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A NEW METHOD TO SOLVE DUAL SYSTEMS OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS BY LEGENDRE WAVELETS

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Abstract. The method that will be presented, is numerical solution based on the Legendre wavelets for solving dual systems of fractional integro-differential equations (FIDEs). First of all we make the operational matrix of fractional order integration. The application of this matrix is transforming FIDEs to a system of algebric equations. By this changing, we are able to solve it by a simple solution. In this way, the Legendre wavelets and their operator matrix are the most important keys of our solution. After explaining the method we test on some illustrative examples which numerical solutions of these examples demonstrate the validity and applicability of suggested method.

1. Introduction

Nowadays using fractional calculus has valuable usages in some fields of science and engineering. The study of dual systems of FIDEs have many applications in engineering, biomechanice and other scientific divisions. Dual systems of FIDEs also appear in modeling some of chemical and material engineering processes [8, 13, 15]. In most cases obtaining an analytical solution of FIDEs is impossible or so difficult. Thus, various procedures for obtaining approximate solutions of this kind of equations have attracted the attentions of many researchers.

In recent years, several numerical methods have been devoted for solving FIDEs but they are not properly applied to solve dual systems of FIDEs [1, 4, 18]. The greatest information that we can obtain from this case, is studing of papers that have

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been presented by various methods to arrive to an approximate solution. One of these methods is wavelet method [20]. Wavelets are generally a family of oscillatory functions which can be used to obtain approximate solutions of unknown functions [12]. There are many methods for solving FIDEs, by helping of wavelets; for instance take a look at [2, 9, 11, 22]. The application of wavelets is significant in many scientific disciplines, such as time-frequency analysis, signal processing and numerical analysis [3].

This paper is based on Legendre wavelets that are a special type of wavelets that successfully have passed the exams in system analysis, system identification, optimal control and numerical solutions of differential and integral equations. Legendre wavelets are based on Legendre polynomials. From numerical point of view, wavelets have a closer and more accurate approximation than Legendre polynomials [17]. In the study of various methods for numerical solution of systems of FIDEs, we find that the wavelets method has been used less. Therefore, we have chosen the method of the Legendre wavelets for numerical solution of systems of FIDEs. We now apply the Legendre wavelets method to solve the following dual system [21]:

$$\begin{cases} D^r f(x) = u_1(x, f(x), g(x)) + \int_0^x u_2(t, f(t), g(t))dt, \\ D^s g(x) = v_1(x, f(x), g(x)) + \int_0^x v_2(t, f(t), g(t))dt, \end{cases}$$

where $x, t \in [0, 1]$, $r, s \in (0, 1]$, and D^r, D^s display the Caputo derivative operator.

2. Legendre Wavelets and their Functional Properties

2.1. Legendre wavelets. Legendre wavelets are defined on $[0, 1)$ as [10]:

$$\psi_{nm}(x) = \begin{cases} 2^{\frac{k}{2}}\sqrt{m + \frac{1}{2}}L_m(2^k x - \hat{n}), & \frac{\hat{n}-1}{2^k} \leq x < \frac{\hat{n}+1}{2^k}, \\ 0, & \text{otherwise}, \end{cases}$$

where $n = 1, 2, \ldots, 2^{k-1}$, $\hat{n} = 2n-1$, $m = 0, 1, 2, \ldots, M-1$, $k, M \in \mathbb{N}$, m is the degree of the Legendre polynomials and $L_m(x)$ are the well-known Legendre polynomials of order m that are defined on the interval $[-1, 1]$ and satisfy the following recursive formula

$$\begin{aligned} &L_0(x) = 1, \quad L_1(x) = x, \\ &L_{m+1}(x) = \Big(\frac{2m+1}{m+1}\Big)xL_m(x) - \Big(\frac{m}{m+1}\Big)L_{m-1}(x), \quad m = 1, 2, \ldots \end{aligned}$$

2.2. Function approximation. The Legendre wavelet series representation of the function $f(x)$ defined over $[0, 1)$ is given by

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{nm}\psi_{nm}(x) = A^T\Psi(x), \tag{2.1}$$

where $a_{nm} = \langle f(x), \psi_{nm}(x)\rangle$ and $\langle \cdot, \cdot\rangle$ denotes the inner product. If the infinite series in (2.1) is finited, (2.1) can be written as

$$f(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{nm}\psi_{nm}(x) = A^T \Psi(x), \tag{2.2}$$

where A and $\Psi(x)$ are $2^{k-1}M \times 1$ matrices given by

$$A = \Big[a_{10}, a_{11}, \ldots, a_{1(M-1)}, a_{20}, a_{21}, \ldots, a_{2(M-1)}, \ldots, a_{2^{k-1}0}, a_{2^{k-1}1}, \ldots, a_{2^{k-1}(M-1)}\Big]^T$$

$$\Psi(x) = \Big[\psi_{10}(x), \psi_{11}(x), \ldots, \psi_{1(M-1)}(x), \psi_{20}(x), \psi_{21}(x), \ldots, \psi_{2(M-1)}(x), \ldots,$$
$$\psi_{2^{k-1}0}(x), \psi_{2^{k-1}1}(x), \ldots, \psi_{2^{k-1}(M-1)}(x)\Big]^T.$$

For simplicity, (2.2) can be rewritten as

$$f(x) \approx \sum_{i=1}^{n'} a_i\psi_i(x) = A_{n'}^T \Psi_{n'}(x) = \widehat{f}(x),$$

where $a_i = a_{nm}$, $\psi_i = \psi_{nm}$, $n' = 2^{k-1}M$, $i = M(n-1)+m+1$. Obtain the collocation points as

$$x_i = \frac{i - 0.5}{n'}, \quad i = 1, 2, \cdots, 2^{k-1}M.$$

We define the Legendre wavelets matrix as

$$\phi_{n'\times n'} = \left[\Psi\left(\frac{1}{2n'}\right), \Psi\left(\frac{3}{2n'}\right), \Psi\left(\frac{5}{2n'}\right), \ldots, \Psi\left(\frac{i-0.5}{n'}\right)\right].$$

3. Operational Matrix of the Integration for Legendre Wavelets

3.1. Preliminaries and natations.

In this section, we first present some definitions and basic concepts that have the most applications in this paper [19].

Definition 3.1. The Reimann-Liouville fractional integral operator of order $\gamma \geq 0$ is a function defined as

$$I^\gamma f(x) = \begin{cases} \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_0^x (x-t)^{\gamma-1} f(t)dt, & \gamma > 0, \\ f(x), & \gamma = 0, \end{cases}$$

where $\Gamma(\gamma)$ is the gamma function as

$$\Gamma(\gamma) = \int_0^\infty t^{\gamma-1}e^{-t}dt.$$

Definition 3.2. The Caputo fractional derivative of order $\gamma > 0$ is defined as

$$D^\gamma f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\gamma)} \displaystyle\int_0^x (x-t)^{n-\gamma-1} f^{(n)}(t)dt, & \gamma > 0,\ n-1 < \gamma < n, \\ \dfrac{d^{(n)} f(x)}{dx^n}, & \gamma = n, \end{cases}$$

where $x \in [0, \infty)$, and $n = 1, 2, 3, \ldots$

For $x > 0$ the Caputo derivative and Reimann-Liouville integral operator have the following relationships

$$\begin{aligned} D^\gamma I^\gamma f(x) =& f(x), \\ I^\gamma D^\gamma f(x) =& f(x) - \sum_{m=0}^{n-1} \frac{f^{(m)}(0^+)}{m!} x^m, \quad n-1 < \gamma < n. \end{aligned} \tag{3.1}$$

3.2. **Operational matrix of the fractional integration.** Here the main goal is to get the fractional-order Legendre wavelets operational matrix of integration. For this purpose, we have to define the set of Block puls functions (BPFs) as follows [16]

$$b_i(x) = \begin{cases} 1, & \frac{i-1}{n'} \leq x < \frac{i}{n'}, \\ 0, & \text{otherwise}, \end{cases}$$

where $i = 1, 2, \ldots, n'$, and $n' = 2^{k-1}M$.

