - C \\ tP_4 - D & tP_5 - E & tP_6 - F \\ tP_7 - G & tP_8 - H & tP_9 - K \end{pmatrix},$$

$$\mathfrak{D}(\mathbf{A}_P(t)) = \mathfrak{D}(A) \times \mathfrak{D}(E) \times \mathfrak{D}(K), \text{ for } t \in \mathbb{C}.$$

In view of Definition 3.1, the entries of our studied model of matrix pencil $\mathbf{A}_P(t)$ obey to the serial of assumptions, those are fundamental to our findings. Based up, we will express them as well.

(\mathcal{H}_1) The diagonal operators pencils of $\mathbf{A}_P(t), tP_1 - A, tP_5 - E$ and $tP_9 - K$ are assume closed with densely defined domain $\mathfrak{D}(tP_1 - A) := \mathfrak{D}(A), \mathfrak{D}(tP_5 - E) := \mathfrak{D}(E)$ and $\mathfrak{D}(tP_9 - K) := \mathfrak{D}(K)$ having non empty resolvent sets $\varrho_{P_1}(A), \varrho_{P_5}(E)$ and $\varrho_{P_9}(K)$, i.e., $\varrho_{P_1}(A) \neq \emptyset, \varrho_{P_5}(E) \neq \emptyset$ and $\varrho_{P_9}(K) \neq \emptyset$, respectively.

(\mathcal{H}_2) For $t \in \varrho_{P_1}(A)$, the entries D and G satisfy the following assumptions:

(i)

$$\mathfrak{D}(tP_4 - D) := \mathfrak{D}(D)$$

\subset

$$\mathfrak{D}(tP_1 - A) := \mathfrak{D}(A)$$

\subset

$$\mathfrak{D}(tP_7 - G) := \mathfrak{D}(G),$$

(ii) $D(tP_1 - A)^{-1} \in \mathcal{L}(\mathbf{U}_1, \mathbf{U}_2)$ and $G(tP_1 - A)^{-1} \in \mathcal{L}(\mathbf{U}_1, \mathbf{U}_3)$.

(\mathcal{H}_3) For $t \in \varrho_{P_5}(E)$, the entries B and H adhere to the following presumptions:

(i)

$$\mathfrak{D}(tP_2 - B) := \mathfrak{D}(B)$$

⊂

$$\mathfrak{D}(tP_5 - E) := \mathfrak{D}(E)$$

⊂

$$\mathfrak{D}(tP_8 - H) := \mathfrak{D}(H),$$

(ii) $B(tP_5 - E)^{-1} \in \mathcal{L}(\mathbf{U}_2, \mathbf{U}_1)$ and $H(tP_5 - E)^{-1} \in \mathcal{L}(\mathbf{U}_2, \mathbf{U}_3)$.

(\mathcal{H}_4) For $t \in \varrho_{P_9}(K)$, the operators C and F check the following claims:

(i)

$$\mathfrak{D}(tP_3 - C) := \mathfrak{D}(C)$$

⊂

$$\mathfrak{D}(tP_9 - K) := \mathfrak{D}(K)$$

⊂

$$\mathfrak{D}(tP_6 - F) := \mathfrak{D}(F),$$

(ii) $C(tP_9 - K)^{-1} \in \mathcal{L}(\mathbf{U}_3, \mathbf{U}_1)$ and $F(tP_9 - K)^{-1} \in \mathcal{L}(\mathbf{U}_3, \mathbf{U}_2)$.

From what proceed, the following terminologies will be introduced as:

- $\mathbb{X}_1(t) := (tP_4 - D)(tP_1 - A)^{-1} \in \mathcal{L}(\mathbf{U}_1, \mathbf{U}_2)$,
- $\mathbb{Y}_1(t) := (tP_2 - B)(tP_5 - E)^{-1} \in \mathcal{L}(\mathbf{U}_2, \mathbf{U}_1)$,
- $\mathbb{X}_2(t) := (tP_7 - G)(tP_1 - A)^{-1} \in \mathcal{L}(\mathbf{U}_1, \mathbf{U}_3)$,
- $\mathbb{Y}_2(t) := (tP_3 - C)(tP_9 - K)^{-1} \in \mathcal{L}(\mathbf{U}_3, \mathbf{U}_1)$,
- $\mathbb{X}_3(t) := (tP_8 - H)(tP_5 - E)^{-1} \in \mathcal{L}(\mathbf{U}_2, \mathbf{U}_3)$,
- $\mathbb{Y}_3(t) := (tP_6 - F)(tP_9 - K)^{-1} \in \mathcal{L}(\mathbf{U}_3, \mathbf{U}_2)$,

for $t \in \varrho_{P_1}(A) \cap \varrho_{P_5}(E) \cap \varrho_{P_9}(K)$.

Hence, drawing on Proposition 3.2 in [28], we recall the associated new factorization form of our studied model of matrix pencil.

Lemma 3.1 (New factorization form of operator matrix pencil block 3×3). *Let's $t \in \mathbb{C}$ and assume that the following items hold:*

- (i) (\mathcal{H}_1) – (\mathcal{H}_4) are satisfied,
- (ii) $t \in \varrho_{P_1}(A) \cap \varrho_{P_5}(E) \cap \varrho_{P_9}(K)$.

Thus, $\mathbf{A}_P(t)$ will be expressed as:

$$(3.2) \quad \mathbf{A}_P(t) := \mathbf{V}_P(t)\mathbf{W}_P(t) \\ := \begin{pmatrix} I & \mathbb{Y}_1(t) & \mathbb{Y}_2(t) \\ \mathbb{X}_1(t) & I & \mathbb{Y}_3(t) \\ \mathbb{X}_2(t) & \mathbb{X}_3(t) & I \end{pmatrix} \begin{pmatrix} tP_1 - A & 0 & 0 \\ 0 & tP_5 - E & 0 \\ 0 & 0 & tP_9 - K \end{pmatrix}.$$

In order to provide additional context, we begin by recalling the two distinctive theoretical operators pencil analogue to the characteristic operators associated to our type of matrix pencil form.

Definition 3.2 (Characteristic functions of our matrix pencil $\mathbf{A}_P(t)$). [28, Definition 3.4] Assume that the hypotheses (\mathcal{H}_1) - (\mathcal{H}_4) hold.

(i) For $t \in \varrho_{P_1}(A) \cap \varrho_{P_5}(E) \cap \varrho_{P_9}(K)$ and $1 \in \varrho_I(\mathbb{X}_1(t)\mathbb{Y}_1(t))$, we define the \mathbf{U}_2 -characteristic and \mathbf{U}_3 -characteristic operators pencils functions of our matrix pencil $\mathbf{A}_P(t)$, respectively, as follows:

$$\Delta_{\mathbf{U}_2}(t) := I - \mathbb{X}_1(t)\mathbb{Y}_1(t)$$

and

$$\begin{aligned} \Delta_{\mathbf{U}_3}(t) := & I - \mathbb{X}_2(t)\mathbb{Y}_2(t) \\ & - [\mathbb{X}_3(t) - \mathbb{X}_2(t)\mathbb{Y}_1(t)]\mathbf{R}_I(1, \mathbb{X}_1(t)\mathbb{Y}_1(t)) [\mathbb{Y}_3(t) - \mathbb{X}_1(t)\mathbb{Y}_2(t)]. \end{aligned}$$

(ii) For $t \in \varrho_{P_1}(A) \cap \varrho_{P_5}(E) \cap \varrho_{P_9}(K)$ with $1 \in \varrho_I(\mathbb{Y}_3(t)\mathbb{X}_3(t))$, we define the \mathbf{U}_2 -characteristic and \mathbf{U}_1 -characteristic operators pencils functions of the matrix $\mathbf{A}_P(t)$, respectively, as the functions:

$$\Lambda_{\mathbf{U}_2}(t) := I - \mathbb{Y}_3(t)\mathbb{X}_3(t)$$

and

$$\begin{aligned} \Lambda_{\mathbf{U}_1}(t) := & I - \mathbb{Y}_2(t)\mathbb{X}_2(t) \\ & - [\mathbb{Y}_1(t) - \mathbb{Y}_2(t)\mathbb{X}_3(t)]\mathbf{R}_I(1, \mathbb{Y}_3(t)\mathbb{X}_3(t)) [\mathbb{X}_1(t) - \mathbb{Y}_3(t)\mathbb{X}_2(t)]. \end{aligned}$$

Before moving to keep the key tool for our investigations, we will use the above characteristic operators functions in order to compute the resolvent expressions of our matrix pencil $\mathbf{A}_P(t)$:

- $\mathbb{T}(t) := [\mathbb{X}_3(t) - \mathbb{X}_2(t)\mathbb{Y}_1(t)]\mathbf{R}_I(1, \mathbb{X}_1(t)\mathbb{Y}_1(t))$,
- $\mathbb{Q}(t) := \mathbf{R}_I(1, \mathbb{X}_1(t)\mathbb{Y}_1(t)) [\mathbb{Y}_3(t) - \mathbb{X}_1(t)\mathbb{Y}_2(t)]$,
- $\tilde{\mathbb{Q}}(t) := [\mathbb{Y}_1(t) - \mathbb{Y}_2(t)\mathbb{X}_3(t)]\mathbf{R}_I(1, \mathbb{Y}_3(t)\mathbb{X}_3(t))$,
- $\tilde{\mathbb{T}}(t) := \mathbf{R}_I(1, \mathbb{Y}_3(t)\mathbb{X}_3(t)) [\mathbb{X}_1(t) - \mathbb{Y}_3(t)\mathbb{X}_2(t)]$.

