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DEGENERATE BERNSTEIN FORMS: SOME COMPUTATIONAL FEATURES AND STOCHASTIC LOGIC ASSOCIATED

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Abstract. In this work, we study degenerate Bernstein basis functions (in the sense of T. Kim and D. S. Kim [15, 16]), analyzing several of their algebraic and analytical properties and drawing parallels with their classical counterparts. We introduce degenerate Bernstein forms and establish the necessary conditions under which such a form can be implemented by a stochastic logic of the Qian-Riedel-Rosenberg type.

1. Introduction

The classical Bernstein polynomials, introduced by S. N. Bernstein in 1912 [5, 6], play a central role in approximation theory and have since become fundamental tools in diverse areas of mathematics and computer science. They provide a means of approximating continuous functions on compact intervals [25, 27]. Their close relation with Bézier curves has led to widespread applications in computer-aided geometric design, while their algebraic and probabilistic structure makes them particularly well suited for applications in combinatorics, numerical analysis, and stochastic models (see, for instance, [8–12] and the references therein).

It is well known that any polynomial of degree at most n can be uniquely expressed in terms of the Bernstein basis functions; this representation is called the *Bernstein form* of the polynomial. Bernstein forms are especially useful in combinational circuit design and stochastic logic hardware due to their natural probabilistic interpretations

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(cf. [28–32]). Related matrix approaches for tensorial Bernstein forms have also been studied [38].

Beyond these classical developments, Bernstein polynomials and their variants have been investigated in diverse mathematical contexts, including q-series, complex analysis, p-adic Volkenborn integrals, and algorithmic applications. Following the approach of [36], the authors in [1] introduced new generating functions for the Bernstein basis functions. Since these generating functions involve the classical exponential function, several works [1, 15, 16, 35–37] have replaced it with a degenerate exponential function to explore the algebraic and analytic behavior of the so-called *degenerate Bernstein polynomials* (or degenerate Bernstein basis functions). These polynomials extend the Bernstein basis by introducing a parameter that generates a family of bases with new algebraic and analytic features.

In particular, degenerate Bernstein polynomials were first introduced by T. Kim and D. S. Kim [15] (see also [16]), motivated by a probabilistic viewpoint in which the degeneracy parameter may be interpreted as a "psychological burden" affecting the success probability in binomial trials. Recent work has further developed this probabilistic perspective, including probabilistic formulations of degenerate Bernstein polynomials [17], degenerate binomial and Poisson random variables [19], and related moment/expectation identities involving degenerate Stirling-type numbers and special random variables [20–24]. We also emphasize that, throughout the paper, by "degenerate Bernstein polynomials" (or "degenerate Bernstein basis functions") we refer to the objects introduced in [15, 16].

Although we briefly recall some known properties (to make the paper self-contained and to highlight the analogy with the classical Bernstein basis), the main contributions of this paper concern (i) computational features of degenerate Bernstein forms and (ii) their implementation in Qian-Riedel-Rosenberg-type stochastic logic.

Motivated by recent developments [28–30, 32], this paper has two main objectives. First, we analyze the degenerate Bernstein basis functions introduced by Kim and Kim [15, 16], examining their algebraic and analytic properties while drawing systematic parallels with their classical counterparts. Second, we define and examine the corresponding degenerate Bernstein forms, with the aim of investigating theoretical aspects of a stochastic logic model of the Qian-Riedel-Rosenberg type. In particular, we establish conditions under which a polynomial expressed in such a form can be implemented by a stochastic logic model of the Qian-Riedel-Rosenberg type.

Throughout this paper, let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote, respectively, the sets of natural numbers, non-negative integers, integers, real numbers, and complex numbers. As usual, we denote by $\mathbb{P}_n$ the linear space of polynomials with real coefficients and a degree at most $n \in \mathbb{N}_0$. For $x, \lambda \in \mathbb{R}$ and $n \in \mathbb{Z}$, we use the notation $(x)_{n,\lambda}$ for the generalized falling factorial given by (see e.g., [13, 15, 16]):

$$(x)_{n,\lambda} = \begin{cases} 1, & \text{if } n = 0, \\ \prod_{i=1}^{n}(x-(i-1)\lambda), & \text{if } n \geq 1, \\ 0, & \text{if } n < 0. \end{cases} \tag{1.1}$$

2. Bernstein Forms Revisited

This section is devoted to collecting some structural and computational properties of Bernstein bases and their associated Bernstein forms. As usual, for each $n \in \mathbb{N}_0$ and for $x \in [0,1]$, the Bernstein basis functions are defined as:

$$b_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, \dots, n. \tag{2.1}$$

It is clear that $b_k^n(x) \in \mathbb{P}_n$, for any $n \in \mathbb{N}_0$. Furthermore, in terms of probabilistic modelling, $b_k^n(x)$ indicates the probability of obtaining k successes in n trials of a random process with individual probability of success x in each trial, $k = 0, \dots, n$, $n \in \mathbb{N}$.

More generally, the Bernstein basis functions of degree n over a general interval $[a, b]$ are given by:

$$\check{b}_k^n(x) = \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^n}, \quad k = 0, \dots, n. \tag{2.2}$$

2.1. **Fundamental algebraic and analytic properties.** The following results provide some algebraic and analytic properties of Bernstein basis functions (2.1) (see e.g., [9, 10, 25, 30, 31]).

Theorem 2.1 (Algebraic properties)**.** *Let $\{b_0^n(x), \dots, b_n^n(x)\}$ be the Bernstein basis functions. Then, the following statements hold.*

(*a*) *Symmetry*

$$b_k^n(x) = b_{n-k}^n(1-x), \quad k = 0, \dots, n.$$

(*b*) *Recursion formula*

$$b_k^{n+1}(x) = x b_{k-1}^n(x) + (1-x) b_k^n(x), \quad k = 0, \dots, n,$$

with initial condition $b_{-1}^n(x) := 0$.

(*c*) *Partition of unity*

$$\sum_{k=0}^{n} b_k^n(x) = 1.$$

(*d*) *Basis relations.*

($d.1$) *Bernstein basis functions are expressed in terms of the power basis as*

$$b_k^n(x) = \sum_{j=k}^{n} \binom{n}{j}\binom{j}{k}(-1)^{j-k}x^j, \quad k = 0, \dots, n. \tag{2.3}$$

($d.2$) *Inversion formula*

$$x^k = \sum_{j=k}^{n} \frac{\binom{j}{k}}{\binom{n}{k}} b_j^n(x), \quad k = 0, \dots, n. \tag{2.4}$$

(e) *Scaling the independent variable*

$$b_k^n(rx) = \sum_{j=k}^{n} b_k^j(r) b_j^n(x), \quad k = 0, \dots, n.$$

(f) *Degree elevation*

$$b_k^n(x) = \sum_{j=k}^{k+r} \frac{\binom{n}{k}\binom{r}{j-k}}{\binom{n+r}{j}} b_j^{n+r}(x), \quad k = 0, \dots, n, r \in \mathbb{N}.$$

Theorem 2.2 (Analytic properties). *Let $\{b_0^n(x), \dots, b_n^n(x)\}$ be the Bernstein basis functions. Then, the following statements hold.*

(a) *Non negativity. For any $x \in [0,1]$ we have*

$$b_k^n(x) \geq 0, \quad k = 0, \dots, n.$$

(b) *Unimodality. The k-th Bernstein basis function attains its maximum at the point $x_{k,n} = \frac{k}{n}$, $k = 0, \dots, n$.*

(c) *Derivatives*

$$\frac{d}{dx} b_k^n(x) = n(b_{k-1}^{n-1}(x) - b_k^{n-1}(x)),$$

where $b_{-1}^{n-1}(x) := 0$ and $b_n^{n-1}(x) := 0$.

(d) *Integral formula*

$$\int b_k^n(x)dx = \frac{1}{n+1} \sum_{j=k+1}^{n+1} b_j^{n+1}(x), \quad \text{for } 0 \leq k \leq n.$$

With appropriate adjustments, analogous versions of Theorems 2.1 and 2.2 also hold for the Bernstein basis functions defined over any interval $[a,b]$, using their general form given in (2.2).

Notice that the inversion formula (2.4) immediately implies that the set of polynomials $\{b_0^n(x), \dots, b_n^n(x)\}$ is a basis for $\mathbb{P}_n$. Thus, any $p(x) \in \mathbb{P}_n$ can be uniquely expressed in terms of $\{b_0^n(x), \dots, b_n^n(x)\}$ as

$$p(x) = \sum_{k=0}^{n} a_k b_k^n(x), \quad a_k \in \mathbb{R}, k = 0, \dots, n. \tag{2.5}$$

The expression on the right-hand-side of (2.5) is called *Bernstein form* of $p(x)$ and was coined by Farouki and Goodman in [10]. Although Qian et al. have preferred to use the term *Bernstein polynomial of degree n* for it (cf. [28,29]).

Some algebraic and analytic properties of Bernstein forms are collected in the following results (cf. [7,11,12,30]).