The BPFs have two properties which will be used later

$$b_i(x)b_j(x) = \begin{cases} b_i(x), & i = j, \\ 0, & i \neq j, \end{cases}$$

$$\int_0^x b_i(x)b_j(x)dx = \begin{cases} \frac{1}{n'}, & i = j, \\ \\ 0, & i \neq j. \end{cases}$$

Definition 3.3. Let $C = [c_1, c_2, \ldots, c_{n'}]^T$ and $D = [d_1, d_2, \ldots, d_{n'}]^T$ be two matrices $n' \times n'$, then we define that $C \otimes D = [c_1d_1, c_2d_2, \ldots, c_{n'}d_{n'}]^T$.

Lemma 3.1. *Suppose that $g(x)$ and $h(x)$ are two functions defined on $L^2[0, 1]$ as we have $g(x) = G^T B_{n'}(x)$ and $h(x) = H^T B_{n'}(x)$, where $G^T = [g_1, g_2, \ldots, g_{n'}]$, $H^T = [h_1, h_2, \ldots, h_{n'}]$ and $B_{n'}(x) = [b_1, b_2, \ldots, b_{n'}]^T$, then we have*

$$g(x)h(x) \approx G^T B_{n'}(x) H^T B_{n'}(x) = (G^T \otimes H^T) B_{n'}(x), \tag{3.2}$$

$$g(x)^2 \approx (G^T B_{n'}(x))^2 = (G^T)^2 B_{n'}(x). \tag{3.3}$$

Proof. By using the properties of BPFs, the proof is obvious. □

The fractional integration of order γ in Reimann-Liouvill concept can be expressed as [5]

$$I^\gamma B_{n'}(x) \approx R^\gamma B_{n'}(x), \tag{3.4}$$

where R^γ is the BPFs operational matrix with

$$R^\gamma = \frac{1}{n'^\gamma}\frac{1}{\Gamma(\gamma+2)}\begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{n'-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{n'-2} \\ 0 & 0 & 1 & \cdots & \xi_{n'-3} \\ 0 & 0 & 0 & \cdots & \xi_{n'-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

and $\xi_k = (k+1)^{\gamma+1} - 2k^{\gamma+1} + (k-1)^{\gamma+1}$, $k = 1, 2, \ldots, n' - 1$.

We now derive the Legendre wavelets operational matrix of the fractional integration. The integration of Legendre wavelets $\Psi_{n'}(x)$ can be obtained as

$$I\Psi_{n'}(x) = \int_0^x \Psi_{n'}(\tau)d\tau \approx q_{n'\times n'}\Psi_{n'}(x), \tag{3.5}$$

where the n'-square matrix $q_{n'\times n'}$ is called Legendre wavelets operational matrix and $q^\gamma_{n'\times n'}$ is called Legendre wavelets fractional integral operational matrix and achived by

$$I^\gamma \Psi_{n'}(x) \approx q^\gamma_{n'\times n'}\Psi(x)_{n'}, \tag{3.6}$$

the Legendre wavelets can be expanded into n'-set BPFs as

$$\Psi_{n'}(x) \approx \phi_{n'\times n'}B_{n'}(x), \tag{3.7}$$

we get [6] from (3.4), (3.6) and (3.7)

$$\begin{aligned} q^\gamma_{n'\times n'}\Psi_{n'}(x) \approx & I^\gamma\Psi_{n'}(x) \approx I^\gamma \phi_{n'\times n'}B_{n'}(x) = \phi_{n'\times n'}I^\gamma B_{n'}(x) \approx \phi_{n'\times n'}R^\gamma B_{n'}(x) \\ \approx & \phi_{n'\times n'}R^\gamma \phi^{-1}_{n'\times n'}\psi_{n'}(x). \end{aligned}$$

Finally, we conclude from (3.6) $q^\gamma_{n'\times n'} \approx \phi_{n'\times n'}R^\gamma \Psi^{-1}_{n'\times n'}$.

In general, the matrix $\phi_{n'\times n'}$ counted in the below form

$$\phi_{n'\times n'} = \begin{bmatrix} L & 0 & 0 & \cdots & 0 \\ 0 & L & 0 & \cdots & 0 \\ 0 & 0 & L & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & L \end{bmatrix},$$

where L is a $M \times M$ matrix given by [7]

$$L = \begin{bmatrix} \psi_{10}\left(\frac{1}{2n'}\right) & \psi_{10}\left(\frac{3}{2n'}\right) & \cdots & \psi_{10}\left(\frac{i-0.5}{n'}\right) \\ \psi_{11}\left(\frac{1}{2n'}\right) & \psi_{11}\left(\frac{3}{2n'}\right) & \cdots & \psi_{11}\left(\frac{i-0.5}{n'}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{2^{k-1}(M-1)}\left(\frac{1}{2n'}\right) & \psi_{2^{k-1}(M-1)}\left(\frac{3}{2n'}\right) & \cdots & \psi_{2^{k-1}(M-1)}\left(\frac{i-0.5}{n'}\right) \end{bmatrix}.$$

The six basis functions are by

$$\begin{cases}\psi_{10}(x)=\sqrt{2}, & \\ \psi_{11}(x)=\sqrt{6}(4x-1), & 0\le x<\frac{1}{2},\\ \psi_{12}(x)=\sqrt{10}(24x^2-12x+1), & \end{cases}$$

$$\begin{cases}\psi_{20}(x)=\sqrt{2}, & \\ \psi_{21}(x)=\sqrt{6}(4x-1), & \frac{1}{2}\le x<1,\\ \psi_{22}(x)=\sqrt{10}(24x^2-36x+13). & \end{cases}$$

Here, we present the matrices R^{γ}, $\phi_{n'\times n'}$ and q^{γ} for $k=2$, $M=3$, $n=1,2$, $m=0,1,2$, $\gamma=0.6$ and using the collocation points $x_i=\frac{i-0.5}{n'}$, $i=1,2,\dots,n'$, $n'=2^{k-1}M$. Clearly, we have:

$$R^{0.6}=\begin{bmatrix}0.23872 & 0.24622 & 0.17586 & 0.14847 & 0.13201 & 0.12061\\ 0 & 0.23872 & 0.24622 & 0.17586 & 0.14847 & 0.13201\\ 0 & 0 & 0.23872 & 0.24622 & 0.17586 & 0.14847\\ 0 & 0 & 0 & 0.23872 & 0.24622 & 0.17586\\ 0 & 0 & 0 & 0 & 0.23872 & 0.24622\\ 0 & 0 & 0 & 0 & 0 & 0.23872\end{bmatrix},$$

$$\phi_{6\times 6}=\begin{bmatrix}1.41421 & 1.41421 & 1.41421 & 0 & 0 & 0\\ -1.63299 & 0 & 1.63299 & 0 & 0 & 0\\ 0.52705 & -1.58114 & 0.52705 & 0 & 0 & 0\\ 0 & 0 & 0 & 1.41421 & 1.41421 & 1.41421\\ 0 & 0 & 0 & -1.63299 & 0 & 1.63299\\ 0 & 0 & 0 & 0.52705 & -1.58114 & 0.52705\end{bmatrix},$$

$$q^{0.6}=\begin{bmatrix}0.45856 & 0.18277 & -0.02360 & 0.47845 & -0.07337 & 0.01977\\ -0.14723 & 0.15079 & 0.12261 & 0.06705 & -0.03495 & 0.01469\\ -0.05571 & -0.09082 & 0.10681 & -0.04913 & 0.00096 & 0.00190\\ 0 & 0 & 0 & 0.45856 & 0.18277 & -0.02360\\ 0 & 0 & 0 & -0.14723 & 0.15079 & 0.12261\\ 0 & 0 & 0 & -0.05571 & -0.09082 & 0.10681\end{bmatrix}.$$

3.3. **Error analysis.** The following theorem presents the error analysis of the Legendre wavelets approximation function. By increasing values of k and M the error gets closer to zero. As you will see, solved examples confirm this sentence. So, we say surely the mentioned method and its approximation function will be successfully responsive for solving examples of the discussed subject.