Lemma 3.2 (Resolvent expressions of operator matrix pencil block 3×3). [28] Suppose that the operator $\mathbf{A}_P(t)$ satisfies the assumptions (\mathcal{H}_1) - (\mathcal{H}_4) . Then, for $t \in \varrho_{P_1}(A) \cap \varrho_{P_5}(E) \cap \varrho_{P_9}(K)$, we get the following.

(i) If $\Delta_{\mathbf{U}_2}(t)$ and $\Delta_{\mathbf{U}_3}(t)$ are invertible for $1 \in \varrho_I(\mathbb{X}_1(t)\mathbb{Y}_1(t))$, then the formal expression for the resolvent expression of $\mathbf{A}_P(t)$ is as follows:

$$\begin{aligned} \mathbf{R}_P(t, \mathbf{M}) := & \mathbf{R}_I(0, \mathbf{A}_P(t)) \\ (3.3) \quad & := \text{diag}(\mathbf{R}_{P_1}(t, A), \mathbf{R}_{P_5}(t, E), \mathbf{R}_{P_9}(t, K)) + (\mathcal{R}_{ij}(t))_{1 \leq i, j \leq 3}, \end{aligned}$$

where the entries $\mathcal{R}_{ij}(t)$ are provided as follows:

$$\begin{aligned} \mathcal{R}_{11}(t) := & \mathbf{R}_{P_1}(t, A)\mathbb{Y}_1(t)\mathbf{R}_I(0, \Delta_{\mathbf{U}_2}(t))\mathbb{X}_1(t) \\ & + \mathbf{R}_{P_1}(t, A) ([\mathbb{Y}_1(t)\mathbb{Q}(t) - \mathbb{Y}_2(t)]\mathbf{R}_I(0, \Delta_{\mathbf{U}_3}(t)) \times [\mathbb{T}(t)\mathbb{X}_1(t) - \mathbb{X}_2(t)]). \end{aligned}$$

$$\begin{aligned}
\mathcal{R}_{12}(t) &:= \mathbf{R}_{P_1}(t, A) \mathbb{Y}_1(t) \mathbf{R}_I(0, \Delta_{U_2}(t)) \\
&\quad - \mathbf{R}_{P_1}(t, A) ([\mathbb{Y}_1(t) \mathbb{Q}(t) - \mathbb{Y}_2(t)] \mathbf{R}_I(0, \Delta_{U_3}(t)) \mathbb{T}(t)), \\
\mathcal{R}_{13}(t) &:= \mathbf{R}_{P_1}(t, A) ([\mathbb{Y}_1(t) \mathbb{Q}(t) - \mathbb{Y}_2(t)] \mathbf{R}_I(0, \Delta_{U_3}(t))), \\
\mathcal{R}_{21}(t) &:= - \mathbf{R}_{P_5}(t, E) \mathbf{R}_I(0, \Delta_{U_2}(t)) \mathbb{X}_1(t) \\
&\quad - \mathbf{R}_{P_5}(t, E) \mathbb{Q}(t) \mathbf{R}_I(0, \Delta_{U_3}(t)) [\mathbb{T}(t) \mathbb{X}_1(t) - \mathbb{X}_2(t)], \\
\mathcal{R}_{22}(t) &:= - \mathbf{R}_{P_5}(t, E) (I - \mathbf{R}_I(0, \Delta_{U_2}(t))) + \mathbf{R}_{P_2}(t, E) \mathbb{Q}(t) \mathbf{R}_I(0, \Delta_{U_3}(t)) \mathbb{T}(t), \\
\mathcal{R}_{23}(t) &:= \mathbf{R}_{P_5}(t, E) \mathbb{Q}(t) \mathbf{R}_I(0, \Delta_{U_3}(t)), \\
\mathcal{R}_{31}(t) &:= \mathbf{R}_{P_1}(t, A) \mathbf{R}_I(0, \Delta_{U_3}(t)) [\mathbb{T}(t) \mathbb{X}_1(t) - \mathbb{X}_2(t)], \\
\mathcal{R}_{32}(t) &:= - \mathbf{R}_{P_9}(t, K) \mathbf{R}_I(0, \Delta_{E_3}(t)) \mathbb{T}(t), \\
\mathcal{R}_{33}(t) &:= - \mathbf{R}_{P_1}(t, A) (I - \mathbf{R}_I(0, \Delta_{U_3}(t))).
\end{aligned}$$

(ii) If $\Lambda_{U_1}(t)$ and $\Lambda_{U_2}(t)$ are invertible for $1 \in \varrho_I(\mathbb{Y}_3(t) \mathbb{X}_3(t))$, then $\mathbf{A}_P(t)$'s resolvent expression is provided as follows:

$$\begin{aligned}
\mathbf{R}_P(t, \mathbf{M}) &:= \mathbf{R}_I(0, \mathbf{A}_P(t)) \\
(3.4) \quad &:= \text{diag}(\mathbf{R}_{P_1}(t, A), \mathbf{R}_{P_5}(t, E), \mathbf{R}_{P_9}(t, K)) + \left(\tilde{\mathcal{R}}_{ij}(t) \right)_{1 \leq i, j \leq 3},
\end{aligned}$$

with the following the entries $\tilde{\mathcal{R}}_{ij}(t)$:

$$\begin{aligned}
\tilde{\mathcal{R}}_{11}(t) &:= - \mathbf{R}_{P_1}(t, A) (I - \mathbf{R}_I(0, \Lambda_{U_1}(t))), \\
\tilde{\mathcal{R}}_{12}(t) &:= - \mathbf{R}_{P_1}(t, A) \mathbf{R}_I(0, \Lambda_{U_1}(t)) \tilde{\mathbb{Q}}(t), \\
\tilde{\mathcal{R}}_{13}(t) &:= \mathbf{R}_{P_1}(t, A) \mathbf{R}_I(0, \Lambda_{U_1}(t)) [\tilde{\mathbb{Q}}(t) \mathbb{Y}_3(t) - \mathbb{Y}_2(t)], \\
\tilde{\mathcal{R}}_{21}(t) &:= - \mathbf{R}_{P_5}(t, E) \tilde{\mathbb{T}}(t) \mathbf{R}_I(0, \Lambda_{U_1}(t)), \\
\tilde{\mathcal{R}}_{22}(t) &:= \mathbf{R}_{P_5}(t, E) \tilde{\mathbb{T}}(t) \mathbf{R}_I(0, \Lambda_{U_1}(t)) \tilde{\mathbb{Q}}(t) - \mathbf{R}_{P_5}(t, E) (I - \mathbf{R}_I(0, \Lambda_{U_2}(t))), \\
\tilde{\mathcal{R}}_{23}(t) &:= - \mathbf{R}_{P_5}(t, E) \tilde{\mathbb{T}}(t) \mathbf{R}_I(0, \Lambda_{U_1}(t)) [\tilde{\mathbb{Q}}(t) \mathbb{Y}_3(t) - \mathbb{Y}_2(t)] \\
&\quad - \mathbf{R}_{P_5}(t, E) \mathbf{R}_I(0, \Lambda_{U_2}(t)) \mathbb{Y}_3(t), \\
\tilde{\mathcal{R}}_{31}(t) &:= \mathbf{R}_{P_9}(t, K) [\mathbb{X}_3(t) \tilde{\mathbb{T}}(t) - \mathbb{X}_2(t)] \mathbf{R}_I(0, \Lambda_{U_1}(t)), \\
\tilde{\mathcal{R}}_{32}(t) &:= - \mathbf{R}_{P_9}(t, K) [\mathbb{X}_3(t) \tilde{\mathbb{T}}(t) - \mathbb{X}_2(t)] \mathbf{R}_I(0, \Lambda_{U_1}(t)) \tilde{\mathbb{Q}}(t) \\
&\quad - \mathbf{R}_{P_9}(t, K) \mathbb{X}_3(t) \mathbf{R}_I(0, \Lambda_{U_2}(t)), \\
\tilde{\mathcal{R}}_{33}(t) &:= \mathbf{R}_{P_9}(t, K) [\mathbb{X}_3(t) \tilde{\mathbb{T}}(t) - \mathbb{X}_2(t)] \mathbf{R}_I(0, \Lambda_{U_1}(t)) [\tilde{\mathbb{Q}}(t) \mathbb{Y}_3(t) - \mathbb{Y}_2(t)] \\
&\quad + \mathbf{R}_{P_9}(K) \mathbb{X}_3(t) \mathbf{R}_I(0, \Lambda_{U_2}(t)) \mathbb{Y}_3(t).
\end{aligned}$$