Theorem 2.3 (Algebraic properties). *Let* $p(x) = \sum_{k=0}^{n} a_k b_k^n(x)$ *be a Bernstein form. Then, the following properties hold.*

(*a*) *Degree elevation. For each* $r \in \mathbb{N}$, *the Bernstein form of* $p(x)$ *with respect to the Bernstein basis functions* $\{b_0^{n+r}(x), \dots, b_{n+r}^{n+r}(x)\}$ *is given by*

$$p(x) = \sum_{k=0}^{n+r} a_k^{n+r}\, b_k^{n+r}(x), \tag{2.6}$$

where a_k^{n+r} *denotes the coefficient of* $b_k^{n+r}(x)$ *in the expansion* (2.6), *and is given explicitly by*

$$a_k^{n+r} = \sum_{j=\max(0,k-r)}^{\min(n,k)} \frac{\binom{r}{k-j}\binom{n}{j}}{\binom{n+r}{k}}\, a_j, \quad \textit{for } k = 0, \dots, n+r.$$

(*b*) *Degree reduction. Let* $\sum_{k=0}^{n} c_k x^k$ *be the representation of* $p(x)$ *in terms of the power basis. Assume that* $c_n = c_{n-1} = \cdots = c_{n-r+1} = 0$ *and* $c_{n-r} \neq 0$, *for some* $1 \leq r < n$. *Then, the Bernstein form of* $p(x)$ *with respect to the Bernstein basis functions* $\{b_0^{n-r}(x), \dots, b_{n-r}^{n-r}(x)\}$ *is given by*

$$p(x) = \sum_{k=0}^{n-r} a_k^{n-r}\, b_k^{n-r}(x),$$

where the coefficient a_k^{n-r} *is given explicitly by*

$$a_k^{n-r} = \sum_{j=0}^{k} (-1)^{k-j} \frac{\binom{k-j+r-1}{r-1}\binom{n}{j}}{\binom{n-r}{k}}\, a_j, \quad \textit{for } k = 0, \dots, n-r.$$

Theorem 2.4 (Analytic properties). *Let* $p(x) = \sum_{k=0}^{n} a_k b_k^n(x)$ *be a Bernstein form. Then, the following properties hold.*

(*a*) *The Bernstein form verifies that* $p(0) = a_0$ *and* $p(1) = a_n$.

(*b*) *Lower and upper bounds. For* $x \in [0,1]$

$$\min_{0 \leq k \leq n} a_k \leq p(x) \leq \max_{0 \leq k \leq n} a_k.$$

(*c*) *Variation-diminishing property. If* $Z[p(x)]$ *denotes the number of zeros of* $p(x)$ *on the interval* $(0,1)$ *and* $V[a_0, a_1, \dots, a_n]$ *the number of sign changes in its sequence of coefficients, then*

$$Z[p(x)] = V[a_0, a_1, \dots, a_n] - 2K,$$

where $0 \leq K \leq \left\lfloor \frac{n}{2} \right\rfloor$.

(*d*) *Derivative. The Bernstein form of* $p'(x)$ *is given by*

$$p'(x) = \sum_{k=0}^{n-1} D_k^{n-1} b_k^{n-1}(x),$$

where $D_k^{n-1} = n(a_{k+1} - a_k)$ *for* $k = 0, \dots, n-1$.

(*e*) *Integration. The Bernstein form of* $\int p(x)dx$ *is given by*

$$\int p(x)dx = \sum_{k=1}^{n+1} I_k^{n+1} b_k^{n+1}(x),$$

where $I_k^n = \frac{1}{n+1}\sum_{j=0}^{k-1} a_j$ *and* $I_0^{n+1} := 0$.

The definite integral of $p(x)$ *on* $[0,1]$ *is*

$$\int_0^1 p(x)dx = \frac{1}{n+1}\sum_{k=0}^{n} a_k.$$

(*f*) *Coefficients approximation. Assume that* $\deg(p(x)) = n$, *then for any* $\varepsilon > 0$ *there exists a natural number* $M \geq n$ *such that for all integers* $m \geq M$ *and* $k = 0, \dots, m$, *we have*

$$\left|\beta_k^m - p\left(\frac{k}{m}\right)\right| < \varepsilon,$$

where $\beta_0^m, \beta_1^m, \dots, \beta_m^m$ *satisfy* $p(x) = \sum_{k=0}^{m} \beta_k^m b_k^m(x)$.

Thanks to (2.2), it is also possible to consider the Bernstein form of $p(x)$ over any interval $[a,b]$. With appropriate adjustments, analogous versions of Theorems 2.3 and 2.4 also hold for Bernstein forms defined on arbitrary intervals.

We conclude this section with a fundamental property of Bernstein representations - well established in geometric modeling (see [7–9]).

Theorem 2.5 (Convex hull property of Bernstein coefficients)**.** *The Bernstein coefficients of a polynomial* $g(x)$ *over* $[a,b]$ *lie within the convex hull of its values on* $[a,b]$.

Proof. Let $g(x)$ be a polynomial of degree n defined on the interval $[a,b]$. Consider the following equally spaced partition of $[a,b]$:

$$x_k := a + \frac{k}{n}(b-a), \quad k = 0, 1, \dots, n.$$

By (2.2), the Bernstein coefficients of $g(x)$ over $[a,b]$ are given by

$$\tilde{a}_k = g\left(x_k\right) = g\left(a + \frac{k}{n}(b-a)\right).$$

Thus, the Bernstein form of $g(x)$ is

$$g(x) = \sum_{k=0}^{n} g\left(x_k\right) \check{b}_k^n(x),$$

where $\check{b}_k^n(x) \geq 0$ for all $x \in [a,b]$, and $\sum_{k=0}^n \check{b}_k^n(x) = 1$.

This means that for each $x \in [a,b]$, $g(x)$ is a convex combination of the values $g(x_k)$. Hence,

$$\min_{0\leq k\leq n} g(x_k) \leq g(x) \leq \max_{0\leq k\leq n} g(x_k), \quad \text{for all } x \in [a,b].$$

Let $R = \{g(x) : x \in [a,b]\}$, and denote by $\mathrm{conv}(R)$ the convex hull of this set. Since $g(x)$ is continuous on a compact interval, its image is also compact and connected, so

$$\mathrm{conv}(R) = \left[\min_{x\in[a,b]} g(x),\ \max_{x\in[a,b]} g(x)\right].$$

Since each coefficient $\tilde{a}_k = g(x_k)$ corresponds to a value of $g(x)$ at some point $x_k \in [a,b]$, it follows that

$$\tilde{a}_k \in \mathrm{conv}(R), \qquad \text{for all } k = 0, \ldots, n.$$

□

2.2. **Computational aspects of Bernstein forms: numerical stability.** When working with computers, it is important to be cautious not only about the data being entered, but also about the operations being performed. Input data are typically subject to uncertainties, which may arise from measurement errors or from *backward error*, that is, the accumulation of rounding errors during floating-point computations.

Let $\Phi \equiv \{\phi_0(x), \ldots, \phi_n(x)\} \subset \mathbb{P}_n$ be a basis defined over any interval $[a,b]$. Then, there exist unique coefficients $c_0, \ldots, c_n$ such that

$$p(x) = \sum_{k=0}^n c_k \phi_k(x).$$

In the context of numerical computation with polynomials, we assume that each coefficient c_k is subject to a perturbation, denoted by δc_k. As a result, the polynomial is processed not as $p(x)$, but as

$$p(x) + \delta p(x) = \sum_{k=0}^n c_k \phi_k(x) + \sum_{k=0}^n \delta c_k \phi_k(x),$$

where $\delta p(x)$ reflects the total effect of these perturbations. The value of $\delta p(x)$ depends on three main factors.

(a) The value of the independent variable x.
(b) The statistical distributions of the perturbations $\delta c_0, \ldots, \delta c_n$.
(c) The adopted basis Φ in which the polynomial will be expressed.

In order to specifically analyze the factor (c), the influence of the factor (a) will be controlled by assuming that $x \in [a,b]$. To control the factor (b), it is assumed that the coefficient perturbations follow uniform distributions with a fixed magnitude $\varepsilon > 0$ such that

$$|\delta c_k| \leq |c_k| \varepsilon, \quad \text{for all } k = 0, \ldots, n.$$

Thus, the polynomial error due to perturbations can be bounded as follows

$$|\delta p(x)| = \left|\sum_{k=0}^{n} \delta c_k \phi_k(x)\right| \leq \sum_{k=0}^{n} |\delta c_k| \cdot |\phi_k(x)| \leq C_\Phi(p(x))\varepsilon,$$

where

$$C_\Phi(p(x)) := \sum_{k=0}^{n} |c_k \phi_k(x)|$$

is the so-called *condition number of the polynomial* $p(x)$ *at the value* x *on the basis* Φ, which allows to control the influence of the basis choice in the perturbation suffered by the polynomial. Note that $C_\Phi(p(x))$ depends as much on the adopted basis Φ as on the particular polynomial $p(x)$ under consideration.