Theorem 3.1 ([14])**.** *Suppose $f(x)\in C^2[0,1]$ and $\widehat{f}(x)$ is the best approximation of $f(x)$, then we have for these two functions defined in* (2.1) *and* (2.2)*:*

$$\begin{aligned}\|e_f\|_2=\|\text{error}(f(x))\|_2=\|f(x)-\widehat{f}(x)\|_2&=o\left(\frac{1}{M!2^{Mk}}\right),\\ &=\frac{c}{M!2^{Mk}}_{c>0}\ \textit{as } k\to\infty,\ M\to\infty.\end{aligned}$$

4. Numerical Examples

In this section, we are going to solve two numerical examples by using the proposed method in Section 3, Also, we will compare their approximate and exact solution from graphical and numerical point of view. The numerical results show these performance of the mentioned method.

Example 4.1 ([21]).

$$\begin{cases} D^r f(x) = -\frac{1}{2} f^2(x) - g(x) + \frac{1}{2} - \int_0^x g(t) f(t) dt, & 0 < r \leq 1, \\ D^s g(x) = g^2(x) + f^2(x) - \int_0^x g(t) dt, & 0 < s \leq 1, \end{cases} \tag{4.1}$$

with the initial conditions $f(0) = 1$ and $g(0) = 0$. Exact solutions for the above coupled systems when $r = s = 1$ are obtained by $f(x) = \cos x$ and $g(x) = \sin x$, the exact solutions of $f(x)$ and $g(x)$ for $r, s \in (0, 1)$ are unknown.

Let

$$\begin{cases} D^r f(x) \approx A_{n'}^T \Psi_{n'}(x), \\ D^s g(x) \approx E_{n'}^T \Psi_{n'}(x), \end{cases} \tag{4.2}$$

where $A_{n'}^T = [a_1, a_2, a_3, \ldots, a_{n'}]$ and $E_{n'}^T = [e_1, e_2, e_3, \ldots, e_{n'}]$. By using the initial conditions and (3.1), (3.6), (3.7) and (4.2), we have

$$\begin{cases} f(x) = & I^r D^r f(x) + f(0) \approx A_{n'}^T q_{n'\times n'}^r \Psi_{n'}(x) + 1 \\ & \approx A_{n'}^T q_{n'\times n'}^r \phi_{n'\times n'} B_{n'}(x) + [1, \ldots, 1]_{n'\times n'}, \\ g(x) = & I^s D^s g(x) + g(0) \approx E_{n'}^T q_{n'\times n'}^s \Psi_{n'}(x) \approx E_{n'} q_{n'\times n'}^s \phi_{n'\times n'} B_{n'}(x). \end{cases} \tag{4.3}$$

Then, by using (3.2), (3.3), (3.5) and (4.3), we obtain

$$\begin{aligned} f^2(x) \approx & (A_{n'}^T q_{n'\times n'}^r \phi_{n'\times n'})^2 B_{n'}(x) + 2A_{n'}^T q_{n'\times n'}^r \phi_{n'\times n'} B_{n'}(x) + [1, 1, \ldots, 1]_{n'\times n'}, \\ g^2(x) \approx & (E_{n'}^T q_{n'\times n'}^s \phi_{n'\times n'})^2 B_{n'}(x), \end{aligned}$$

(4.4)

$$\begin{aligned} \int_0^x g(t) dt \approx & \int_0^x E_{n'}^T q_{n'\times n'}^s \Psi_{n'}(t) dt \approx E_{n'}^T q_{n'\times n'}^{1+s} \phi_{n'\times n'} B_{n'}(x), \\ g(x) f(x) \approx & (E_{n'}^T q_{n'\times n'}^s \phi_{n'\times n'} B_{n'}(x))(A_{n'}^T q_{n'\times n'}^r \phi_{n'\times n'} B_{n'}(x) + 1) \\ = & (E_{n'}^T q_{n'\times n'}^s \phi_{n'\times n'} \otimes A_{n'}^T q_{n'\times n'}^r \phi_{n'\times n'}) B_{n'}(x) + E_{n'}^T q_{n'\times n'}^s \phi_{n'\times n'} B_{n'}(x), \\ \int_0^x g(t) f(t) dt \approx & \int_0^x (E_{n'}^T q_{n'\times n'}^s \phi_{n'\times n'} \otimes A_{n'}^T q_{n'\times n'}^r \phi_{n'\times n'}) B_{n'}(t) dt \\ & + \int_0^x E_{n'}^T q_{n'\times n'}^s \phi_{n'\times n'} B_{n'}(t) dt \\ = & (E_{n'}^T q_{n'\times n'}^s \phi_{n'\times n'} \otimes A_{n'}^T q_{n'\times n'}^r \phi_{n'\times n'}) \int_0^x B_{n'}(t) dt \\ & + E_{n'}^T q_{n'\times n'}^s \phi_{n'\times n'} \int_0^x B_{n'}(t) dt \end{aligned}$$

$$
\begin{aligned}
(4.5)\qquad &\approx(E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'}\otimes A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'})\int_0^x \phi^{-1}_{n'\times n'}\Psi_{n'}(t)dt\\
&+(E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'})\int_0^x \phi^{-1}_{n'\times n'}\Psi_{n'}(t)dt\\
&\approx(E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'}\otimes A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'})\phi^{-1}_{n'\times n'}q_{n'\times n'}\phi_{n'\times n'}B_{n'}(x)\\
&+E^T_{n'}q^{1+s}_{n'\times n'}\phi_{n'\times n'}B_{n'}(x).
\end{aligned}
$$

By replacing (3.7), (4.2)–(4.4) and (4.5) into (4.1), and also according to the properties of BPFs, we conclude

$$
(4.6)\qquad
\begin{cases}
A^T_{n'}\phi_{n'\times n'} = & -\frac{1}{2}(A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'})^2 - A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'}\\
& -E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'} - (E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'}\otimes A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'}\\
& +E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'})\phi^{-1}_{n'\times n'}q_{n'\times n'}\phi_{n'\times n'},\\
E^T_{n'}\phi_{n'\times n'} = & (E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'})^2 + (A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'})^2 + 2A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'}\\
& -E^T_{n'}q^{1+s}_{n'\times n'}\phi_{n'\times n'} + [1,1,\dots,1]_{1\times n'}.
\end{cases}
$$

(4.6) is now a system of nonlinear algebric equations which is a transformed type of (4.1). It has $2n'$ unknown coefficients, A_i and E_i, which we can find them and the numerical solutions of $f(x)$ and $g(x)$ by solving this system by presented numerical method.

The approximate solutions obtained by using the proposed method and also absolute error value for different values k, M, r, s and x in the Tables $1-3$ have been shown. From Tables 1-3 and Figures 1-5 we can see that by increasing k and M the numerical solutions converge to the exact solutions, specially when $r, s \to 1$.

TABLE 1. Numerical results of the Example 4.1 for $k = 2$, $M = 6$, $n' = 2^{k-1}M = 12$, $i = 1, 2, 3, \dots, n'$, and different values r and s.