We shall utilize these findings to introduce the primary instrument for our research. However, we observe firstly, that $\tilde{\mathbf{A}}_P(t)$ is equal to nothing else $\mathbf{A}_P(t)$ with null entries P_2, P_3, P_4, P_6, P_7 and P_8 . Secondly, for $t \in \varrho_{P_1}(A) \cap \varrho_{P_5}(E) \cap \varrho_{P_9}(K)$ such

that $1 \in \varrho_I(\tilde{X}_1(t)\tilde{Y}_1(t)) \cap \varrho_I(\tilde{Y}_3(t)\tilde{X}_3(t))$, we will use the following terminologies:

$$\begin{aligned} \tilde{\Delta}_{U_2}(t) &:= I - \tilde{X}_1(t)\tilde{Y}_1(t), \\ \tilde{\Delta}_{U_3}(t) &:= I - \tilde{X}_2(t)\tilde{Y}_2(t) \\ &\quad - [\tilde{X}_3(t) - \tilde{X}_2(t)\tilde{Y}_1(t)] \mathbf{R}_I(1, \tilde{X}_1(t)\tilde{Y}_1(t)) [\tilde{Y}_3(t) - \tilde{X}_1(t)\tilde{Y}_2(t)], \\ \tilde{\Lambda}_{U_1}(t) &:= I - \tilde{Y}_3(t)\tilde{X}_3(t), \\ \tilde{\Lambda}_{U_2}(t) &:= I - \tilde{Y}_2(t)\tilde{X}_2(t) \\ &\quad - [\tilde{Y}_1(t) - \tilde{Y}_2(t)\tilde{X}_3(t)] \mathbf{R}_I(1, \tilde{Y}_3(t)\tilde{X}_3(t)) [\tilde{X}_1(t) - \tilde{Y}_3(t)\tilde{X}_2(t)], \end{aligned}$$

where $\tilde{X}_1(t)$ (resp. $\tilde{X}_2(t)$ and $\tilde{X}_3(t)$) is expressed as $X_1(t)$ (resp. $X_2(t)$ and $X_3(t)$) with null entries P_4 (resp. P_7 and P_8) and $\tilde{Y}_1(t)$ (resp. $\tilde{Y}_2(t)$ and $\tilde{Y}_3(t)$) is nothing else as $Y_1(t)$ (resp. $Y_2(t)$ and $Y_3(t)$) with zero entries P_2 (resp. P_3 and P_6). At the same time, we omit the following notations:

- $\hat{T}(t) := [\tilde{X}_3(t) - \tilde{X}_2(t)\tilde{Y}_1(t)] \mathbf{R}_I(1, \tilde{X}_1(t)\tilde{Y}_1(t))$,
- $\hat{Q}(t) := \mathbf{R}_I(1, \tilde{X}_1(t)\tilde{Y}_1(t)) [\tilde{Y}_3(t) - \tilde{X}_1(t)\tilde{Y}_2(t)]$,
- $\hat{Q}(t) := [\tilde{Y}_1(t) - \tilde{Y}_2(t)\tilde{X}_3(t)] \mathbf{R}_I(1, \tilde{Y}_3(t)\tilde{X}_3(t))$,
- $\hat{T}(t) := \mathbf{R}_I(1, \tilde{Y}_3(t)\tilde{X}_3(t)) [\tilde{X}_1(t) - \tilde{Y}_3(t)\tilde{X}_2(t)]$.

Theorem 3.1. (Spectral values of operator pencil $\mathbf{A}_P(t)$ with diagonal domain) Assume that:

- (i) the assumptions $(\mathcal{H}_1) - (\mathcal{H}_6)$ are fulfilled,
- (ii) $t \in \varrho_{P_1}(A) \cap \varrho_{P_5}(E) \cap \varrho_{P_9}(K)$,
- (iii) $\tilde{\Delta}_{U_2}(t)$ (resp. $\tilde{\Lambda}_{U_1}(t)$) and $\tilde{\Delta}_{U_3}(t)$ (resp. $\tilde{\Lambda}_{U_2}(t)$) are invertible for $1 \in \varrho_I(\tilde{X}_1(t)\tilde{Y}_1(t))$ (resp. $1 \in \varrho_I(\tilde{Y}_3(t)\tilde{X}_3(t))$),
- (iv) $P_2, P_3, P_4, P_6, P_7, P_8, \tilde{X}_k(t)$ and $\tilde{Y}_k(t)$, for $k \in \{1, 2, 3\}$ and $t \in \varrho_{P_1}(A) \cap \varrho_{P_5}(E) \cap \varrho_{P_9}(K)$, are compact operators. Then, we obtain:

$$\sigma_{\mathbf{P}, \Phi_*}(\mathbf{A}) = \sigma_{P_1, \Phi_*}(A) \cup \sigma_{P_5, \Phi_*}(E) \cup \sigma_{P_9, \Phi_*}(K),$$

for $(\sigma_{\mathbf{P}, \Phi_*}(\cdot), \sigma_{P_k, \Phi_*}(\cdot)) \in \{(\sigma_{\mathbf{P}, \Phi_l}(\cdot), \sigma_{P_k, \Phi_*}(\cdot)), (\sigma_{\mathbf{P}, \Phi_r}(\cdot), \sigma_{P_k, \Phi_r}(\cdot)), (\sigma_{\mathbf{P}, \Phi}(\cdot), \sigma_{P_k, \Phi}(\cdot))\}$, and $\sigma_{\mathbf{P}, W_*}(\mathbf{A}) \subseteq \sigma_{P_1, W_*}(A) \cup \sigma_{P_5, W_*}(E) \cup \sigma_{P_9, W_*}(K)$, for

$(\sigma_{\mathbf{P}, W_*}(\cdot), \sigma_{P_k, W_*}(\cdot)) \in \{(\sigma_{\mathbf{P}, W_l}(\cdot), \sigma_{P_k, W_l}(\cdot)), (\sigma_{\mathbf{P}, W_r}(\cdot), \sigma_{P_k, W_r}(\cdot)), (\sigma_{\mathbf{P}, W}(\cdot), \sigma_{P_k, W}(\cdot))\}$,

$k \in \{1, 5, 9\}$. Additionally, since ${}^C\sigma_{P_1, \Phi_*}(A)$, ${}^C\sigma_{P_5, \Phi_*}(E)$ and ${}^C\sigma_{P_9, \Phi_*}(K)$ are connected, one obtains:

$$\sigma_{\mathbf{P}, W_*}(\mathbf{A}) = \sigma_{P_1, W_*}(A) \cup \sigma_{P_5, W_*}(E) \cup \sigma_{P_9, W_*}(K), \quad \text{for } * \in \{l, r\}.$$

Proof. The strategy of the proof of our investigation is formulated to provide firstly the link between the relative spectra of \mathbf{A} or the essential spectra of the pencil matrix $\mathbf{A}_P(t)$ and $\tilde{\mathbf{A}}_P(t)$. Secondly, subsequent link in terms of the relative essential spectra between $\tilde{\mathbf{A}}_P(t)$ and those diagonal entries will be derived.

Indeed, our finding will be proved by steps.

Step 1. For the sake of simplest, we omit the following expression of our matrix pencil $\mathbf{A}_{\mathbf{P}}(t)$ as the following form:

$$\mathbf{A}_{\mathbf{P}}(t) := \widetilde{\mathbf{A}}_{\mathbf{P}_d}(t) + \widetilde{\mathbb{P}}(t),$$

with

$$\widetilde{\mathbf{A}}_{\mathbf{P}_d}(t) := \begin{pmatrix} tP_1 - A & -B & -C \\ -D & tP_5 - E & F \\ -G & -H & tP_9 - K \end{pmatrix} \quad \text{and} \quad \widetilde{\mathbb{P}}(t) := \begin{pmatrix} 0 & tP_2 & tP_3 \\ tP_4 & 0 & tP_6 \\ tP_7 & tP_8 & 0 \end{pmatrix}.$$

For $t \in \varrho_{P_1}(A) \cap \varrho_{P_5}(E) \cap \varrho_{P_9}(K)$, the compactness criterion of $\widetilde{\mathbb{P}}(t)$ on $\prod_{k=1}^3 \mathbf{U}_k$ will be derived easily from the compactness criteria given by item **(iv)**. Hence, from [12, Theorem 2.2] and [13, Corollary 3.1], we obtain that:

$$(3.5) \quad \sigma_*(\mathbf{A}_{\mathbf{P}}(t)) = \sigma_*\left(\widetilde{\mathbf{A}}_{\mathbf{P}_d}(t)\right),$$

where $\sigma_*(\cdot)$ represents the left (resp. right) Fredholm, Fredholm, left (resp. right) Weyl and Weyl essential spectra of an operator pencil (who is defined as the set of all $t \in \mathbb{C}$ under which such operator pencil is no left (resp. right) Fredholm, Fredholm, left (resp. right) Weyl and Weyl operator). Obviously, this result follows immediately from the intrinsic properties of compactness and the fundamental relationships between these essential spectra.