In the results presented throughout this section, the concept of a *non-negative basis* plays a crucial role. We follow mainly, the approach of [10]. Given a basis $\Phi \equiv \{\phi_0(x), \ldots, \phi_n(x)\} \subset \mathbb{P}_n$ defined over any interval $[a, b]$, it is said to be a *non-negative basis* on $[a, b]$ if it satisfies

$$\phi_k(x) \geq 0, \quad \text{for all } x \in [a, b] \text{ and } k = 0, \ldots, n.$$

Definition 2.1. Let $\Phi, \Psi \subset \mathbb{P}_n$ be two non-negative bases on $[a, b]$. We say that the basis Φ is systematically more stable than the basis Ψ when $C_\Phi(p(x)) \leq C_\Psi(p(x))$ for every $p(x) \in \mathbb{P}_n$.

By considering the set of all non-negative bases for $\mathbb{P}_n$ on $[a, b]$, denoted by $\mathscr{P}_n$, it is possible to define a partial order on this set by means of the following rule.

Given $\Phi, \Psi \in \mathscr{P}_n$, we say that $\Phi \precsim \Psi$ if for some non-negative $(n + 1) \times (n + 1)$ matrix A, we have

$$\Psi^T = A\Phi^T. \tag{2.7}$$

It is not difficult to prove that $\Phi \precsim \Psi$ and $\Psi \precsim \Phi$ simultaneously hold, if and only if, under suitable ordering, the elements of Φ are positive scalar multiples of the corresponding elements of Ψ. In such a case, we write $\Phi \sim \Psi$, indicating that Φ is *similar* to Ψ. If $\Phi \precsim \Psi$ but $\Phi \nsim \Psi$, we write $\Phi \prec \Psi$, meaning that Φ *precedes* Ψ. Hence, $\Phi \precsim \Psi$, if and only if, Φ precedes or is similar to Ψ.

Hence, $(\mathscr{P}_n, \precsim)$ is a partially ordered set, but not totally ordered set, since there may exist bases $\Phi, \Psi \in \mathscr{P}_n$ such that neither the matrix A defined by (2.7) nor its inverse is non-negative; no precedence relation can be established between such bases. In this case, the bases are said to be *incomparable.*

This leads to the definition of a *minimal basis.* A non-negative basis $\Phi \in \mathscr{P}_n$ is said to be minimal if there exists no basis $\Psi \in \mathscr{P}_n$ such that $\Psi \prec \Phi$. In general, more than one minimal basis may exist in $\mathscr{P}_n$.

As discussed in [10], Definition 2.1 can be characterized through the partial ordering of bases defined by non-negative transition matrices (2.7).

Theorem 2.6. *Let* $\Phi, \Psi \in \mathscr{P}_n$ *be two non-negative bases. Then, for every polynomial* $p(x) \in \mathbb{P}_n$*, the following equivalence holds:*

$$\Phi \precsim \Psi \quad \Leftrightarrow \quad C_\Phi(p(x)) \leq C_\Psi(p(x)). \tag{2.8}$$

For instance, if Φ is Bernstein basis functions (2.1) and Ψ is the power basis, then the identities (2.3), (2.4) and Theorem 2.6 imply that $C_\Phi(p(x)) < C_\Psi(p(x))$ for every polynomial $p(x) \in \mathbb{P}_n$. Therefore, the Bernstein basis functions (2.1) is strictly systematically more stable than the power basis. This fact is related to the following more general result presented in [10].

Theorem 2.7. *The Bernstein basis functions* (2.1) *is a minimal element of* $\mathscr{P}_n$ *and is the only minimal element for which the basis functions have no roots in* $(0, 1)$*.*

Definition 2.2. A basis $\Phi \subset \mathbb{P}_n$ on $[a, b]$ is optimally stable if $C_\Phi(p(x))$ is the least possible condition number for all $p(x) \in \mathbb{P}_n$ on $[a, b]$. That is:

$$C_\Phi(p(x)) \leq C_\Psi(p(x)), \quad \text{for every } \Psi \in \mathscr{P}_n \text{ and } p(x) \in \mathbb{P}_n.$$

Definition 2.2 gives a stronger condition than being minimal under $\precsim$. This condition means being minimal in the total order of condition numbers, not just in the partial ordering of bases defined by non-negative transition matrices (2.7). It is also worth noting that every optimally stable basis is minimal basis.

However, not all the optimally stable bases are used in practice because optimal stability alone may not be sufficient for the adoption of a basis in practical applications. In addition to being optimally stable, a basis must also admit efficient algorithms for interpolation, approximation, root-finding, and similar tasks in order to be considered for adoption. In this sense, the Bernstein basis functions (2.1) is the only known optimally stable basis that is widely used due to its variety of algorithms that address the diverse computational requirements.

We refer the interested reader to [2, 3] for recent developments and applications of Bernstein polynomials in approximation theory and numerical analysis, as well as for an alternative generalization of the condition number for polynomials.

3. Degenerate Bernstein Forms: Computational Features

This section is devoted to the study of degenerate Bernstein basis functions and their associated forms, focusing on several structural properties that extend some of the results presented in Section 2.

We begin with the following definition.

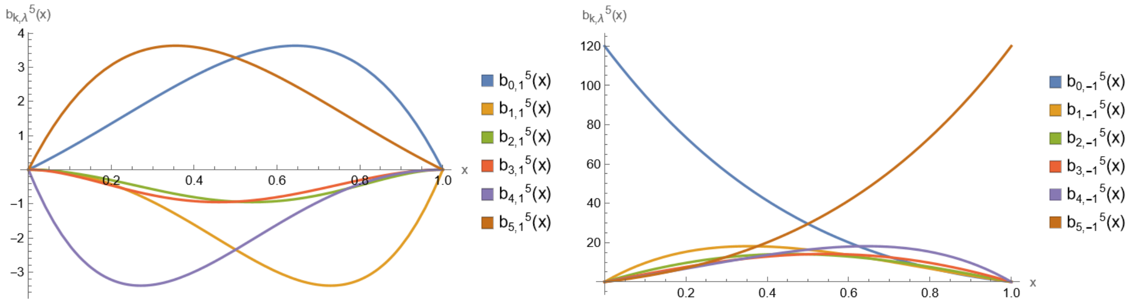
Definition 3.1. For each $\lambda, x \in \mathbb{R}$, the degenerate Bernstein basis functions are defined as:

$$b^n_{k,\lambda}(x) := \binom{n}{k} (x)_{k,\lambda} (1 - x)_{n-k,\lambda}, \quad k = 0, \ldots, n, \text{ and } n \in \mathbb{N}_0, \tag{3.1}$$

where $(x)_{k,\lambda}$ denotes the generalized falling factorial (1.1).

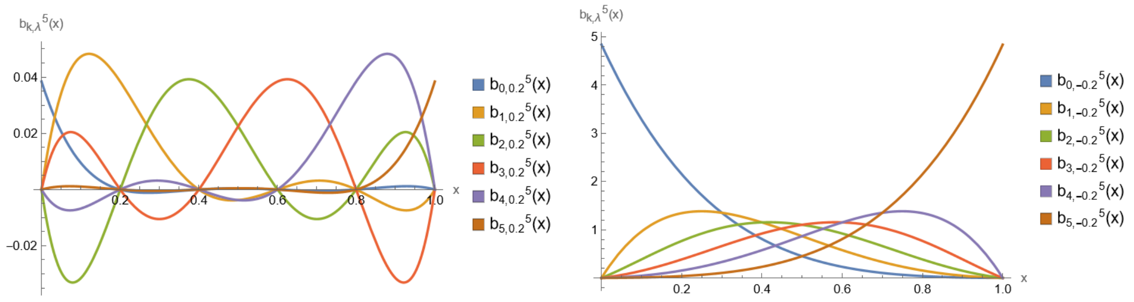
The degenerate Bernstein basis functions (also called degenerate Bernstein polynomials) were first introduced by T. Kim and D. S. Kim [15, 16], motivated by a probabilistic viewpoint in which the degeneracy parameter may be interpreted as a “psychological burden” affecting the success probability in binomial trials.

It is clear that $b^n_{k,\lambda}(x) \in \mathbb{P}_n$, for any $n \in \mathbb{N}_0$ and each $\lambda \in \mathbb{R}$. Furthermore, from (1.1) and (3.1) we have $b^n_{k,0}(x) = b^n_k(x)$ for all $k, n \in \mathbb{N}$ and $x \in \mathbb{R}$. In Figure 1 we plot some degenerate Bernstein basis functions along the interval $[0, 1]$ for certain values of the parameter λ.



(A) Degenerate Bernstein basis functions of degree 5 and $\lambda = 1$

(B) Degenerate Bernstein basis functions of degree 5 and $\lambda = -1$



(C) Degenerate Bernstein basis functions of degree 5 and $\lambda = 0.2$

(D) Degenerate Bernstein basis functions of degree 5 and $\lambda = -0.2$

FIGURE 1. Degenerate Bernstein basis functions of degree 5 for $\lambda = \pm 1, \pm 0.2$.