$x_i = \frac{i-0.5}{n'}$	$r = 0.7,\ s = 0.7$ $f(x),\ g(x)$	$r = 0.8,\ s = 0.8$ $f(x),\ g(x)$	$r = 0.9,\ s = 0.9$ $f(x),\ g(x)$	$r = 1,\ s = 1$ $f(x),\ g(x)$	Ecact solution $f(x),\ g(x)$
$x_1 = 0.04167$	0.98729, 0.11185	0.99339, 0.08097	0.99660, 0.05817	0.99827, 0.41546	0.99913, 0.04166
$x_2 = 0.12500$	0.95593, 0.24847	0.97390, 0.19959	0.98486, 0.15830	0.99136, 0.12435	0.99220, 0.12467
$x_3 = 0.20833$	0.91290, 0.34846	0.94327, 0.29703	0.96398, 0.24904	0.97758, 0.20629	0.97838, 0.20683
$x_4 = 0.29167$	0.86384, 0.42851	0.90475, 0.38190	0.93530, 0.33342	0.95704, 0.28680	0.95777, 0.28755
$x_5 = 0.37500$	0.81126, 0.49298	0.86008, 0.45627	0.89968, 0.41195	0.92988, 0.36531	0.93051, 0.36627
$x_6 = 0.45833$	0.75698, 0.54381	0.81063, 0.52091	0.85786, 0.48458	0.89628, 0.44128	0.89679, 0.44245
$x_7 = 0.54167$	0.70249, 0.58196	0.75760, 0.57604	0.81054, 0.55109	0.85648, 0.51418	0.85685, 0.51556
$x_8 = 0.62500$	0.64909, 0.60799	0.70209, 0.62168	0.75844, 0.61113	0.81076, 0.58351	0.81096, 0.58510
$x_9 = 0.70833$	0.59794, 0.62222	0.64516, 0.65768	0.70224, 0.66435	0.75944, 0.64878	0.75945, 0.65057
$x_{10} = 0.79167$	0.55009, 0.62485	0.58783, 0.68380	0.64269, 0.71035	0.70287, 0.70954	0.70266, 0.71153
$x_{11} = 0.87500$	0.50650, 0.61613	0.53112, 0.69978	0.58050, 0.74871	0.64145, 0.76535	0.64100, 0.76754
$x_{12} = 0.95833$	0.46802, 0.59639	0.47601, 0.70530	0.51646, 0.77900	0.57560, 0.81583	0.57488, 0.81823

TABLE 2. Numerical results of the Example 4.1 for $k = 6$, $M = 3$, $n' = 2^{k-1}M = 96$, $i = 1, 2, 3, \ldots, n'$, and different values r and s

$x_i = \frac{i-0.5}{n'}$	$r = 0.85,\ s = 0.85$ $f(x),\ g(x)$	$r = 0.9,\ s = 0.9$ $f(x),\ g(x)$	$r = 0.95,\ s = 0.95$ $f(x),\ g(x)$	$r = 1,\ s = 1$ $f(x),\ g(x)$	Ecact solution $f(x),\ g(x)$
$x_8 = 0.07813$	$0.99150, 0.12072$	$0.99393, 0.10459$	$0.99568, 0.09043$	$0.99694, 0.07804$	$0.99695, 0.07805$
$x_{16} = 0.16146$	$0.97100, 0.22197$	$0.97769, 0.19983$	$0.98292, 0.17943$	$0.98698, 0.16075$	$0.98699, 0.16076$
$x_{24} = 0.24479$	$0.94160, 0.31225$	$0.95307, 0.28786$	$0.96250, 0.26449$	$0.97018, 0.24234$	$0.97019, 0.24235$
$x_{32} = 0.32813$	$0.90481, 0.39397$	$0.22105, 0.36986$	$0.93492, 0.34583$	$0.94664, 0.32226$	$0.94665, 0.32227$
$x_{40} = 0.41146$	$0.86177, 0.46779$	$0.88241, 0.44602$	$0.90064, 0.42322$	$0.91653, 0.39993$	$0.91654, 0.39995$
$x_{48} = 0.49479$	$0.81349, 0.53383$	$0.83787, 0.51619$	$0.86010, 0.49628$	$0.88006, 0.47483$	$0.88007, 0.47485$
$x_{56} = 0.57813$	$0.76090, 0.59198$	$0.78813, 0.58007$	$0.81378, 0.56461$	$0.83748, 0.54643$	$0.83749, 0.54645$
$x_{64} = 0.66146$	$0.70490, 0.64202$	$0.73389, 0.63733$	$0.76217, 0.62776$	$0.78910, 0.61424$	$0.78910, 0.61427$
$x_{72} = 0.74479$	$0.64638, 0.68365$	$0.67585, 0.68758$	$0.70579, 0.68530$	$0.73523, 0.67779$	$0.73523, 0.67782$
$x_{80} = 0.82813$	$0.58625, 0.71652$	$0.61477, 0.73042$	$0.64520, 0.73680$	$0.67626, 0.73663$	$0.67626, 0.73666$
$x_{88} = 0.91146$	$0.52539, 0.74023$	$0.55138, 0.76544$	$0.58097, 0.78185$	$0.61260, 0.79036$	$0.61259, 0.79040$
$x_{96} = 0.99979$	$0.46474, 0.75438$	$0.48647, 0.79219$	$0.51373, 0.82002$	$0.54469, 0.83861$	$0.54468, 0.83865$

TABLE 3. Absolute error relevant to Tables 1 and 2 when $r = s = 1$

x_i	e_f	e_g	x_i	e_f	e_g
x_1	$8.6311e-04$	$1.0824e-04$	x_8	$1.3409e-05$	$3.1776e-06$
x_2	$8.4048e-04$	$3.2442e-04$	x_{16}	$1.2897e-05$	$6.5587e-06$
x_3	$7.9451e-04$	$5.3968e-04$	x_{24}	$1.2015e-05$	$9.9213e-06$
x_4	$7.2457e-04$	$7.5337e-04$	x_{32}	$1.0753e-05$	$1.3255e-05$
x_5	$6.3022e-04$	$9.6491e-04$	x_{40}	$9.1071e-06$	$1.6552e-05$
x_6	$5.1122e-04$	$1.1739e-03$	x_{48}	$7.0736e-06$	$1.9807e-05$
x_7	$3.6749e-04$	$1.3801e-03$	x_{56}	$4.6529e-06$	$2.3020e-05$
x_8	$1.9913e-04$	$1.5838e-03$	x_{64}	$1.8473e-06$	$2.6198e-05$
x_9	$6.3200e-06$	$1.7858e-03$	x_{72}	$1.3399e-06$	$2.9357e-05$
x_{10}	$2.1069e-04$	$1.9875e-03$	x_{80}	$4.9047e-06$	$3.2527e-05$
x_{11}	$4.5169e-04$	$2.1913e-03$	x_{88}	$8.8442e-06$	$3.5751e-05$
x_{12}	$7.1658e-04$	$2.4006e-03$	x_{96}	$1.3158e-05$	$3.9088e-05$

Example 4.2 ([21]).

$$
\begin{cases} D^r f(x) = \dfrac{1}{3}g(x)f(x) - g(x) + 1 - \int_0^x [g(t) - 2f(t)]dt, & 0 < r \leq 1, \\ D^s g(x) = \dfrac{1}{3}g(x)f(x) + \dfrac{1}{2}f^2(x) + 2f(x) - \int_0^x [g(t) + f(t)]dt, & 0 < s \leq 1, \end{cases} \tag{4.7}
$$

with the initial conditions $f(0) = 0$ and $g(0) = 0$, exact solutions for the above coupled systems when $r = s = 1$ are obtained by $f(x) = x$ and $g(x) = x^2$. The exact solutions