Subsequently, we will leveraging in the next step the above described result within the context of matrix framework. In other terms, we highlights some spectral links between the matrix pencil $\widetilde{\mathbf{A}}_{\mathbf{P}_d}(t)$ and those diagonal operators pencil entries $tP_1 - A$, $tP_5 - E$ and $tP_9 - K$.

Step 2. We will proof that:

$$\sigma_*\left(\widetilde{\mathbf{A}}_{\mathbf{P}_d}(t)\right) = \sigma_*\left(\widetilde{\mathbf{A}}_{d\mathbf{P}_d}(t)\right) = \bigcup_{k=1}^3 \sigma_*(\mathbb{D}_k(t)),$$

where:

$$\widetilde{\mathbf{A}}_{d\mathbf{P}_d}(t) := \text{diag}(tP_1 - A, tP_5 - E, tP_9 - K) := \text{diag}(\mathbb{D}_1(t), \mathbb{D}_2(t), \mathbb{D}_3(t)).$$

In fact, we derive from the use of items **(ii)** and **(iv)** that $\widetilde{\mathbf{A}}_{\mathbf{P}_d}(t)$ is invertible. Consequently, an explicit formulation of its resolvent $\mathbf{R}_{\mathbf{I}}\left(0, \widetilde{\mathbf{A}}_{\mathbf{P}_d}(t)\right)$ is provided as follows, in accordance with Lemma 3.2:

$$(3.6) \quad \mathbf{R}_{\mathbf{I}}\left(0, \widetilde{\mathbf{A}}_{\mathbf{P}_d}(t)\right) := \mathbf{R}_{\mathbf{I}}\left(0, \widetilde{\mathbf{A}}_{d\mathbf{P}_d}(t)\right) + (\mathcal{Q}_{ij}(t))_{1 \leq i, j \leq 3}$$

and

$$(3.7) \quad \mathbf{R}_{\mathbf{I}}\left(0, \widetilde{\mathbf{A}}_{\mathbf{P}_d}(t)\right) := \mathbf{R}_{\mathbf{I}}\left(0, \widetilde{\mathbf{A}}_{d\mathbf{P}_d}(t)\right) + \left(\widetilde{\mathcal{Q}}_{ij}(t)\right)_{1 \leq i, j \leq 3},$$

where

$$\mathcal{Q}_{11}(t) := \mathbf{R}_{P_1}(t, A) \widetilde{\mathbb{Y}}_1(t) \mathbf{R}_{\mathbf{I}}\left(0, \widetilde{\Delta}_{\mathbf{U}_2}(t)\right) \widetilde{\mathbb{X}}_1(t)$$

$$\begin{aligned}
& + \mathbf{R}_{P_1}(t, A) \left(\left[\tilde{\mathbf{Y}}_1(t) \hat{\mathbf{Q}}(t) - \tilde{\mathbf{Y}}_2(t) \right] \mathbf{R}_I \left(0, \tilde{\Delta}_{\mathbf{U}_3}(t) \right) \times \left[\hat{\mathbf{T}}(t) \tilde{\mathbf{X}}_1(t) - \tilde{\mathbf{X}}_2(t) \right] \right), \\
\mathcal{Q}_{12}(t) & := \mathbf{R}_{P_1}(t, A) \tilde{\mathbf{Y}}_1(t) \mathbf{R}_I \left(0, \tilde{\Delta}_{\mathbf{U}_2}(t) \right) \\
& \quad - \mathbf{R}_{P_1}(t, A) \left(\left[\tilde{\mathbf{Y}}_1(t) \hat{\mathbf{Q}}(t) - \tilde{\mathbf{Y}}_2(t) \right] \mathbf{R}_I \left(0, \tilde{\Delta}_{\mathbf{U}_3}(t) \right) \hat{\mathbf{T}}(t) \right), \\
\mathcal{Q}_{13}(t) & := \mathbf{R}_{P_1}(t, A) \left(\left[\tilde{\mathbf{Y}}_1(t) \hat{\mathbf{Q}}(t) - \tilde{\mathbf{Y}}_2(t) \right] \mathbf{R}_I \left(0, \tilde{\Delta}_{\mathbf{U}_3}(t) \right) \right), \\
\mathcal{Q}_{21}(t) & := - \mathbf{R}_{P_5}(t, E) \mathbf{R}_I \left(0, \tilde{\Delta}_{\mathbf{U}_2}(t) \right) \tilde{\mathbf{X}}_1(t) \\
& \quad - \mathbf{R}_{P_5}(t, E) \hat{\mathbf{Q}}(t) \mathbf{R}_I \left(0, \tilde{\Delta}_{\mathbf{U}_3}(t) \right) \left[\hat{\mathbf{T}}(t) \tilde{\mathbf{X}}_1(t) - \tilde{\mathbf{X}}_2(t) \right], \\
\mathcal{Q}_{22}(t) & := - \mathbf{R}_{P_5}(t, E) \left(I - \mathbf{R}_I \left(0, \tilde{\Delta}_{\mathbf{U}_2}(t) \right) \right) + \mathbf{R}_{P_2}(t, E) \hat{\mathbf{Q}}(t) \mathbf{R}_I \left(0, \tilde{\Delta}_{\mathbf{U}_3}(t) \right) \hat{\mathbf{T}}(t), \\
\mathcal{Q}_{23}(t) & := \mathbf{R}_{P_5}(t, E) \hat{\mathbf{Q}}(t) \mathbf{R}_I \left(0, \tilde{\Delta}_{\mathbf{U}_3}(t) \right), \\
\mathcal{Q}_{31}(t) & := \mathbf{R}_{P_1}(t, A) \mathbf{R}_I \left(0, \tilde{\Delta}_{\mathbf{U}_3}(t) \right) \left[\hat{\mathbf{T}}(t) \tilde{\mathbf{X}}_1(t) - \tilde{\mathbf{X}}_2(t) \right], \\
\mathcal{Q}_{32}(t) & := - \mathbf{R}_{P_9}(t, K) \mathbf{R}_I \left(0, \tilde{\Delta}_{\mathbf{E}_3}(t) \right) \hat{\mathbf{T}}(t), \\
\mathcal{Q}_{33}(t) & := - \mathbf{R}_{P_1}(t, A) \left(I - \mathbf{R}_I \left(0, \tilde{\Delta}_{\mathbf{U}_3}(t) \right) \right),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathcal{Q}}_{11}(t) & := - \mathbf{R}_{P_1}(t, A) \left(I - \mathbf{R}_I \left(0, \tilde{\Lambda}_{\mathbf{U}_1}(t) \right) \right), \\
\tilde{\mathcal{Q}}_{12}(t) & := - \mathbf{R}_{P_1}(t, A) \mathbf{R}_I \left(0, \tilde{\Lambda}_{\mathbf{U}_1}(t) \right) \hat{\mathbf{Q}}(t), \\
\tilde{\mathcal{Q}}_{13}(t) & := \mathbf{R}_{P_1}(t, A) \mathbf{R}_I \left(0, \tilde{\Lambda}_{\mathbf{U}_1}(t) \right) \hat{\mathbf{Q}}(t) \tilde{\mathbf{Y}}_3(t) - \tilde{\mathbf{Y}}_2(t), \\
\tilde{\mathcal{Q}}_{21}(t) & := - \mathbf{R}_{P_5}(t, E) \hat{\mathbf{T}}(t) \mathbf{R}_I \left(0, \tilde{\Lambda}_{\mathbf{U}_1}(t) \right), \\
\tilde{\mathcal{Q}}_{22}(t) & := \mathbf{R}_{P_5}(t, E) \hat{\mathbf{T}}(t) \mathbf{R}_I \left(0, \tilde{\Lambda}_{\mathbf{U}_1}(t) \right) \hat{\mathbf{Q}}(t) - \mathbf{R}_{P_5}(t, E) \left(I - \mathbf{R}_I \left(0, \tilde{\Lambda}_{\mathbf{U}_2}(t) \right) \right), \\
\tilde{\mathcal{Q}}_{23}(t) & := - \mathbf{R}_{P_5}(t, E) \hat{\mathbf{T}}(t) \mathbf{R}_I \left(0, \tilde{\Lambda}_{\mathbf{U}_1}(t) \right) \hat{\mathbf{Q}}(t) \tilde{\mathbf{Y}}_3(t) - \tilde{\mathbf{Y}}_2(t) \\
& \quad - \mathbf{R}_{P_5}(t, E) \mathbf{R}_I \left(0, \tilde{\Lambda}_{\mathbf{U}_2}(t) \right) \tilde{\mathbf{Y}}_3(t), \\
\tilde{\mathcal{Q}}_{31}(t) & := \mathbf{R}_{P_9}(t, K) \left[\tilde{\mathbf{X}}_3(t) \hat{\mathbf{T}}(t) - \tilde{\mathbf{X}}_2(t) \right] \mathbf{R}_I \left(0, \tilde{\Lambda}_{\mathbf{U}_1}(t) \right), \\
\tilde{\mathcal{Q}}_{32}(t) & := - \mathbf{R}_{P_9}(t, K) \left[\tilde{\mathbf{X}}_3(t) \hat{\mathbf{T}}(t) - \tilde{\mathbf{X}}_2(t) \right] \mathbf{R}_I \left(0, \tilde{\Lambda}_{\mathbf{U}_1}(t) \right) \hat{\mathbf{Q}}(t) \\
& \quad - \mathbf{R}_{P_9}(t, K) \tilde{\mathbf{X}}_3(t) \mathbf{R}_I \left(0, \tilde{\Lambda}_{\mathbf{U}_2}(t) \right), \\
\tilde{\mathcal{Q}}_{33}(t) & := \mathbf{R}_{P_9}(t, K) \left[\tilde{\mathbf{X}}_3(t) \hat{\mathbf{T}}(t) - \tilde{\mathbf{X}}_2(t) \right] \mathbf{R}_I \left(0, \tilde{\Lambda}_{\mathbf{U}_1}(t) \right) \left[\hat{\mathbf{Q}}(t) \tilde{\mathbf{Y}}_3(t) - \tilde{\mathbf{Y}}_2(t) \right] \\
& \quad + \mathbf{R}_{P_9}(t, K) \tilde{\mathbf{X}}_3(t) \mathbf{R}_I \left(0, \tilde{\Lambda}_{\mathbf{U}_2}(t) \right) \tilde{\mathbf{Y}}_3(t),
\end{aligned}$$