Definition 3.2. For a fixed degree $n \in \mathbb{N}_0$, we say that $\lambda \in \mathbb{R}$ is *n-admissible* if

$$(1)_{n,\lambda} = \prod_{j=0}^{n-1} (1 - j\lambda) \neq 0 \quad \text{equivalently} \quad \lambda \notin \left\{ \frac{1}{j} : j = 1, \dots, n-1 \right\}. \tag{3.2}$$

It is clear that for $n = 0$ or $n = 1$ the condition (3.2) is satisfied for any $\lambda \in \mathbb{R}$, thus every $\lambda \in \mathbb{R}$ is 0-admissible or 1-admissible, respectively. However for $n \geq 2$ the condition (3.2) is strictly necessary in order to obtain the well definition of the λ-admissibility.

When λ is not n-admissible, some identities below may become singular and the family (3.1) may become linearly dependent (for instance, for $\lambda = 1$ and $n \geq 2$ all $b^n_{k,1}(x)$ share the common factor $x(1-x)$). In the statements where a *basis* (and uniqueness of representation) is required, we will explicitly assume (3.2).

For each $\lambda \in \mathbb{R}$ the generalized falling factorials $\{(x)_{0,\lambda}, \ldots, (x)_{n,\lambda}\} \subset \mathbb{P}_n$ form a basis for $\mathbb{P}_n$. Moreover, if λ is n-admissible in the sense of (3.2), then the degenerate Bernstein basis functions (3.1) also form a basis for $\mathbb{P}_n$. In order to prove this first fact we recall the definition of signed Stirling numbers of first kind and Stirling numbers of second kind, respectively, in the sense of [13].

Definition 3.3. The signed Stirling numbers of the first kind are defined by the recurrence relation

$$s(n+1,k) = s(n,k-1) - n\,s(n,k),$$

where $s(n,k)$ is defined for $n,k \geq 1$, with boundary conditions

$$s(0,0) = 1, \quad s(n,n) = 1, \quad s(n,0) = 0, \text{ for } n > 0, \quad s(0,k) = 0, \text{ for } k > 0.$$

Definition 3.4. The Stirling numbers of the second kind are defined by the recurrence relation

$$S(n+1,k) = k\,S(n,k) + S(n,k-1),$$

where $S(n,k)$ is defined for $n,k \geq 1$, with boundary conditions

$$\begin{aligned} S(0,0) =& 1, \quad S(n,0) = 0, \text{ for } n > 0, \quad S(0,k) = 0, \text{ for } k > 0, \\ S(n,n) =& 1, \text{ for } n \geq 0. \end{aligned}$$

Proposition 3.1. *For each $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}_0$ the following identities are satisfied:*

$$(x)_{n,\lambda} = \sum_{k=0}^{n} \lambda^{n-k} s(n,k) x^k, \tag{3.3}$$

where $s(n,k)$ denotes the signed Stirling number of first kind

$$x^n = \sum_{k=0}^{n} \lambda^{n-k} S(n,k) (x)_{k,\lambda}, \tag{3.4}$$

where $S(n,k)$ denotes the Stirling number of second kind

Proof. Without loss of generality, we assume that $n \in \mathbb{N}$. Since $(x)_{n,\lambda} \in \mathbb{P}_n$, there exist unique coefficients $\{a^n_0, \ldots, a^n_n\}$ such that $(x)_{n,\lambda} = \sum_{k=0}^n a^n_k x^k$. Notice that these coefficients are obtained from expanding and rearranging the terms of the generalized falling factorial (1.1). Then, it is clear that when the expansion in the power basis is done, the coefficient corresponding to the power x^k will be $a^n_k = \lambda^{n-k} c^n_k$ for some value of c^n_k. Hence,

$$(x)_{n,\lambda} = \sum_{k=0}^{n} \lambda^{n-k} c^n_k x^k\,.$$

Using the generalized falling factorial property

$$(x)_{n+1,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-n\lambda) = (x)_{n,\lambda}(x-n\lambda)\,,$$

we have

$$(x)_{n+1,\lambda} = \left(\sum_{k=0}^{n} \lambda^{n-k} c_k^n x^k\right)(x - n\lambda) = \sum_{k=0}^{n} \lambda^{n-k} c_k^n x^{k+1} - n \sum_{k=0}^{n} \lambda^{n+1-k} c_k^n x^k$$
$$= c_n^n x^{n+1} + \sum_{k=1}^{n} (c_{k-1}^n - n c_k^n)\lambda^{n+1-k} x^k + c_0^n \lambda^{n+1} = \sum_{k=0}^{n+1} c_k^{n+1} \lambda^{n+1-k} x^k\,.$$

Thus, the coefficients c_k^n satisfy the recurrence relation $c_k^{n+1} = c_{k-1}^n - n c_k^n$, for all $n, k \geq 1$. However, from the expansion of $(x)_{n,\lambda}$ it is also clear that $c_0^n = 0$ and $c_n^n = 1$ for every $n > 0$, and the particular case $(x)_{0,\lambda} = 1$ implies $c_0^0 = 1$ and $c_k^0 = 0$ for all $k > 0$. Therefore, we see that the coefficients c_k^n are indeed the signed Stirling numbers of first kind since they satisfy the same relations, i.e, $c_k^n = s(n,k)$. Thus, (3.3) is satisfied.

Similarly, if we seek for the coefficients $\{\alpha_0^n, \ldots, \alpha_n^n\}$ such that $x^n = \sum_{k=0}^{n} \alpha_k^n \, (x)_{k,\lambda}$. Since $(x)_{n,\lambda}$ is the unique polynomial of degree n belonging to $\{(x)_{0,\lambda}, \ldots, (x)_{n,\lambda}\}$, matching coefficients we have $\alpha_n^n = 1$. Moreover, the expansion (3.3) implies that it contributes to the power x^{n-1} with the coefficient $\lambda s(n, n-1)$, whose exponent for the power of λ is 1, that must be cancelled by the generalized falling factorial $(x)_{n-1,\lambda}$ because it is the only remaining polynomial of degree $n-1$ or greater with not fixed coefficient yet. Therefore, $\alpha_{n-1}^n = -\lambda s(n, n-1)$.

For the power x^{n-2} notice that $(x)_{n,\lambda}$ contributes with a term whose exponent for the power of λ is 2, whereas $(x)_{n-1,\lambda}$ will do it with a term whose exponent for the power of λ equal to 1 but then the contribution of $-\lambda s(n, n-1)(x)_{n-1,\lambda}$ will be one power wit exponent 2 in λ, which then implies that $\alpha_{n-2}^n = \lambda^2 c_{n-2}^n$, where c_{n-2}^n is some value.

Following this reasoning, assume that it still holds for $k = m+1 < n$ the property $\alpha_{m+1}^n = \lambda^{n-(m+1)} c_{m+1}^n$ for some value c_{m+1}^n. It will be computed the coefficient α_m^n. Notice that independently, each $(x)_{k,\lambda}$ contributes to the power x^m with $k > m$ with a term whose power of λ is $k - m$. Therefore, each $\alpha_k \, (x)_{k,\lambda} = \lambda^{n-k} c_k^n \, (x)_{k,\lambda}$ with $k > m$ will contribute to x^m with a term of power of λ equal to $n - m$. Thus, $\alpha_m^n = \lambda^{n-m} c_m^n$ for some c_m^n.

It is possible to write then $x^n = \sum_{k=0}^{n} \lambda^{n-k} c_k^n \, (x)_{k,\lambda}$. Hence, it remains to be proven that the c_k^n coefficients are the Stirling numbers of second kind. For this, it suffices to see that the identity $(x)_{k+1,\lambda} = (x)_{k,\lambda}(x - k\lambda)$ implies $x(x)_{k,\lambda} = (x)_{k+1,\lambda} + k\lambda(x)_{k,\lambda}$. Hence,

$$x^{n+1} = x \sum_{k=0}^{n} \lambda^{n-k} c_k^n (x)_{k,\lambda} = \sum_{k=1}^{n+1} \lambda^{n+1-k} c_{k-1}^n (x)_{k,\lambda} + \sum_{k=0}^{n} k\lambda^{n+1-k} c_k^n (x)_{k,\lambda}$$
$$= c_n^n (x)_{n+1,\lambda} + \sum_{k=1}^{n} (c_{k-1}^n + k c_k^n)\lambda^{n+1-k} (x)_{k,\lambda} = \sum_{k=0}^{n+1} c_k^{n+1} \lambda^{n+1-k} (x)_{k,\lambda}.$$