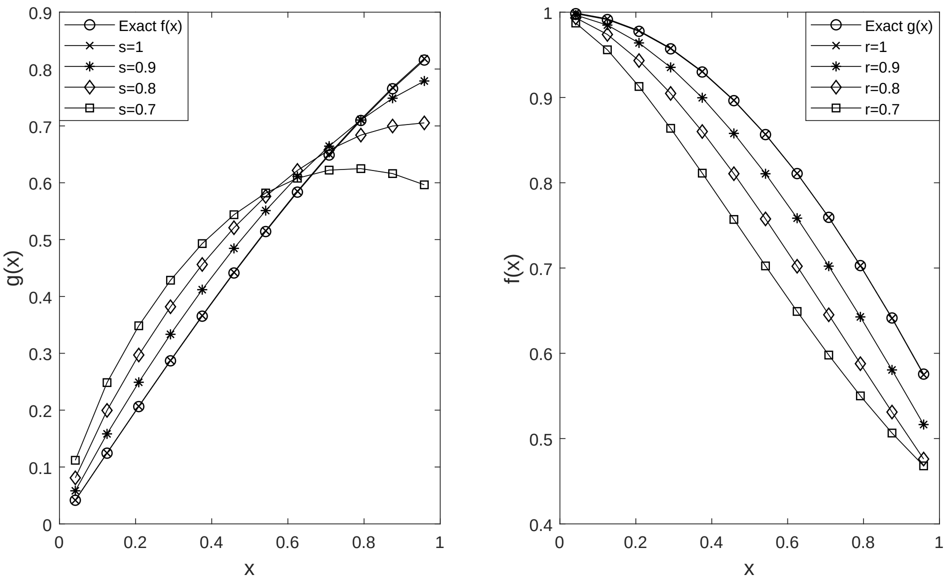

FIGURE 1. Numerical solution for different values of r and s when $k = 2$, $M = 6$ and $n' = 12$.

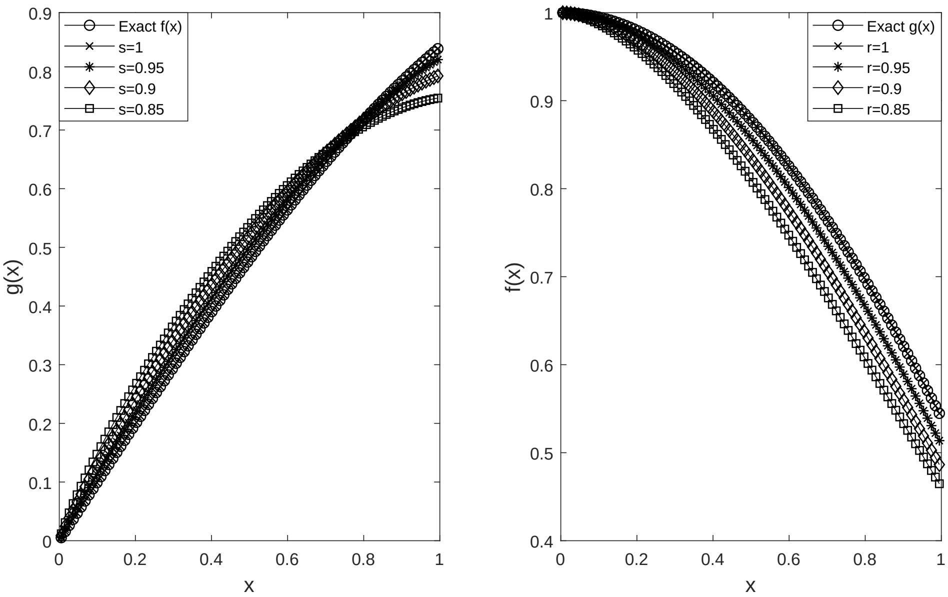

FIGURE 2. Numerical solution for different values of r and s when $k = 6$, $M = 3$ and $n' = 96$.

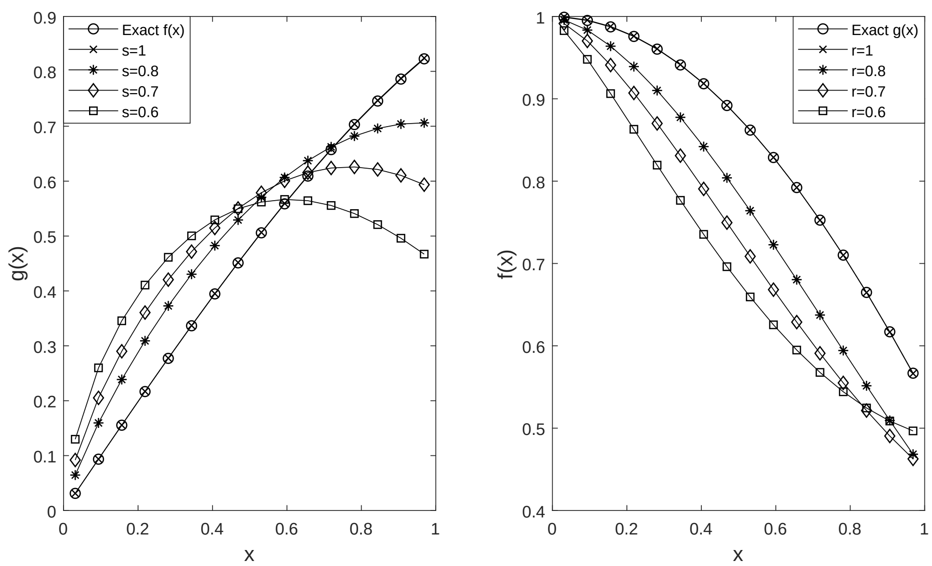

FIGURE 3. Numerical solution for different values of r and s when $k = 3$, $M = 4$ and $n' = 16$.

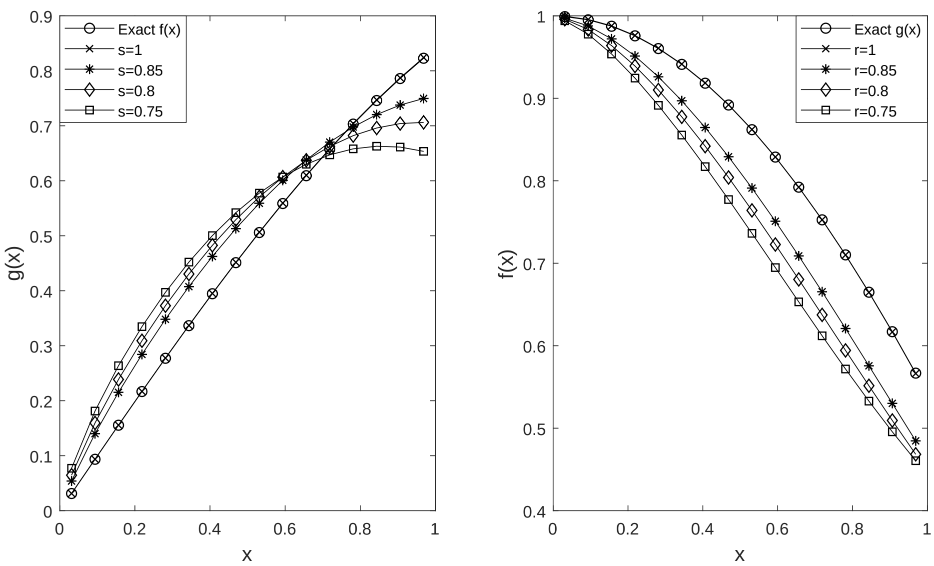

FIGURE 4. Numerical solution for different values of r and s when $k = 3$, $M = 4$ and $n' = 16$.