respectively. Consequently, from the compactness criterion of perturbation of the operators $\tilde{X}_k(t)$ and $\tilde{Y}_k(t)$, for $k \in \{1, 2, 3\}$, we show that

$$(3.8) \quad \mathbf{R}_I(0, \tilde{\mathbf{A}}_{\mathbf{P}_d}(t)) - \mathbf{R}_I(0, \tilde{\mathbf{A}}_{d\mathbf{P}_d}(t)) \in \mathcal{K} \left(\prod_{k=1}^3 \mathbf{U}_k \right)$$

and

$$(3.9) \quad \mathbf{R}_I(0, \tilde{\mathbf{A}}_{\mathbf{P}_d}(t)) - \mathbf{R}_I(0, \tilde{\mathbf{A}}_{d\mathbf{P}_d}(t)) \in \mathcal{K} \left(\prod_{k=1}^3 \mathbf{U}_k \right),$$

respectively, follow from the intrinsic properties of the class of compact operators, which it is a closed two-sided ideal of the set of bounded operators.

Henceforth, according Eqs. (3.8) and (3.9) to Theorem 2.3 in [12], Theorem 3.3 in [24] and Corollary 4 in [33], we omit that the filling in $\sigma_* \left(\tilde{\mathbf{A}}_{\mathbf{P}_d}(t) \right)$ should take place in $\sigma_* \left(\tilde{\mathbf{A}}_{d\mathbf{P}_d}(t) \right)$ for $\sigma_*(\cdot) \in \{ \sigma_{\Phi_l}(\cdot), \sigma_{\Phi_r}(\cdot), \sigma_{\Phi}(\cdot), \sigma_{W_l}(\cdot), \sigma_{W_r}(\cdot), \sigma_W(\cdot) \}$, with

$$i \left(\tilde{\mathbf{A}}_{\mathbf{P}_d}(t) \right) := i \left(\tilde{\mathbf{A}}_{d\mathbf{P}_d}(t) \right) = \sum_{k=1}^3 i(\mathbb{D}_k(t)),$$

in contrast that $\tilde{\mathbf{A}}_{d\mathbf{P}_d}(t)$ is a diagonal operator matrix. Yielding that

$$\sigma_* \left(\tilde{\mathbf{A}}_{\mathbf{P}_d}(t) \right) = \sigma_* \left(\tilde{\mathbf{A}}_{d\mathbf{P}_d}(t) \right) = \bigcup_{k=1}^3 \sigma_*(\mathbb{D}_k(t)), \quad \sigma_*(\cdot) \in \{ \sigma_{\Phi_l}(\cdot), \sigma_{\Phi_r}(\cdot), \sigma_{\Phi}(\cdot) \},$$

and

$$\sigma_* \left(\tilde{\mathbf{A}}_{\mathbf{P}_d}(t) \right) \subseteq \bigcup_{k=1}^3 \sigma_*(\mathbb{D}_k(t)), \quad \sigma_*(\cdot) \in \{ \sigma_{W_l}(\cdot), \sigma_{W_r}(\cdot), \sigma_W(\cdot) \}.$$

Consequently, the connect criteria of the component

$${}^C \sigma_{P_l, \Phi_*}(A), {}^C \sigma_{P_s, \Phi_*}(E), {}^C \sigma_{P_\emptyset, \Phi_*}(K), \quad * \in \{l, r\},$$

inherit the property of relative left-right Weyl spectra of our matrix form as:

$$\sigma_* \left(\tilde{\mathbf{A}}_{\mathbf{P}_d}(t) \right) = \bigcup_{k=1}^3 \sigma_*(\mathbb{D}_k(t)), \quad \sigma_*(\cdot) \in \{ \sigma_{W_l}(\cdot), \sigma_{W_r}(\cdot) \},$$

follow from Lemma 4.1 in [13].

Subsequently, the relative Weyl spectrum type of $\tilde{\mathbf{A}}_{\mathbf{P}_d}(t)$, $\sigma_W \left(\tilde{\mathbf{A}}_{\mathbf{P}_d}(t) \right)$, may be obvious from the specific subsets within $\sigma_{W_l} \left(\tilde{\mathbf{A}}_{\mathbf{P}_d}(t) \right) \cup \sigma_{W_r} \left(\tilde{\mathbf{A}}_{\mathbf{P}_d}(t) \right)$. \square

So, our approach is iterated using a model of partial differential-algebraic equations in the final part.

4. EXAMPLE OF PARTIAL DIFFERENTIAL-ALGEBRAIC EQUATIONS

Partial differential-algebraic equations (DAEs) arise in various fields such as physics, engineering and economics. Operator pencils are often used to model partial differential-algebraic equations which can be viewed as coupled equations of partial differential equations subject to linear constraints. Keeping this in mind, the concept of addressing these challenges within the context of partial differential-algebraic equations becomes an appealing alternative to attend our theoretical findings. More precisely, in this part, we consider infinite-dimensional partial differential-algebraic equations modeled as:

$$(DAEs) \frac{d}{dt} Sz(t) = E_{\mathbf{D}} z(t), \quad Sz(0) = Sz_0, \quad z(t) := (z_1(t), z_2(t), z_3(t))^t,$$

to validate our theoretical framework, where:

$$S := \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \quad \text{and} \quad E_{\mathbf{D}} := \begin{pmatrix} T_{\mathbf{D}_1} & K_{c12} & K_{c13} \\ K_{c21} & T_{\mathbf{D}_2} & K_{c23} \\ K_{c31} & K_{c32} & T_{\mathbf{D}_3} \end{pmatrix}.$$

Hence, with the above mentioned partial differential-algebraic equations (DAEs), we associate our operator matrix pencil:

$$\begin{aligned} \mathcal{A}(\lambda) &:= \lambda S - E_{\mathbf{D}} \quad (\text{on } \mathcal{U} \times \mathcal{U} \times \mathcal{U}), \\ &= (\mathcal{A}_{kk}(\lambda))_{1 \leq k \leq 3}, \end{aligned}$$

with

◊ the diagonal entries $\mathcal{A}_{kk}(\lambda) := \lambda S_{kk} - T_{\mathbf{D}_k}$, $k \in \{1, \dots, 3\}$, are expressed as:

$$\mathcal{A}_{kk}(\lambda) : \mathfrak{D}(\mathcal{A}_{kk}(\lambda)) \subseteq \mathcal{U} \rightarrow \mathcal{U}, \quad z_k \mapsto (\mathcal{A}_{kk}(\lambda))z_k,$$

$$(x, \xi) \mapsto (\mathcal{A}_{kk}(\lambda))z_k(x, \xi) := \lambda \eta_k(\xi) z_k(x, \xi) - \xi \frac{\partial z_k(x, \xi)}{\partial x} - \sigma_k(\xi) z_k(x, \xi),$$

for $(x, \xi) \in (-a, a) \times (-1, 1)$, on the domain $\mathfrak{D}(\mathcal{A}_{kk}(\lambda)) = \{z_k \in \mathcal{U} \cap \mathbf{S} : z_k^i = \mathbf{D}_k z_k^o\}$,