Therefore, the coefficients c_k^n satisfy the recurrence relation $c_k^{n+1} = c_{k-1}^n + k c_k^n$, for all $n, k \geq 1$. However, since the leading coefficient of the polynomial $(x)_{n+1,\lambda}$ is already 1 then, $c_n^n = 1$ for each $n > 0$. Moreover, observe from the last equality that $c_0^n = 0$ for all $n > 0$. Lastly, the particular case $x^0 = 1 = \alpha_0^0(x)_{0,\lambda} + \sum_{k=1}^n c_k^0(x)_{k,\lambda}$ gives directly that $c_0^0 = 1$ and $c_k^0 = 0$ for all $k > 0$. Thus, one sees that the coefficients c_k^n are indeed the Stirling numbers of second kind since they satisfy the same relations, i.e., $c_k^n = S(n,k)$. Then, (3.4) is satisfied. ☐

We have two straightforward consequences from Proposition 3.1. First, identity (3.4) say us that the set $\{(x)_{0,\lambda}, \dots, (x)_{n,\lambda}\}$ is a basis for $\mathbb{P}_n$. Second, the identity (3.4) allow us to deduce an explicit expression for the degenerate Stirling numbers of first kind in the sense of [13] in terms of the signed Stirling number of first kind, as follows. According to [13] the degenerate Stirling numbers of first kind $s_\lambda(n,k)$ are defined as the coefficients in the expansion

$$(x)_{n,\lambda} = \sum_{k=0}^{n} s_\lambda(n,k) x^k \,.$$

Then, the comparison between this expansion and (3.3) yields

$$s_\lambda(n,k) = \lambda^{n-k} s(n,k), \quad \text{for all } n,k \in \mathbb{N}_0 \text{ and } \lambda \in \mathbb{R}.$$

Related identities for generalized/degenerated Stirling numbers and Bell-type polynomials, can be found for instance in [21]. However, in order to avoid ambiguities it is worth to mention that degenerated Stirling numbers given in [21] are different from the degenerated Stirling numbers given in [13], however both authors used the same name for these numbers.

Another important and useful result for this section is the following [13, 15, 16].

Theorem 3.1 (Binomial theorem for generalized falling factorials)**.** *Given arbitrary real values for the parameters x, y and λ, the k-th generalised falling factorial satisfies that*

$$(x+y)_{k,\lambda} = \sum_{s=0}^{k} \binom{k}{s} (x)_{s,\lambda} (y)_{k-s,\lambda} \,. \tag{3.5}$$

With these ideas in mind, we can proceed to study the main structural properties of degenerated Bernstein basis functions and their associated degenerate Bernstein forms.

Theorem 3.2. *Let $\{b_{0,\lambda}^n(x), \dots, b_{n,\lambda}^n(x)\}$ be the degenerate Bernstein basis functions. Then, the following properties are satisfied.*

(*a*) *Symmetry*

$$b_{k,\lambda}^n(x) = b_{n-k,\lambda}^n(1-x), \quad k = 0, \dots, n.$$

(*b*) *Recursion*

$$b_{k,\lambda}^{n+1}(x) = (x-(k-1)\lambda) b_{k-1,\lambda}^n(x) + (1-x-(n-k)\lambda) b_{k,\lambda}^n(x), \quad k = 0, \dots, n,$$

with initial condition $b_{-1,\lambda}^n(x) := 0$.

(*c*) *Degenerate partition of unity*

$$\sum_{k=0}^{n} b_{k,\lambda}^{n}(x) = (1)_{n,\lambda}. \tag{3.6}$$

(*d*) *Basis relations.*

(*d*.1) *Degenerate Bernstein basis functions are expressed in terms of the generalized falling factorial basis as*

$$b_{k,\lambda}^{n}(x) = \sum_{j=k}^{n} \binom{n}{k}\binom{n-k}{j-k}(-1)^{j-k}(1-j\lambda)_{n-j,\lambda}(x)_{j,\lambda}, \quad k=0,\dots,n. \tag{3.7}$$

(*d*.2) *Inversion type formula*

$$(x)_{k,\lambda} = \frac{1}{(1-k\lambda)_{n-k,\lambda}} \sum_{j=k}^{n} \frac{\binom{j}{k}}{\binom{n}{k}} b_{j,\lambda}^{n}(x), \quad (1-k\lambda)_{n-k,\lambda} \neq 0, k=0,\dots,n. \tag{3.8}$$

(*e*) *Degree elevation. For each* $n \in \mathbb{N}_0$ *we have*

$$b_{k,\lambda}^{n}(x) = \frac{1}{(1-n\lambda)_{r,\lambda}} \sum_{j=k}^{k+r} \frac{\binom{n}{k}\binom{r}{j-k}}{\binom{n+r}{j}} b_{j,\lambda}^{n+r}(x), \quad k=0,\dots,n, r \in \mathbb{N}_0, (1-n\lambda)_{r,\lambda} \neq 0. \tag{3.9}$$

(*f*) [15, Theorem 2.1] *Generating function. The degenerate Bernstein basis functions can be expressed by means of the following generating function and series:*

$$\frac{1}{k!}(x)_{k,\lambda} t^{k} (1+\lambda t)^{\frac{(1-x)}{\lambda}} = \sum_{n=k}^{+\infty} b_{k,\lambda}^{n}(x) \frac{t^{n}}{n!}, \quad k \in \mathbb{N}_0.$$

Proof. Since (*a*) and (*b*) are straightforward consequences of (3.1), and the interested reader may find a complete proof of (*f*) in [15], we focus our efforts on the proof of (*c*), (*d*) and (*e*).

Using (3.1) and (3.5) we obtain that

$$\sum_{k=0}^{n} b_{k,\lambda}^{n}(x) = \sum_{k=0}^{n} \binom{n}{k} (x)_{k,\lambda}(1-x)_{n-k,\lambda} = (x+(1-x))_{n,\lambda} = (1)_{n,\lambda},$$

hence, (3.6) holds.

Again, by (3.1) and (3.5), we deduce that

$$\begin{aligned} b_{k,\lambda}^{n}(x) =& \binom{n}{k}(x)_{k,\lambda}(-1)^{n-k}\left((x-k\lambda)+(-1+(n-1)\lambda)\right)_{n-k,\lambda} \\ =& \binom{n}{k}(x)_{k,\lambda}(-1)^{n-k}\left(\sum_{j=k}^{n}\binom{n-k}{j-k}(x-\lambda k)_{j-k,\lambda}(-1+(n-1)\lambda)_{n-j,\lambda}\right) \\ =& \sum_{j=k}^{n}\binom{n}{k}\binom{n-k}{j-k}(-1)^{n-k}(x)_{j,\lambda}(-1+(n-1)\lambda)_{n-j,\lambda}. \end{aligned} \tag{3.10}$$

Now, from the well-known identities

$$\begin{aligned}(-x)_{k,\lambda} =&(-1)^k x(x+\lambda)\cdots(x+(k-1)\lambda) = (-1)^k (x)_{k,-\lambda},\\ (x)_{k,-\lambda} =&x(x+\lambda)\cdots(x+(k-1)\lambda) = (x+(k-1)\lambda)_{k,\lambda},\end{aligned}$$

it follows that $(-1+(n-1)\lambda)_{n-j,\lambda} = (-1+j\lambda)_{n-j,-\lambda} = (-1)^{n-j}(1-j\lambda)_{n-j,\lambda}$. Hence, the right-hand-side of (3.10) becomes

$$\begin{aligned}&\sum_{j=k}^{n}\binom{n}{k}\binom{n-k}{j-k}(-1)^{2n-k-j}(x)_{j,\lambda}(1-j\lambda)_{n-j,\lambda}\\ =&\sum_{j=k}^{n}\binom{n}{k}\binom{n-k}{j-k}(-1)^{j-k}(x)_{j,\lambda}(1-j\lambda)_{n-j,\lambda},\end{aligned}$$

and thus (3.7) follows.

Applying (3.5), we deduce that

$$\begin{aligned}(x)_{k,\lambda}(1-k\lambda)_{n-k,\lambda} &= (x)_{k,\lambda}(x-k\lambda+(1-x))_{n-k,\lambda}\\ &= (x)_{k,\lambda}\sum_{s=0}^{n-k}\binom{n-k}{s}(x-\lambda k)_{s,\lambda}(1-x)_{n-k-s,\lambda}\\ &= (x)_{k,\lambda}\sum_{j=k}^{n}\binom{n-k}{j-k}(x-\lambda k)_{j-k,\lambda}(1-x)_{n-j,\lambda}\\ &= \sum_{j=k}^{n}\frac{\binom{j}{k}}{\binom{n}{k}} b^n_{j,\lambda}(x),\end{aligned}$$

which proves (3.8), whenever $(1-k\lambda)_{n-k,\lambda} \neq 0,\ k=0,\dots,n$.