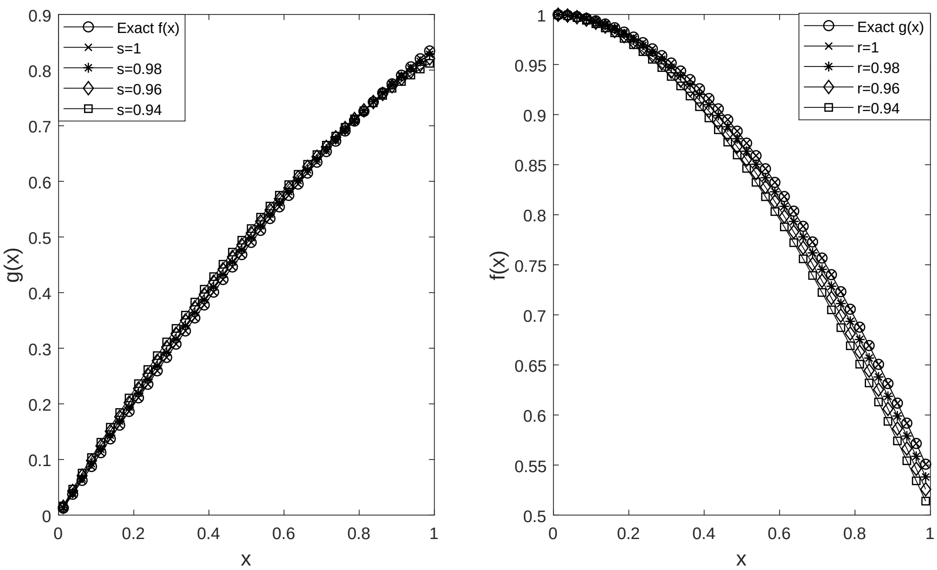

FIGURE 5. Numerical solution for different values of r and s when $k = 4$, $M = 5$ and $n' = 40$.

of $f(x)$ and $g(x)$ for $r, s \in (0, 1)$ are unknown. Let

$$\begin{cases} D^r f(x) \approx A_{n'}^T \Psi_{n'}(x), \\ D^s g(x) \approx E_{n'}^T \Psi_{n'}(x), \end{cases} \tag{4.8}$$

where $A_{n'}^T = [a_1, a_2, a_3, \ldots, a_{n'}]$ and $E_{n'}^T = [e_1, e_2, e_3, \ldots, e_{n'}]$.

By using the initial conditions and (3.2), (3.6), (3.7) and (4.8) we have

$$\begin{cases} f(x) = I^r D^r f(x) + f(0) \approx A_{n'}^T q_{n' \times n'}^r \Psi_{n'}(x) \approx A_{n'}^T q_{n' \times n'}^r \phi_{n' \times n'} B_{n'}(x), \\ g(x) = I^s D^s g(x) + g(0) \approx E_{n'}^T q_{n' \times n'}^s \Psi_{n'}(x) \approx E_{n'}^T q_{n' \times n'}^s \phi_{n' \times n'} B_{n'}(x). \end{cases} \tag{4.9}$$

So, by using (3.2), (3.3), (3.5) and (4.9), we obtain

$$\begin{aligned} g(x) f(x) \approx& (E_{n'}^T q_{n' \times n'}^s \phi_{n' \times n'} B_{n'}(x))(A_{n'}^T q_{n' \times n'}^r \phi_{n' \times n'} B_{n'}(x)) \\ =& (E_{n'}^T q_{n' \times n'}^s \phi_{n' \times n'} \otimes A_{n'}^T q_{n' \times n'}^r \phi_{n' \times n'}) B_{n'}(x), \end{aligned} \tag{4.10}$$

$$f^2(x) \approx (A_{n'}^T q_{n' \times n'}^r \phi_{n' \times n'} B_{n'}(x))^2 = (A_{n'}^T q_{n' \times n'}^r \phi_{n' \times n'})^2 B_{n'}(x), \tag{4.11}$$

$$\begin{aligned} \int_0^x f(t) dt \approx& \int_0^x A_{n'}^T q_{n' \times n'}^r \Psi_{n'}(t) dt = A_{n'}^T q_{n' \times n'}^r \int_0^x \Psi_{n'}(t) dt \\ \approx& A_{n'}^T q_{n' \times n'}^r q_{n' \times n'}^1 \Psi(x) \approx A_{n'}^T q_{n' \times n'}^{1+r} \phi_{n' \times n'} B_{n'}(x), \end{aligned} \tag{4.12}$$

$$\int_0^x g(t) dt \approx \int_0^x E_{n'}^T q_{n' \times n'}^s \Psi_{n'}(t) dt \approx E_{n'}^T q_{n' \times n'}^{1+s} \phi_{n' \times n'} B_{n'}(x). \tag{4.13}$$

By replacing (3.7) and (4.8)–(4.13) into (4.7), we obtain

$$(4.14)\quad \begin{cases} A^T_{n'}\phi_{n'\times n'}B_{n'}(x) = & \frac{1}{3}(E_{n'}q^s_{n'\times n'}\phi_{n'\times n'} \otimes A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'})B_{n'}(x) \\ & -E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'}B_{n'}(x) + [1,1,1,\ldots,1]_{1\times n'}B_{n'}(x) \\ & -E^T_{n'}q^{1+s}_{n'\times n'}\phi_{n'\times n'}B_{n'}(x) + 2A^T_{n'}q^{1+r}_{n'\times n'}\phi_{n'\times n'}B_{n'}(x), \\ E^T_{n'}\phi_{n'\times n'} = & \frac{1}{3}(E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'} \otimes A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'})B_{n'}(x) \\ & +\frac{1}{2}(A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'})^2 B_{n'}(x) + 2A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'}B_{n'}(x) \\ & -E^T_{n'}q^{1+s}_{n'\times n'}\phi_{n'\times n'}B_{n'}(x) - A^T_{n'}q^{1+r}_{n'\times n'}\phi_{n'\times n'}B_{n'}(x). \end{cases}$$

According to the properties of BPFs and (4.14) we have

$$\begin{cases} A^T_{n'}\phi_{n'\times n'} = & \frac{1}{3}(E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'} \otimes A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'}) - E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'} \\ & +[1,1,1,\ldots,1]_{1\times n'} - E^T_{n'}q^{1+s}_{n'\times n'}\phi_{n'\times n'} + 2A^T_{n'}q^{1+r}_{n'\times n'}\phi_{n'\times n'}, \\ E^T_{n'}\phi_{n'\times n'} = & \frac{1}{3}(E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'} \otimes A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'}) + \frac{1}{2}(A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'})^2 \\ & +2A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'} - E^T_{n'}q^{1+s}_{n'\times n'}\phi_{n'\times n'} - A^T_{n'}q^{1+r}_{n'\times n'}\phi_{n'\times n'}. \end{cases}$$

TABLE 4. Numerical results of the Example 4.2 for $k = 2$, $M = 6$, $n' = 12$ and different values r and s.

$x_i = \frac{i-0.5}{n'}$	$r = 0.7,\ s = 0.7$ $f(x),\ g(x)$	$r = 0.8,\ s = 0.8$ $f(x),\ g(x)$	$r = 0.9,\ s = 0.9$ $f(x),\ g(x)$	$r = 1,\ s = 1$ $f(x),\ g(x)$	Ecact solution $f(x),\ g(x)$
$x_1 = 0.04167$	0.11157, 0.02570	0.08101, 0.01331	0.05826, 0.00681	0.04162, 0.00345	0.04167, 0.00174
$x_2 = 0.12500$	0.24734, 0.08992	0.19985, 0.05289	0.15887, 0.03048	0.12485, 0.01729	0.12500, 0.01563
$x_3 = 0.20833$	0.34747, 0.17865	0.29849, 0.11560	0.25105, 0.07284	0.20808, 0.04499	0.20833, 0.04340
$x_4 = 0.29167$	0.43049, 0.28039	0.38662, 0.19511	0.33858, 0.13148	0.29130, 0.08656	0.29167, 0.0851
$x_5 = 0.37500$	0.50203, 0.39010	0.46725, 0.28816	0.42265, 0.20501	0.37452, 0.14198	0.37500, 0.14063
$x_6 = 0.45833$	0.56500, 0.50429	0.54192, 0.39228	0.50386, 0.29232	0.45773, 0.21126	0.45833, 0.21007
$x_7 = 0.54167$	0.62123, 0.62019	0.61160, 0.50541	0.58259, 0.39247	0.54094, 0.29439	0.54167, 0.29340
$x_8 = 0.62500$	0.67207, 0.73553	0.67698, 0.62570	0.65908, 0.50455	0.62415, 0.39137	0.62500, 0.39063
$x_9 = 0.70833$	0.71856, 0.84840	0.73855, 0.75142	0.73351, 0.62770	0.70736, 0.50221	0.70833, 0.50174
$x_{10} = 0.79167$	0.76163, 0.95723	0.79675, 0.88091	0.80600, 0.76104	0.79057, 0.62689	0.79167, 0.62674
$x_{11} = 0.87500$	0.80212, 1.06071	0.85194, 1.01260	0.87667, 0.90368	0.87377, 0.76541	0.87500, 0.76563
$x_{12} = 0.95833$	0.84082, 1.15789	0.90445, 1.14498	0.94558, 1.05471	0.95697, 0.91778	0.95833, 0.91840