◊ the off-diagonal operators pencils $\mathcal{A}_{kj}(\lambda) := \lambda S_{kj} - K_{ckj}$, $k \neq j \in \{1, \dots, 3\}$ are defined as follows

$$\mathcal{A}_{kj}(\lambda) : \mathcal{U} \rightarrow \mathcal{U}, \quad u \mapsto \mathcal{A}_{kj}(\lambda)u,$$

$$(x, \xi) \mapsto (\mathcal{A}_{kj}(\lambda)u)(x, \xi) = \lambda S_{kj}u(x, \xi) - \int_{-1}^1 \kappa_{ckj}(x, \xi, \xi') u(x, \xi') d\xi',$$

under the following terminologies:

- $\mathcal{U} := L_p((-a, a) \times (-1, 1), dx d\xi)$, $a > 0$ and $p > 1$,
- $\mathcal{U}^o := L_p(\{-a\} \times (-1, 0), |\xi| d\xi) \times L_p(\{a\} \times (0, 1), |\xi| d\xi)$,
- $\mathcal{U}^i := L_p(\{-a\} \times (0, 1), |\xi| d\xi) \times L_p(\{a\} \times (-1, 0), |\xi| d\xi)$,
- $\mathbf{S} := \left\{ u \in \mathcal{U} : v \frac{\partial u}{\partial x} \in \mathcal{U} \right\}$: defines the partial Sobolev space,
- $\eta_k(\cdot)$: defines the coefficient of multiplication of the operator S_{kk} ,
- $\sigma_k(\cdot) \in L^\infty(-1, 1)$: represents the collision frequency are in $L^\infty(-1, 1)$,

- (z^i, z^o) : express the trace functions corresponding to the incoming and the outgoing fluxes, respectively, for any function z ,
- $\kappa_{ckj} : (-a, a) \times (-1, 1) \times (-1, 1) \rightarrow \mathbb{R}$: referred to as the scattering kernel of the collision operator K_{ckj} ,
- $\mathbf{D}_k \in \mathcal{L}(\mathcal{U}^o, \mathcal{U}^i)$,
- $S_{kj} \in \mathcal{L}(\mathcal{U})$, $k \neq j \in \{1, \dots, 3\}$.

Hence, keeping with the above descriptive example, it is known also as three groups of transport operator pencils with reflective boundary conditions modeled as $z^i = \mathbf{D}z^o$, for $\mathbf{D} \in \mathcal{L}(\mathcal{U}^o \times \mathcal{U}^o \times \mathcal{U}^o, \mathcal{U}^i \times \mathcal{U}^i \times \mathcal{U}^i)$. Strictly speaking, it is represented as a perturbation of a closed linear operator pencil

$$\lambda S_d - T_{\mathbf{D}} := \text{diag}(\lambda S_{11} - T_{\mathbf{D}_1}, \lambda S_{22} - T_{\mathbf{D}_2}, \lambda S_{33} - T_{\mathbf{D}_3}),$$

with bounded linear operator matrix pencil

$$\mathbf{P}_c = \begin{pmatrix} 0 & \lambda M_2 - K_{c12} & \lambda M_3 - K_{c13} \\ \lambda M_4 - K_{c21} & 0 & \lambda M_6 - K_{c23} \\ \lambda M_7 - K_{c31} & \lambda M_8 - K_{c32} & 0 \end{pmatrix},$$

on the following domain:

$$\begin{aligned} \mathfrak{D}(A(\lambda)) &:= \mathfrak{D}(\lambda S_d - T_{\mathbf{D}}) = \bigcap_{k=1}^3 \mathfrak{D}(\lambda S_k - T_{\mathbf{D}_k}) \\ &= \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \middle| \begin{array}{l} z_1 \in \mathcal{U} \cap \mathbf{S} \\ z_2 \in \mathcal{U} \cap \mathbf{S} \\ z_3 \in \mathcal{U} \cap \mathbf{S} \end{array} : \begin{array}{l} z_1^i = \mathbf{D}_1 z_1^o \\ z_2^i = \mathbf{D}_2 z_2^o \\ z_3^i = \mathbf{D}_3 z_3^o \end{array} \right\} \\ &= \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \middle| \begin{array}{l} z_1 \in \mathcal{U} \cap \mathbf{S} \\ z_2 \in \mathcal{U} \cap \mathbf{S} \\ z_3 \in \mathcal{U} \cap \mathbf{S} \end{array} : \begin{pmatrix} z_1^i \\ z_2^i \\ z_3^i \end{pmatrix} = \begin{pmatrix} \mathbf{D}_1 & 0 & 0 \\ 0 & \mathbf{D}_2 & 0 \\ 0 & 0 & \mathbf{D}_3 \end{pmatrix} \begin{pmatrix} z_1^o \\ z_2^o \\ z_3^o \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbf{S} \times \mathbf{S} \times \mathbf{S} : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}^i = \mathbf{D} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}^o \right\}. \end{aligned}$$

A remarkable and seamless alignment with our theoretical framework will be structured below as:

- $\mathbf{U}_k \curvearrowright \mathcal{U}$, for $1 \leq k \leq 3$,
- $tP_1 - A \curvearrowright \lambda S_{11} - T_{\mathbf{D}_1}$, $\mathfrak{D}(A) := \mathfrak{D}(T_{\mathbf{D}_1})$,
- $tP_2 - B \curvearrowright \lambda S_{12} - K_{c12}$,
- $tP_3 - C \curvearrowright \lambda S_{13} - K_{c13}$,
- $tP_4 - D \curvearrowright \lambda S_{21} - K_{c21}$,
- $tP_5 - E \curvearrowright \lambda S_{22} - T_{\mathbf{D}_2}$, $\mathfrak{D}(E) := \mathfrak{D}(T_{\mathbf{D}_2})$,
- $tP_6 - F \curvearrowright \lambda S_{23} - K_{c23}$,
- $tP_7 - G \curvearrowright \lambda S_{31} - K_{c31}$,
- $tP_8 - H \curvearrowright \lambda S_{32} - K_{c32}$,
- $tP_9 - K \curvearrowright \lambda S_{33} - T_{\mathbf{D}_3}$, $\mathfrak{D}(K) := \mathfrak{D}(T_{\mathbf{D}_3})$,

- $\mathbf{A_P}(t) \curvearrowright \mathcal{A}(\lambda)$, ; with $\mathfrak{D}(\mathcal{A}(\lambda)) := \prod_{k=1}^3 \mathfrak{D}(\lambda S_{kk} - T_{\mathbf{D}_k})$.

Remark 4.1. • Based on the works of M. Kharroubi [31], S. Charfi et l. in [12] and I. Walha in [34], one has $\lambda S_{kk} - T_{\mathbf{D}_k}$, $k \in \{1, 2, 3\}$, is a closed, densely defined linear operator. Therefore, the validity of our finding presented in Section 3 by assumption (\mathcal{H}_1) is satisfied for this example.

• Furthermore, drawing from the boundedness property of the collision operators $K_{c_{kj}}$, $1 \leq k \neq j \leq 3$, and for $\lambda \in \bigcap_{k=1}^3 \varrho_{S_{kk}}(T_{\mathbf{D}_k})$, we have:

$$(4.1) \quad K_{c_{jk}} \mathbf{R}_{S_{kk}}(\lambda, T_{\mathbf{D}_k}) \leq \frac{\|K_{c_{jk}}\|}{\mu_k^* \operatorname{Re}(\lambda)} \left(p^{(\frac{-1}{p})} \frac{\|\mathbf{D}_k\|}{1 - e^{-2a} \mu_k^* \operatorname{Re}(\lambda) \|\mathbf{D}_k\|} + 1 \right),$$

for $\operatorname{Re}(\lambda) > 0$, such that $\mu_k^* \operatorname{Re}(\lambda) + \lambda_k^* > 0$, where the reals λ_k^* and μ_k^* are defined, respectively, as:

$$\begin{aligned} \lambda_k^* &:= \inf_{\xi \in (-1,1)} \sigma_k(\xi) = 0, \\ \mu_k^* &:= \inf_{\xi \in (-1,1)} \eta_k(\xi) > 0, \quad \text{for } k \in \{1, 2, 3\}. \end{aligned}$$

Thus, the conditions imposed on the entries of the matrix operator \mathbf{P}_c fulfill the assumptions within the scope of this example, namely denoted as (\mathcal{H}_2) - (\mathcal{H}_4) , in Section 3.

Subsequently, we are interested to prove our first main outcome via the regularity concept of the collision operator within the sense of M. Mokhtar Kharroubi as defined in [31]. This theorem will guide us to a sufficient condition that guarantees our desired criteria for perturbations in the aforementioned example.