Finally, using the identity $(1-n\lambda)_{r,\lambda} = ((x-k\lambda)+(1-x-n\lambda+k\lambda))_{r,\lambda}$, (3.1) and (3.5) we get

$$\begin{aligned}b^n_{k,\lambda}(x)(1-n\lambda)_{r,\lambda} &= b^n_{k,\lambda}(x)\left((x-k\lambda)+(1-x-n\lambda+k\lambda)\right)_{r,\lambda}\\ &= b^n_{k,\lambda}(x)\left(\sum_{j=k}^{k+r}\binom{r}{j-k}(x-k\lambda)_{j-k,\lambda}(1-x-(n-k)\lambda)_{r-(j-k),\lambda}\right)\\ &= \sum_{j=k}^{k+r}\binom{n}{k}\binom{r}{j-k}(x)_{j,\lambda}(1-x)_{n+r-j,\lambda} = \sum_{j=k}^{k+r}\frac{\binom{n}{k}\binom{r}{j-k}}{\binom{n+r}{j}} b^{n+r}_{j,\lambda}(x),\end{aligned}$$

from which (3.9) follows, whenever $(1-n\lambda)_{r,\lambda} \neq 0$. □

Theorem 3.3 (Derivative and integral formulas)**.** *Let* $\{b^n_{0,\lambda}(x),\dots,b^n_{n,\lambda}(x)\}$ *be the degenerate Bernstein basis functions and assume that* λ *is* $(n-1)$*-admissible in the sense of* (3.2)*, with* $n>2$*. Then,*

$$\frac{d}{dx}b^n_{k,\lambda}(x) = \binom{n}{k}\sum_{j=0}^{n-1}A_{j,k}(\lambda)b^{n-1}_{j,\lambda}(x), \quad k=1,\dots,n-1, \tag{3.11}$$

where

$$A_{j,k}(\lambda)=\begin{cases}\sum_{s=0}^{j}\sum_{l=0}^{j-s}\frac{\binom{k-1-s}{l}(-\lambda)_{k-1-s-l,\lambda}}{(1-(n-k+s+l)\lambda)_{k-s-l-1,\lambda}}\cdot\frac{\binom{k-s-l-1}{j-s-l}}{\binom{n-1}{j}}, & 0\le j\le k-1,\\ -\sum_{s=0}^{n-1-j}\sum_{l=0}^{n-1-j-s}\frac{\binom{n-k-1-s}{l}(-\lambda)_{n-k-1-s-l,\lambda}}{(1-(s+l+k)\lambda)_{n-1-s-l-k,\lambda}}\cdot\frac{\binom{n-1-s-l-k}{j-k}}{\binom{n-1}{j}}, & k\le j\le n-1.\end{cases}$$

If λ *is* $(n+1)$*-admissible in the sense of* (3.2)*, with* $n\in\mathbb{N}$*, then*

$$\int b_{k,\lambda}^{n}(x)dx=\sum_{j=0}^{n+1}B_{j,k}^{n+1}(\lambda)b_{j,\lambda}^{n+1}(x),\quad k=0,\dots,n, \tag{3.12}$$

where

$$B_{j,k}^{n+1}(\lambda)=\sum_{s=0}^{j}\sum_{l=\max(s-1,0)}^{n}\sum_{m=\max(k,l)}^{n}\frac{\binom{n}{k}\binom{n-k}{m-k}\binom{j}{s}(-1)^{m-k}(1-m\lambda)_{n-m,\lambda}\lambda^{m+1-s}}{\binom{n+1}{s}(1-s\lambda)_{n+1-s,\lambda}}\times\frac{s(m,l)S(l+1,s)}{(l+1)}.$$

Proof. It is clear that

$$\frac{d}{dx}b_k^n(x)=\binom{n}{k}\left(\frac{d}{dx}((x)_{k,\lambda})(1-x)_{n-k,\lambda}+(x)_{k,\lambda}\frac{d}{dx}((1-x)_{n-k,\lambda})\right),$$

thus, our problem is reduced to compute separately the derivatives $\frac{d}{dx}((x)_{k,\lambda})$ and $\frac{d}{dx}((1-x)_{n-k,\lambda})$ to be substituted later in the above expression. Since

$$(x)_{s,\lambda}(x-(s+1)\lambda)_{k-1-s,\lambda}=x(x-\lambda)\cdots\frac{d}{dx}((x-s\lambda))\cdots(x-(k-1)\lambda),$$

for $s=0\dots,k-1$. Then, using (3.5), we obtain

$$\frac{d}{dx}((x)_{k,\lambda})(1-x)_{n-k,\lambda}=\sum_{s=0}^{k-1}\sum_{l=0}^{k-1-s}\frac{\binom{k-1-s}{l}}{\binom{n-k+s+l}{s+l}}(-\lambda)_{k-1-s-l,\lambda}b_{s+l,\lambda}^{n-k+s+l}(x).$$

By (3.9) we can express the sum on the right of above identity in the form

$$S_1:=\sum_{s=0}^{k-1}\sum_{l=0}^{k-1-s}\frac{\binom{k-1-s}{l}(-\lambda)_{k-1-s-l,\lambda}}{\binom{n-k+s+l}{s+l}(1-(n-k+s+l)\lambda)_{k-s-l-1,\lambda}}\times\sum_{j=s+l}^{k-1}\frac{\binom{n-k+s+l}{s+l}\binom{k-s-l-1}{j-s-l}}{\binom{n-1}{j}}b_{j,\lambda}^{n-1}(x).$$

Similarly, using (3.5) we get

$$(x)_{k,\lambda}\frac{d}{dx}((1-x)_{n-k,\lambda})=-\sum_{s=0}^{n-k-1}\sum_{l=0}^{n-k-1-s}\frac{\binom{n-k-1-s}{l}}{\binom{s+l+k}{k}}(-\lambda)_{n-k-1-s-l,\lambda}b_{k,\lambda}^{s+l+k}(x),$$

and again by (3.9), we can express the sum on the right of previous identity as follows

$$S_2 := -\sum_{s=0}^{n-k-1}\sum_{l=0}^{n-k-1-s}\frac{\binom{n-k-1-s}{l}(-\lambda)_{n-k-1-s-l,\lambda}}{(1-(s+l+k)\lambda)_{n-1-s-l-k,\lambda}}\sum_{j=k}^{n-1-s-l}\frac{\binom{n-1-s-l-k}{j-k}}{\binom{n-1}{j}}b_{j,\lambda}^{n-1}(x)\,.$$

Therefore, by rearranging the terms of the sums S_1 and S_2, and making suitable substitutions (3.11) follows.

Now by (3.7), (3.3), (3.4) and (3.8), for $k = 0,\dots,n$,

$$\int b_{k,\lambda}^{n}(x)dx = \sum_{j=0}^{n+1}\left(\sum_{s=0}^{j}\sum_{l=\max(s-1,0)}^{n}\sum_{m=\max(k,l)}^{n}\frac{\binom{n}{k}\binom{n-k}{m-k}\binom{j}{s}(-1)^{m-k}(1-m\lambda)_{n-m,\lambda}\lambda^{m+1-s}}{\binom{n+1}{s}(1-s\lambda)_{n+1-s,\lambda}}\cdot\frac{s(m,l)S(l+1,s)}{(l+1)}\right)b_{j,\lambda}^{n+1}(x),$$

and so (3.12) follows. ☐

From Proposition 3.1 and inversion type formula (3.8) we can deduce that the set $\Phi_\lambda \equiv \{b_{0,\lambda}^n(x),\dots,b_{n,\lambda}^n(x)\}$ is a basis for $\mathbb{P}_n$ whenever λ is n-admissible in the sense of (3.2). Thus, for each fixed n-admissible $\lambda \in \mathbb{R}$, any $p(x) \in \mathbb{P}_n$ can be uniquely expressed in terms of Φ_λ as

$$p(x) = \sum_{k=0}^{n} a_k b_{k,\lambda}^n(x), \quad a_k \in \mathbb{R}, k = 0,\dots,n. \tag{3.13}$$

We call the expression on the right-hand-side of (3.13) *degenerate Bernstein form* of $p(x)$.

The degree elevation and reduction relations for degenerate Bernstein forms are given in the next theorem.