TABLE 5. Numerical results of the Example 4.2 for $k = 6$, $M = 3$, $n' = 96$, for $i = 8, 16, 24, \ldots, 96$, and different values r and s.

$x_i = \frac{i-0.5}{n'}$	$r = 0.85$, $s = 0.85$, $f(x),\ g(x)$	$r = 0.90$, $s = 0.90$, $f(x),\ g(x)$	$r = 0.95$, $s = 0.95$, $f(x),\ g(x)$	$r = 1$, $s = 1$, $f(x),\ g(x)$	Exact solution $f(x),\ g(x)$
$x_8 = 0.07813$	$0.12072, 0.01712$	$0.10466, 0.01221$	$0.09051, 0.00867$	$0.07812, 0.00613$	$0.07813, 0.00610$
$x_{16} = 0.16146$	$0.22252, 0.05872$	$0.20055, , 0.45011$	$0.18013, 0.03434$	$0.16146, 0.02609$	$0.16146, 0.02607$
$x_{24} = 0.24479$	$0.31477, 0.11879$	$0.29056, 0.09507$	$0.26714, 0.07568$	$0.24479, 0.05995$	$0.24479, 0.05992$
$x_{32} = 0.32813$	$0.40070, 0.19461$	$0.37657, 0.16072$	$0.35221, 0.13191$	$0.32812, 0.10769$	$0.32813, 0.10767$
$x_{40} = 0.41146$	$0.48172, 0.28420$	$0.45943, 0.24076$	$0.43578, 0.20252$	$0.41145, 0.16932$	$0.41146, 0.16930$
$x_{48} = 0.49479$	$0.55863, 0.38597$	$0.53961, 0.33418$	$0.51805, 0.28702$	$0.49478, 0.24484$	$0.49479, 0.24482$
$x_{56} = 0.57813$	$0.63198, 0.49845$	$0.61741, 0.44006$	$0.59916, 0.38499$	$0.57811, 0.33424$	$0.57813, 0.33423$
$x_{64} = 0.66146$	$0.70212, 0.62030$	$0.69306, 0.55752$	$0.67920, 0.49602$	$0.66144, 0.43754$	$0.66146, 0.43753$
$x_{72} = 0.74479$	$0.76936, 0.75018$	$0.76670, 0.68568$	$0.78823, 0.61968$	$0.74478, 0.55472$	$0.74479, 0.55471$
$x_{80} = 0.82813$	$0.83390, 0.88680$	$0.83845, 0.82367$	$0.83628, 0.75556$	$0.82811, 0.68579$	$0.82813, 0.68579$
$x_{88} = 0.91146$	$0.89595, 1.02884$	$0.90839, 0.97059$	$0.91339, 0.90321$	$0.91144, 0.83075$	$0.91146, 0.83076$
$x_{96} = 0.99479$	$0.95568, 1.17499$	$0.97659, 1.12551$	$0.98958, 1.06215$	$0.99477, 0.98960$	$0.99479, 0.98961$

TABLE 6. Absolute error relevant to Tables 4 and 5 when $r = s = 1$

x_i	e_f	e_g	x_i	e_f	e_g
x_1	$4.9582e-05$	$1.7179e-03$	x_8	$1.4528e-06$	$2.6604e-05$
x_2	$1.5141e-04$	$1.6686e-03$	x_{16}	$3.0835e-06$	$2.5657e-05$
x_3	$2.5823e-04$	$1.5927e-03$	x_{24}	$4.7883e-06$	$2.4283e-05$
x_4	$3.6945e-04$	$1.4885e-03$	x_{32}	$6.5581e-06$	$2.2451e-05$
x_5	$4.8451e-04$	$1.3539e-03$	x_{40}	$8.3847e-06$	$2.0132e-05$
x_6	$6.0291e-04$	$1.1872e-03$	x_{48}	$1.0261e-05$	$1.7294e-05$
x_7	$7.2421e-04$	$9.8598e-04$	x_{56}	$1.2179e-05$	$1.3901e-05$
x_8	$8.4799e-04$	$7.4813e-04$	x_{64}	$1.4134e-05$	$9.9173e-06$
x_9	$9.7387e-04$	$4.7122e-04$	x_{72}	$1.6120e-05$	$5.3045e-06$
x_{10}	$1.1015e-03$	$1.5269e-04$	x_{80}	$1.8131e-05$	$2.1047e-08$
x_{11}	$1.2305e-03$	$2.1017e-04$	x_{88}	$2.0162e-05$	$5.9769e-06$
x_{12}	$1.3606e-03$	$6.2023e-04$	x_{96}	$2.2208e-05$	$1.2736e-05$

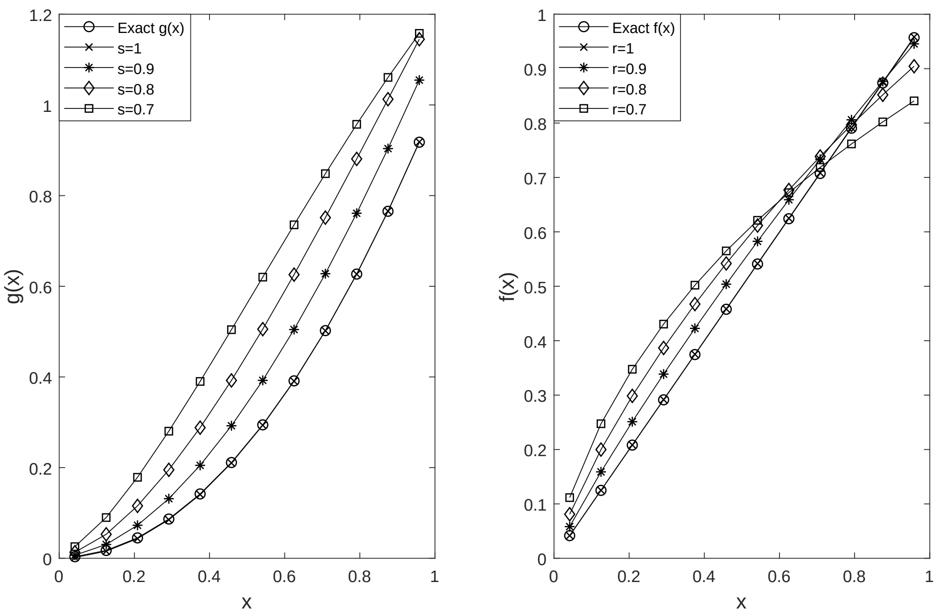

FIGURE 6. Numerical solution for different values of r and s, when $k = 2$, $M = 6$ and $n' = 12$.