Lemma 4.1. *Let us assume for $\lambda \in \bigcap_{k=1}^3 \varrho_{S_{kk}}(T_{\mathbf{D}_k})$, that*

(i) *the collision operators $K_{c_{21}}$, $K_{c_{31}}$ and $K_{c_{32}}$ are regulars, respectively, then we have:*

$$\begin{aligned} \tilde{\mathbf{X}}_1(\lambda) &:= -K_{c_{21}} \mathbf{R}_{S_{11}}(\lambda, T_{\mathbf{D}_1}) \in \mathcal{K}(\mathcal{U}), \\ \tilde{\mathbf{X}}_2(\lambda) &:= -K_{c_{31}} \mathbf{R}_{S_{11}}(\lambda, T_{\mathbf{D}_1}) \in \mathcal{K}(\mathcal{U}), \\ \tilde{\mathbf{X}}_3(\lambda) &:= -K_{c_{32}} \mathbf{R}_{S_{22}}(\lambda, T_{\mathbf{D}_2}) \in \mathcal{K}(\mathcal{U}), \end{aligned}$$

(ii) *the collision operators $K_{c_{12}}$, $K_{c_{13}}$ and $K_{c_{23}}$ are regulars, hence we get*

$$\begin{aligned} \tilde{\mathbf{Y}}_1(\lambda) &:= -K_{c_{12}} \mathbf{R}_{S_{22}}(\lambda, T_{\mathbf{D}_2}) \in \mathcal{K}(\mathcal{U}), \\ \tilde{\mathbf{Y}}_2(\lambda) &:= -K_{c_{13}} \mathbf{R}_{S_{33}}(\lambda, T_{\mathbf{D}_3}) \in \mathcal{K}(\mathcal{U}), \\ \tilde{\mathbf{Y}}_3(\lambda) &:= -K_{c_{23}} \mathbf{R}_{S_{33}}(\lambda, T_{\mathbf{D}_3}) \in \mathcal{K}(\mathcal{U}). \end{aligned}$$

Proof. The proof of these required results can be easily derived from the use of Lemma 4.1 in [34]. □

To proceed further and keeping with the associated terminologies with our theoretical framework, we list thoughtfully the following structured notations:

$$\begin{aligned} \tilde{\Delta}_{\mathbf{U}_2}(t) \curvearrowright \tilde{\Delta}_{2,u}(\lambda) &:= I - K_{c_{21}} \mathbf{R}_{S_{11}}(\lambda, T_{\mathbf{D}_1}) K_{c_{12}} \mathbf{R}_{S_{22}}(\lambda, T_{\mathbf{D}_2}), \\ \tilde{\Delta}_{\mathbf{U}_3}(t) \curvearrowright \tilde{\Delta}_{3,u}(\lambda) &:= I - K_{c_{31}} \mathbf{R}_{S_{11}}(\lambda, T_{\mathbf{D}_1}) K_{c_{13}} \mathbf{R}_{S_{33}}(\lambda, T_{\mathbf{D}_3}) \\ &\quad - (-K_{c_{32}} \mathbf{R}_{S_{22}}(\lambda, T_{\mathbf{D}_2}) - K_{c_{31}} \mathbf{R}_{S_{11}}(\lambda, T_{\mathbf{D}_1}) K_{c_{12}} \mathbf{R}_{S_{22}}(\lambda, T_{\mathbf{D}_2})) \\ &\quad \times \mathbf{R}_I \left(1, \tilde{\mathbb{X}}_1(\lambda) \tilde{\mathbb{Y}}_1(\lambda) \right) \\ &\quad \times (-K_{c_{23}} \mathbf{R}_{S_{33}}(\lambda, T_{\mathbf{D}_3}) - K_{c_{21}} \mathbf{R}_{S_{11}}(\lambda, T_{\mathbf{D}_1}) K_{c_{13}} \mathbf{R}_{S_{33}}(\lambda, T_{\mathbf{D}_3})), \\ \tilde{\Lambda}_{\mathbf{U}_1}(t) \curvearrowright \tilde{\Lambda}_{1,u}(\lambda) &:= I - K_{c_{23}} \mathbf{R}_{S_{33}}(\lambda, T_{\mathbf{D}_3}) K_{c_{32}} \mathbf{R}_{S_{22}}(\lambda, T_{\mathbf{D}_2}). \\ \tilde{\Lambda}_{\mathbf{U}_2}(t) \curvearrowright \tilde{\Lambda}_{2,u}(\lambda) &:= I - K_{c_{13}} \mathbf{R}_{S_{33}}(\lambda, T_{\mathbf{D}_3}) K_{c_{31}} \mathbf{R}_{S_{11}}(\lambda, T_{\mathbf{D}_1}) \\ &\quad - (-K_{c_{12}} \mathbf{R}_{S_{22}}(\lambda, T_{\mathbf{D}_2}) - K_{c_{13}} \mathbf{R}_{S_{33}}(\lambda, T_{\mathbf{D}_3}) K_{c_{32}} \mathbf{R}_{S_{22}}(\lambda, T_{\mathbf{D}_2})) \\ &\quad \times \mathbf{R}_I \left(1, \tilde{\mathbb{Y}}_3(\lambda) \tilde{\mathbb{X}}_3(\lambda) \right) \\ &\quad \times (-K_{c_{21}} \mathbf{R}_{S_{11}}(\lambda, T_{\mathbf{D}_1}) - K_{c_{23}} \mathbf{R}_{S_{33}}(\lambda, T_{\mathbf{D}_3}) K_{c_{32}} \mathbf{R}_{S_{22}}(\lambda, T_{\mathbf{D}_2})), \end{aligned}$$

for $\lambda \in \bigcap_{k=1}^3 \varrho_{S_{kk}}(T_{\mathbf{D}_k})$ with $1 \in \varrho_I \left(\tilde{\mathbb{X}}_1(\lambda) \tilde{\mathbb{Y}}_1(\lambda) \right) \cap \varrho_I \left(\tilde{\mathbb{Y}}_3(\lambda) \tilde{\mathbb{X}}_3(\lambda) \right)$.

Building upon the findings presented in the previous lemma, it is obvious to reach the following statement.

Corollary 4.1. *Let $\lambda \in \bigcap_{k=1}^3 \varrho_{S_{kk}}(T_{\mathbf{D}_k})$ such that $\text{Re}(\lambda) > 0$. Since $K_{c_{kj}}$ is assumed to be regular, then the following items hold true:*

- (i) $1 \in \varrho_I \left(\tilde{\mathbb{X}}_1(\lambda) \tilde{\mathbb{Y}}_1(\lambda) \right) \cap \varrho_I \left(\tilde{\mathbb{Y}}_3(\lambda) \tilde{\mathbb{X}}_3(\lambda) \right)$,
- (ii) $\tilde{\Delta}_{2,u}(\lambda)$ and $\tilde{\Lambda}_{1,u}(\lambda)$ are invertible.

Furthermore, for $1 \in \varrho_I \left(\tilde{\mathbb{X}}_1(\lambda) \tilde{\mathbb{Y}}_1(\lambda) \right)$ and $1 \in \varrho_I \left(\tilde{\mathbb{Y}}_3(\lambda) \tilde{\mathbb{X}}_3(\lambda) \right)$, we have:

- (iii) $\tilde{\Delta}_{3,u}(\lambda)$ and $\tilde{\Lambda}_{2,u}(\lambda)$ are invertible too, respectively.

Proof. Consider $\lambda \in \bigcap_{k=1}^3 \varrho_{S_{kk}}(T_{\mathbf{D}_k}) \cap \mathbb{E}$ (here $\mathbb{E} := \{t \in \mathbb{C} : \text{Re}(t) > 0\}$).