Theorem 3.4. *Let* $p(x) = \sum_{k=0}^{n} a_k\, b_{k,\lambda}^n(x)$ *be a degenerate Bernstein form. Then the following properties hold.*

(*a*) *Degree elevation. Let* $r \in \mathbb{N}_0$ *and assume that* λ *is* $(n+r)$*-admissible in the sense of* (3.2) (*in particular,* $(1-n\lambda)_{r,\lambda} \neq 0$). *The degenerate Bernstein form of* $p(x)$ *can be expressed in terms of the degenerate Bernstein basis functions of degree* $n+r$ *as*

$$p(x) = \sum_{j=0}^{n+r} a_{j,\lambda}^{n+r}\, b_{j,\lambda}^{n+r}(x),$$

where the coefficients $a_{0,\lambda}^{n+r},\dots,a_{n+r,\lambda}^{n+r}$ *are given by*

$$a_{j,\lambda}^{n+r} = \sum_{k=\max(0,j-r)}^{\min(n,j)}\frac{1}{(1-n\lambda)_{r,\lambda}}\cdot\frac{\binom{n}{k}\binom{r}{j-k}}{\binom{n+r}{j}}a_{k,\lambda}^{n}, \quad j = 0,\dots,n+r.$$

(*b*) *Degree reduction. Let* $r \in \mathbb{N}_0$ *and assume that* λ *is* n*-admissible in the sense of* (3.2)*, with* $n > 1$*. Let* $g(x) \in \mathbb{P}_n$ *with* $\deg(g(x)) = n$*. If*

$$g(x) = \sum_{k=0}^{n+r} a_{k,\lambda}^{n+r}\, b_{k,\lambda}^{n+r}(x)$$

is its representation in the degenerate Bernstein basis of degree $n+r$*, then its coefficients in the degenerate Bernstein basis of degree* n *are given by*

$$a_{j,\lambda}^{n} = \sum_{m=0}^{j} \sum_{l=m}^{n} \sum_{t=l}^{n+r} \sum_{k=0}^{t} C_{j,m,l,t,k}^{(n,r,\lambda)}\, a_{k,\lambda}^{n+r}, \quad j = 0, \ldots, n,$$

where

$$C_{j,m,l,t,k}^{(n,r,\lambda)} = \binom{n+r}{k}\binom{n+r-k}{t-k}(-1)^{t-k}(1-t\lambda)_{n+r-t,\lambda}\, s(t,l)\, \frac{\lambda^{t-m} S(l,m)}{(1-m\lambda)_{n-m,\lambda}} \cdot \frac{\binom{j}{m}}{\binom{n}{m}}.$$

Proof. From the degree elevation property (3.9), we obtain

$$p(x) = \sum_{j=0}^{n+r} \left(\sum_{k=\max(0,j-r)}^{\min(n,j)} \frac{1}{(1-n\lambda)_{r,\lambda}} \cdot \frac{\binom{n}{k}\binom{r}{j-k}}{\binom{n+r}{j}} a_{k,\lambda}^{n} \right) b_{j,\lambda}^{n+r}(x).$$

On the other hand, it is known that

$$p(x) = \sum_{j=0}^{n+r} a_{j,\lambda}^{n+r}\, b_{j,\lambda}^{n+r}(x).$$

By comparing both representations of $p(x)$, we conclude that

$$a_{j,\lambda}^{n+r} = \sum_{k=\max(0,j-r)}^{\min(n,j)} \frac{1}{(1-n\lambda)_{r,\lambda}} \cdot \frac{\binom{n}{k}\binom{r}{j-k}}{\binom{n+r}{j}} a_{k,\lambda}^{n}, \quad j = 0, \ldots, n+r.$$

Now, let $r \in \mathbb{N}$. From (3.3), (3.7), (3.4), and the inversion-type formula (3.8), it follows that

$$p(x) = \sum_{j=0}^{n} \sum_{m=0}^{j} \sum_{l=m}^{n} \sum_{t=l}^{n+r} \sum_{k=0}^{t} C_{j,m,l,t,k}^{(n,r,\lambda)}\, a_{k,\lambda}^{n+r}\, b_{j,\lambda}^{n}(x),$$

where

$$C_{j,m,l,t,k}^{(n,r,\lambda)} = \binom{n+r}{k}\binom{n+r-k}{t-k}(-1)^{t-k}(1-t\lambda)_{n+r-t,\lambda}\, s(t,l)\, \frac{\lambda^{t-m} S(l,m)}{(1-m\lambda)_{n-m,\lambda}} \cdot \frac{\binom{j}{m}}{\binom{n}{m}}.$$

Therefore, the coefficients of $p(x) = \sum_{j=0}^{n} a_{j,\lambda}^{n} b_{j,\lambda}^{n}(x)$ are given by

$$a_{j,\lambda}^{n} = \sum_{m=0}^{j} \sum_{l=m}^{n} \sum_{t=l}^{n+r} \sum_{k=0}^{t} C_{j,m,l,t,k}^{(n,r,\lambda)}\, a_{k,\lambda}^{n+r}, \quad j = 0, \ldots, n.$$

□

Notice that when λ is n-admissible in the sense of (3.2), from (3.6) it follows that the normalized degenerate Bernstein basis functions

$$\hat{b}^n_{k,\lambda}(x) = \frac{1}{(1)_{n,\lambda}} b^n_{k,\lambda}(x), \quad k = 0, \ldots, n, \tag{3.14}$$

form a partition of unity, i.e.,

$$\sum_{k=0}^{n} \hat{b}^n_{k,\lambda}(x) = 1, \qquad \text{for all } x \in \mathbb{R}.$$

However, as expected, neither the degenerate Bernstein basis functions nor their normalizations generally satisfy the non negativity condition. In fact, a detailed analysis of the values λ for which $b^n_{k,\lambda}(x)$ remains non-negative on $[0,1]$ is provided in [14, Theorem 3.7]. From this result, it follows that the most general condition ensuring

$$b^n_{k,\lambda}(x) \geq 0, \qquad \text{for all } x \in [0,1]$$

is simply that $\lambda \leq 0$.

Therefore, under the condition $\lambda \leq 0$, the degenerate Bernstein basis functions are non-negative on $[0,1]$. This property allows us to derive bounds and end-point evaluations for degenerate Bernstein forms, as stated in the following theorem.

Theorem 3.5. *Let* $p(x) = \sum_{k=0}^{n} a_k b^n_{k,\lambda}(x)$ *be a degenerate Bernstein form. Then, for* $x \in [0,1]$ *and* $\lambda \leq 0$ *the following properties hold.*

(*a*) *Lower and upper bounds.*

$$(1)_{n,\lambda} \min_k a_k \leq p(x) \leq (1)_{n,\lambda} \max_k a_k. \tag{3.15}$$

(*b*) *End-point values. The degenerate Bernstein polynomial verifies that* $p(0) = a_0(1)_{n,\lambda}$ *and* $p(1) = a_n(1)_{n,\lambda}$.

Proof. Since $x \in [0,1]$ and $\lambda \leq 0$, it follows that $b^n_{k,\lambda}(x) \geq 0$ for $k = 0, \ldots, n$. Hence, by (3.6), we obtain

$$\sum_{k=0}^{n} a_k b^n_{k,\lambda}(x) \leq \max_k a_k \sum_{k=0}^{n} b^n_{k,\lambda}(x) = (1)_{n,\lambda} \max_k a_k.$$

Similarly,

$$\sum_{k=0}^{n} a_k b^n_{k,\lambda} \geq \min_k a_k \sum_{k=0}^{n} b^n_{k,\lambda} = (1)_{n,\lambda} \min_k a_k.$$

Combining these inequalities, we obtain (3.15).

For the end-point values, consider first $x = 0$. Then,

$$p(0) = \sum_{k=0}^{n} a_k b^n_{k,\lambda}(0) = a_0 b^n_{0,\lambda}(0) = a_0(1)_{n,\lambda}.$$

Similarly, for $x = 1$, we have

$$p(1) = \sum_{k=0}^{n} a_k b^n_{k,\lambda}(1) = a_n b^n_{n,\lambda}(1) = a_n(1)_{n,\lambda}.$$

□

We conclude this section with the following result, which establishes a connection between the power basis, the Bernstein basis functions, and their degenerate counterparts in terms of their condition numbers. This relation illustrates the comparative stability of these bases under perturbations.

Theorem 3.6. *For* $\lambda \leq 0$*, let us consider* $\Phi = \{1, x, \ldots, n\}$*,* $\Phi_\lambda = \{(x)_{0,\lambda}, \ldots, (x)_{n,\lambda}\}$*,* $\Psi = \{b_0^n(x), \ldots, b_n^n(x)\}$ *and* $\Psi_\lambda = \{b_{0,\lambda}^n(x), \ldots, b_{n,\lambda}^n(x)\}$ *be the power basis, the generalized falling factorial basis, the Bernstein basis functions and the degenerate Bernstein basis functions, respectively. Then, for every polynomial* $p(x) \in \mathbb{P}_n$ *with* $x \in [0,1]$*, the following relations hold:*

(*a*) $C_{\Phi_\lambda}(p(x)) > C_{\Psi_\lambda}(p(x))$,
(*b*) $C_{\Phi_\lambda}(p(x)) > C_{\Phi}(p(x))$,
(*c*) $C_{\Phi_{\lambda_1}}(p(x)) > C_{\Phi_{\lambda_2}}(p(x))$, *whenever* $\lambda_1 < \lambda_2 \leq 0$,
(*d*) $C_{\Psi_\lambda}(p(x)) > C_{\Psi}(p(x))$.

Proof. Inequality $C_{\Phi_\lambda}(p(x)) > C_{\Psi_\lambda}(p(x))$ is a straightforward consequence of (3.7) and (3.8).

Since $\Phi, \Phi_\lambda \in \mathscr{P}_n$ (on the interval $[0,1]$), by (3.3) and the condition $\lambda \leq 0$ we deduce that

$$(x)_{n,\lambda} = \sum_{k=0}^{n} (-1)^{2n-2k} |\lambda|^{n-k} |s(n,k)| \, x^k = \sum_{k=0}^{n} |\lambda|^{n-k} |s(n,k)| \, x^k,$$

which implies that $C_\Phi(p(x)) \precsim C_{\Phi_\lambda}(p(x))$, or equivalently $C_\Phi(p(x)) \leq C_{\Phi_\lambda}(p(x))$, by (2.8).