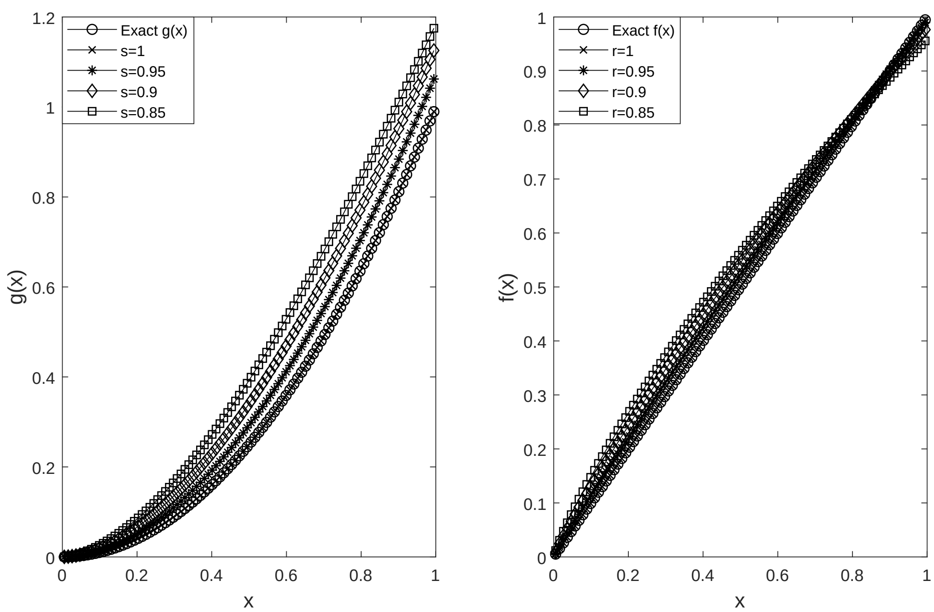

FIGURE 7. Numerical solution for different values of r and s when $k = 6$, $M = 3$ and $n' = 96$.

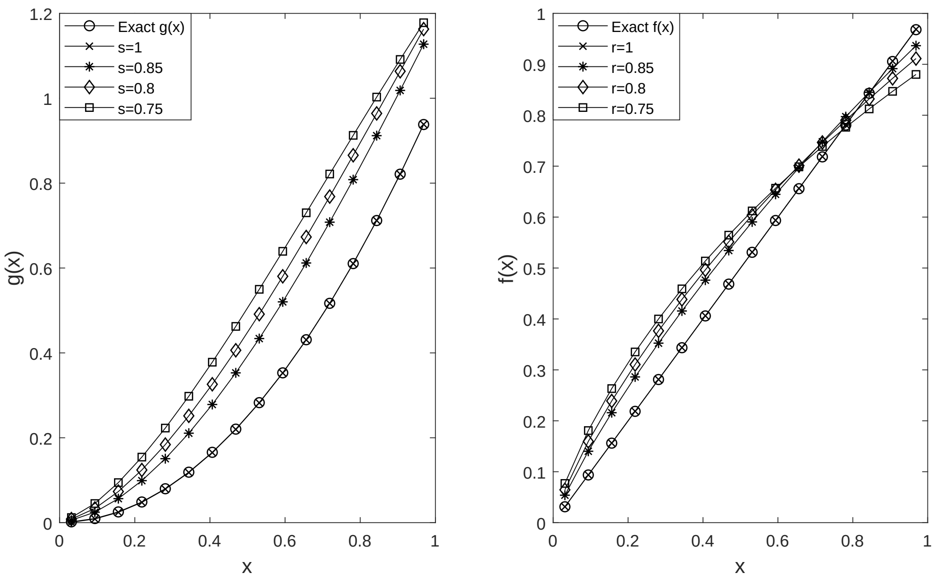

FIGURE 8. Numerical solution for different values of r and s when $k = 3$, $M = 4$ and $n' = 16$.

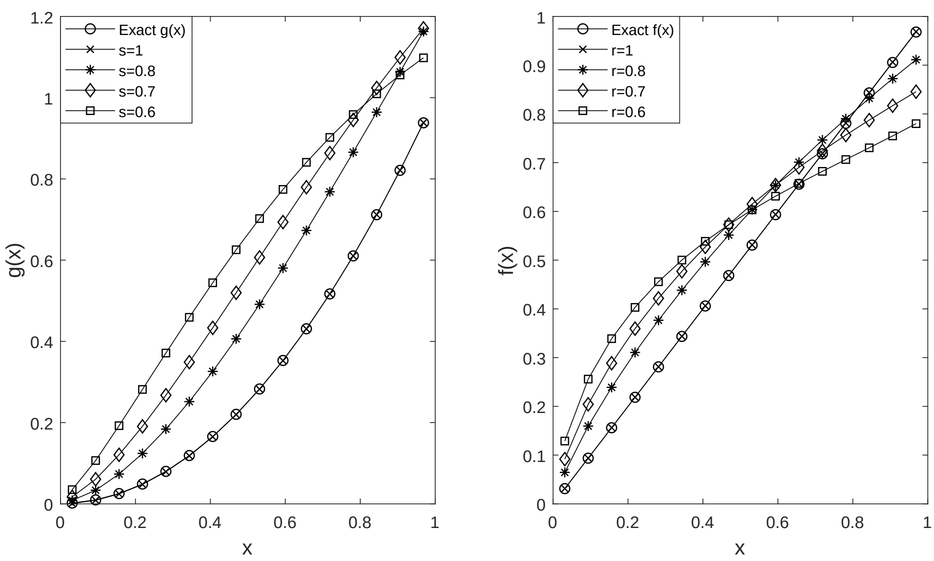

FIGURE 9. Numerical solution for different values of r and s when $k = 3$, $M = 4$ and $n' = 16$.

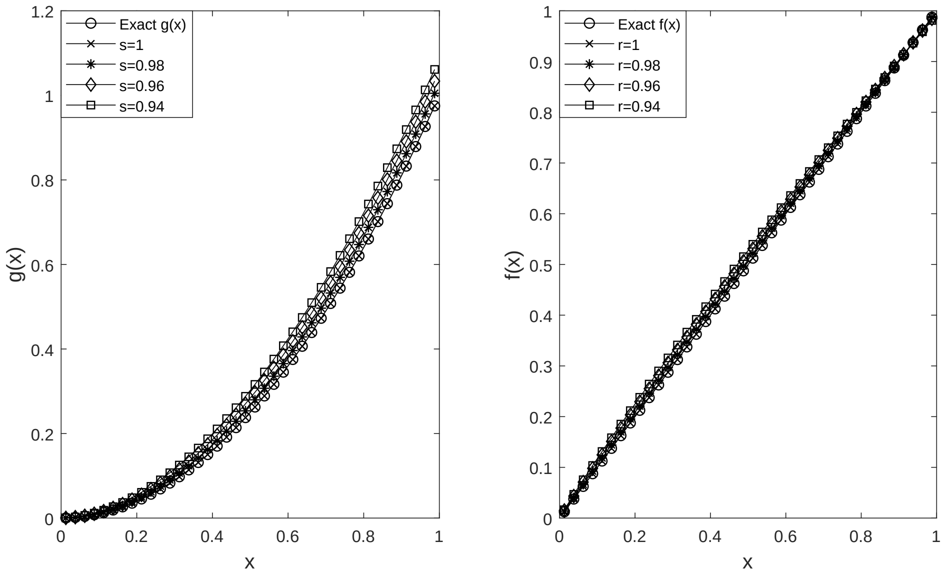

FIGURE 10. Numerical solution for different values of r and s when $k = 4$, $M = 5$ and $n' = 40$.

5. Conclusion

The main purpose of the presented article is introducing Legendre wavelets method for resolving coupled systems of FIDEs. As you saw, the numerical results obtained here, confirm its high accuracy degree.

The most noticeable profit of the mentioned method is converting complicated equations to simple ones, like we performed on examples. One of the best benefits of this procedure is having high exactness that you may have been recognized it according to the tables and figures.

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