(i) Bearing up Eq. (4.1) with the regularity assumption of $K_{c_{kj}}$, we omit the following inequality for the compact operators $\tilde{\mathbb{X}}_1(\lambda) \tilde{\mathbb{Y}}_1(\lambda)$ and $\tilde{\mathbb{Y}}_3(\lambda) \tilde{\mathbb{X}}_3(\lambda)$:

$$\begin{aligned} \left\| \tilde{\mathbb{X}}_1(\lambda) \tilde{\mathbb{Y}}_1(\lambda) \right\| &\leq \|K_{c_{21}}\| \cdot \|K_{c_{12}}\| \\ &\quad \times \prod_{k=1}^2 \frac{1}{\mu_k^* \text{Re}(\lambda)} \left(p^{\left(\frac{-1}{p}\right)} \frac{\|\mathbf{D}_k\|}{1 - e^{-2a} \mu_k^* \text{Re}(\lambda) \|\mathbf{D}_k\|} + 1 \right) \end{aligned}$$

and

$$\begin{aligned} \left\| \tilde{\mathbb{Y}}_3(\lambda) \tilde{\mathbb{X}}_3(\lambda) \right\| &\leq \|K_{c_{23}}\| \cdot \|K_{c_{32}}\| \\ &\quad \times \prod_{k=2}^3 \frac{1}{\mu_k^* \text{Re}(\lambda)} \left(p^{\left(\frac{-1}{p}\right)} \frac{\|\mathbf{D}_k\|}{1 - e^{-2a} \mu_k^* \text{Re}(\lambda) \|\mathbf{D}_k\|} + 1 \right). \end{aligned}$$

Consequently, we obtain:

$$\lim_{\operatorname{Re} \lambda \rightarrow +\infty} r_\sigma \left(\tilde{\mathbb{X}}_1(\lambda) \tilde{\mathbb{Y}}_1(\lambda) \right) = 0 \quad \text{and} \quad \lim_{\operatorname{Re} \lambda \rightarrow +\infty} r_\sigma \left(\tilde{\mathbb{Y}}_3(\lambda) \tilde{\mathbb{X}}_3(\lambda) \right) = 0.$$

Thus, assert that there exists enough large scalar $\lambda_0 \in \mathbb{E}$ for which the operators $I - \tilde{\mathbb{X}}_1(\lambda) \tilde{\mathbb{Y}}_1(\lambda)$ and $I - \tilde{\mathbb{Y}}_3(\lambda) \tilde{\mathbb{X}}_3(\lambda)$ are invertible. Hence full, Gohberg-Smul'yan theorem [21, Theorem 11.4] yields that the operator $I - \tilde{\mathbb{X}}_1(\lambda) \tilde{\mathbb{Y}}_1(\lambda)$ (resp. $I - \tilde{\mathbb{Y}}_3(\lambda) \tilde{\mathbb{X}}_3(\lambda)$) is invertible for all $\lambda \in \mathbb{E}$ except for a countable subset contained in \mathbb{E} . This implies that $1 \in \varrho_I \left(\tilde{\mathbb{X}}_1(\lambda) \tilde{\mathbb{Y}}_1(\lambda) \right) \cap \varrho_I \left(\tilde{\mathbb{Y}}_3(\lambda) \tilde{\mathbb{X}}_3(\lambda) \right)$ and so the outcome.

(ii) The required results of this item follows easily from the last item.

(iii) Assume further that $1 \in \varrho_I \left(\tilde{\mathbb{X}}_1(\lambda) \tilde{\mathbb{Y}}_1(\lambda) \right)$ and $1 \in \varrho_I \left(\tilde{\mathbb{Y}}_3(\lambda) \tilde{\mathbb{X}}_3(\lambda) \right)$. Thus, we obtain in view of Eq. (4.1):

$$\begin{aligned} \|I - \tilde{\Delta}_{3,u}(\lambda)\| &\leq \frac{\|K_{c_{31}}\|}{\mu_1^* \operatorname{Re}(\lambda)} \left(p^{\left(\frac{-1}{p}\right)} \frac{\|\mathbf{D}_1\|}{1 - e^{-2a} \mu_1^* \operatorname{Re}(\lambda) \|\mathbf{D}_1\|} + 1 \right) \mathbb{W}_1 \\ &\quad + \frac{\|K_{c_{32}}\|}{\mu_2^* \operatorname{Re}(\lambda)} \left(p^{\left(\frac{-1}{p}\right)} \frac{\|\mathbf{D}_2\|}{1 - e^{-2a} \mu_2^* \operatorname{Re}(\lambda) \|\mathbf{D}_2\|} + 1 \right) \mathbb{W}_2 \end{aligned}$$

and

$$\begin{aligned} \|I - \tilde{\Lambda}_{2,u}(\lambda)\| &\leq \frac{\|K_{c_{13}}\|}{\mu_3^* \operatorname{Re}(\lambda)} \left(p^{\left(\frac{-1}{p}\right)} \frac{\|\mathbf{D}_3\|}{1 - e^{-2a} \mu_3^* \operatorname{Re}(\lambda) \|\mathbf{D}_3\|} + 1 \right) \tilde{\mathbb{W}}_1 \\ &\quad + \frac{\|K_{c_{12}}\|}{\mu_2^* \operatorname{Re}(\lambda)} \left(p^{\left(\frac{-1}{p}\right)} \frac{\|\mathbf{D}_2\|}{1 - e^{-2a} \mu_2^* \operatorname{Re}(\lambda) \|\mathbf{D}_2\|} + 1 \right) \tilde{\mathbb{W}}_2, \end{aligned}$$

for bounded operators \mathbb{W}_k and $\tilde{\mathbb{W}}_k$, $k \in \{1, 2\}$. Therefore, the fact that $r_\sigma(\cdot) \leq \|\cdot\|$, reveals that:

$$\lim_{\operatorname{Re} \lambda \rightarrow +\infty} r_\sigma \left(I - \tilde{\Delta}_{3,u}(\lambda) \right) = 0 \quad \text{and} \quad \lim_{\operatorname{Re} \lambda \rightarrow +\infty} r_\sigma \left(I - \tilde{\Lambda}_{2,u}(\lambda) \right) = 0.$$

These results highlight the existence of enough large scalar $\lambda_0 \in \mathbb{E}$ under which the operators $\tilde{\Delta}_{3,u}(\lambda)$ and $\tilde{\Lambda}_{2,u}(\lambda)$ are invertible. Consequently, according to Theorem 11.4 in [21], we conclude that the operator $\tilde{\Delta}_{3,u}(\lambda)$ and $\tilde{\Lambda}_{2,u}(\lambda)$ are too invertible for all $\lambda \in \mathbb{E}$ except for a countable subset contained in \mathbb{E} . Hence, we derive our claim. \square

The spectral values of our matrix framework will be inferred in the subsequent theorem.

Theorem 4.1. Consider the following for $\lambda \in \bigcap_{k=1}^3 \varrho_{S_{kk}}(T_{\mathbf{D}_k})$ satisfying that $\operatorname{Re}(\lambda) > 0$:

- (i) the operators \mathbf{D}_k , $k \in \{1, 2, 3\}$, and S_{kj} , $1 \leq k \neq j \leq 3$ are compact,
- (ii) the collision operator $K_{c_{kj}}$ is regular, for $1 \leq k \neq j \leq 3$.

Then, we obtain:

$$\sigma_*(\mathcal{A}(\lambda)) = \sigma_{S,*}(E_{\mathbf{D}}) := \bigcup_{1 \leq k \leq 3} \sigma_{S_{kk},*}(T_{\mathbf{D}_k}) = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq 0\},$$

for $* \in \{\Phi_l, \Phi_r, \Phi, \mathcal{W}_l, \mathcal{W}_r, \mathcal{W}\}$.

Proof. Eq. (2.1) will guide us to express the picture of the eigenvalues associated to the operator $\lambda S_{kk} - T_{\mathbf{D}_k}$ as:

$$\sigma_{S_{kk},*}(T_{\mathbf{D}_k}) := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq 0\}, \quad * \in \{\Phi_+, \Phi_-, \Phi_l, \Phi_r, \Phi, \mathcal{W}_l, \mathcal{W}_r, \mathcal{W}\},$$

in view of the outcome developed by I. Walha in [34]. By leveraging this result within our matrix framework, we show thanks to Theorem 3.1, Lemma 4.1 and Corollary 4.1 that the filling in some of the holes in $\sigma_*(\mathcal{A}(\lambda))$ should take place in $\bigcup_{1 \leq k \leq 3} \sigma_{S_{kk},*}(T_{\mathbf{D}_k})$. \square

At the end of this study, we present a summary of our contribution in the following conclusion.

5. CONCLUSION

This study aims to make a significant contribution to the analysis problem of relatively essential spectra of an unbounded operator matrix pencil. In particular, it appears that the study's best use of the framework of the unbounded block 3×3 of operator matrix pencil defined with diagonal domain, is to derive new methodologies and arguments that enable both an accurate description of its eigenvalues and an intriguing description of its invertibility. Therefore, it is important to note that our primary focus in addressing this novel type of operator matrix was not a synthetic new model of an unbounded 3×3 block operator matrix, but we were interested in offering a suitable idea that would result in an effective tool for describing the eigenvalues of certain physical systems and enriching the theoretical landscape of operator matrix pencil. Additionally, partial differential algebraic equations are presented as a physical example of our findings to help clarify the practical consequences of our theoretical discoveries.

In conclusion, this study provides insightful theoretical information that will likely impact future research in mathematics and other scientific fields. It extends the study of the eigenvalues of the operator matrix pencil block 3×3 , which could have an effect on many different applications and offer new ideas for further exploration in mathematical theory fields.

To offer the readers the idea to develop other evolution of our findings, the following questions will be asked.

- (1) Can you find more alternative criteria of perturbations than the above founded in our contributions which cover our investigations and loss the generality?
- (2) Do the results we have established hold true in another context of matrix framework?

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