But from (3.4) and (2.8) it follows that $C_{\Phi_\lambda}(p(x)) \nleq C_\Phi(p(x))$. Consequently, $C_\Phi(p(x)) < C_{\Phi_\lambda}(p(x))$.

Now, assume $\lambda_1 < \lambda_2 \leq 0$. In order to prove the inequality in part (*c*), we consider any polynomial $p(x) \in \mathbb{P}_n$ and its power form $\sum_{k=0}^{n} a_k x^x$. Then the application of (3.4) yields the following representations of $p(x)$ with respect to the generalized falling factorial bases Φ_{λ_1} and Φ_{λ_2}, respectively,

$$p(x) = \sum_{k=0}^{n} \Lambda_{k,m} (x)_{k,\lambda_m}, \quad m = 1, 2, \tag{3.16}$$

with

$$\Lambda_{k,m} = \sum_{r=k}^{n} a_r \lambda_m^r S(r,k), \quad m = 1, 2, \ k = 0, \ldots, n.$$

Since $\lambda_1 < \lambda_2 \leq 0$ implies that $(x)_{k,\lambda_2} < (x)_{k,\lambda_1}$, for all $x \in [0,1]$, $k = 2, \ldots, n$, and clearly $(x)_{0,\lambda_m} = 1$ and $(x)_{1,\lambda_m} = x$, for $m = 1, 2$, we obtain

$$(x)_{k,\lambda_2} < (x)_{k,\lambda_1}, \quad \text{for all } x \in [0,1], \ k = 0, \ldots, n. \tag{3.17}$$

In view of (3.16), (3.17), and the inequality $0 \leq |\lambda_2| < |\lambda_1|$, we deduce that

$$C_{\Phi_{\lambda_2}}(p(x)) = \sum_{k=0}^{n} |\Lambda_{k,2}| \, (x)_{k,\lambda_2} = \sum_{k=0}^{n} \left| \sum_{r=k}^{n} a_r \lambda_2^r S(r,k) \right| (x)_{k,\lambda_2}.$$

Now, since $S(r,k) \geq 0$ and $|\lambda_2| < |\lambda_1|$, the sequence $\{\lambda_2^r S(r,k)\}_{r=k}^{n}$ is a componentwise contraction of $\{\lambda_1^r S(r,k)\}_{r=k}^{n}$. Hence,

$$\left| \sum_{r=k}^{n} a_r \lambda_2^r S(r,k) \right| \leq \left| \sum_{r=k}^{n} a_r \lambda_1^r S(r,k) \right| = |\Lambda_{k,1}|,$$

for each $k = 0, \ldots, n$. Combining this with (3.17), we obtain

$$|\Lambda_{k,2}|(x)_{k,\lambda_2} < |\Lambda_{k,1}|(x)_{k,\lambda_1}, \quad \text{for each } k = 0, \ldots, n.$$

Summing over k yields $C_{\Phi_{\lambda_2}}(p(x)) < C_{\Phi_{\lambda_1}}(p(x))$, which proves part (c).

Finally, for part (d), notice that each factor of $b_{k,\lambda}^n(x)$ can be written as a Bernstein form of degree one. Indeed, for $\mu_1 \geq 0$ and $\mu_2 \geq 1$ we have

$$\begin{aligned} x + \mu_1 &= \mu_1 b_0^1(x) + (1+\mu_1)\, b_1^1(x), \\ \mu_2 - x &= \mu_2 b_0^1(x) + (\mu_2 - 1)\, b_1^1(x). \end{aligned}$$

Since products of Bernstein forms with non-negative coefficients are again Bernstein forms of higher degree with non-negative coefficients, it follows that every $b_{k,\lambda}^n(x)$ can be expressed as a non-negative linear combination of the Bernstein basis functions Ψ. Therefore, we have $\Psi \precsim \Psi_\lambda$, and in view of (2.8), this implies $C_{\Psi_\lambda}(p(x)) \geq C_{\Psi}(p(x))$. The inequality is strict whenever $\lambda < 0$, because in that case the degenerate Bernstein basis functions differ from the classical ones, and thus the expansions cannot coincide. Hence, $C_{\Psi_\lambda}(p(x)) > C_{\Psi}(p(x))$, which proves part (d). ☐

4. Qian-Riedel-Rosenberg-type Stochastic Logic and Degenerate Bernstein Forms

The main idea behind the combinational circuits design with polynomial arithmetic of Qian et al. [28–30] consist of the following.

(1) Take advantage, in a suitable way, of the redundancy provided by stochastic computing for choosing binary sequences $x \in \{0,1\}^N$ corresponding to the value p_x, in order to make an association between x and a certain N-tuple of independent random variables $X = (X_1, \ldots, X_N)$, where each component X_k has Bernoulli distribution with some parameter $p_k \in [0,1]$.

(2) Given a Boolean function $y = f(x_1, \ldots, x_N)$ implementing a combinational circuit, use the association aforementioned for inducing a stochastic circuit implemented by a function of the form $Y = F(X_1, \ldots, X_N)$ (see for instance, [26]).

The passage of the Boolean function $y = f(x_1, \ldots, x_N)$ to the function $Y = F(X_1, \ldots, X_N)$ is called stochastic logic or stochastic logic in the sense of Qian-Riedel-Rosenberg [28–30] and the following property holds.

Theorem 4.1. (*cf.* [28, Theorem 1]]). *Given a Boolean function* $f : \{0,1\}^N \to \{0,1\}$. *Stochastic logic yields a polynomial in* N *variables* $\hat{F}$ *given by*

$$\hat{F}(a_1,\ldots,a_N) = \sum_{i_1=0}^{1} \cdots \sum_{i_N=0}^{1} \left(\alpha_{i_1 \ldots i_N} \prod_{k=1}^{N} a_k^{i_k} \right), \tag{4.1}$$

where the coefficients $\alpha_{i_1 \ldots i_N}$ *are integers. Moreover, for each* $y = f(x_1,\ldots,x_N)$ *we have*

$$p_Y = \hat{F}(p_{X_1}, p_{X_2}, \ldots, p_{X_N}) = \sum_{i_1=0}^{1} \cdots \sum_{i_N=0}^{1} \left(\alpha_{i_1 \ldots i_N} \prod_{k=1}^{N} p_{X_k}^{i_k} \right). \tag{4.2}$$

If we preassign some variables of the polynomial $\hat{F}(a_1,\ldots,a_N)$ given in (4.1) as constant values in $[0,1]$ and the rest of variables is taken equal to one variable t, then $\hat{F}$ becomes a polynomial in one variable and real coefficients $g(t)$. Thus, different boolean functions f and preassigned variables will give rise to different polynomials $g(t)$. For particular combinational circuits whose stochastic logic yields a multivariate polynomial as in (4.1) and for their corresponding associated polynomials $g(t)$, the authors of [28] proposed representations of $g(t)$ in terms of certain families of Bernstein forms.

As mentioned in [32], the remarkable papers [15, 16] suggest that the fundamental properties and identities satisfied by the degenerate Bernstein basis functions could be used to define a special model of stochastic logic. In this section we explore such a model.

Given $n \in \mathbb{N}$ and (x, p_x) a stochastic number with $x \in \{0,1\}^n$. For each $k = 1,2,\ldots,n$ we choose $p_k \in [0,1]$ and consider discrete and independent random variables X_k having Bernoulli distribution with parameter p_k, i.e., $X_k \sim Be(p_k)$ (cf. [31] and the references therein). Since $x_k \in \{0,1\}$, each probability density function is given by $P\{X_k = x_k\} = p_k^{x_k}(1-p_k)^{1-x_k}$.

We define

$$p_{X_k} := P\{X_k = 1\} = p_k \quad \text{and} \quad 1 - p_{X_k} := P\{X_k = 0\} = 1 - p_k, \quad k = 1,2,\ldots,n.$$

Suppose that we have a combinational circuit $y = f(x_1, x_2, \ldots, x_n)$ consisting of a decoding block[1] and a multiplexing block[2], which transform the n inputs $\{x_1,\ldots,x_n\} \in \{0,1\}$ as follows: If k out of the inputs $\{x_1,\ldots,x_n\}$ of the decoding block are logical 1, then s_k is set to 1 and the other outputs are set to 0, $0 \le k \le n$. So, the output of the decoding block is $(s_0,\ldots,s_n)$. The outputs of the decoding block are fed into the multiplexing block, as shown in Figure 2 and act as the selecting signals (control

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[1] In general, a decoding block is a combinatorial circuit which has n inputs and m outputs, with $m \le 2^n$. A typical application of the decoding blocks is to generate keyboard codes for introducing data into the computer from a keyboard [33].

[2] A multiplexing block is a combinatorial circuit which has n inputs, m control inputs with $m \le n$, and a unique output. Essentially, multiplexing blocks generalized to a multiplexer circuit [33].

inputs). The data signals (inputs) of the multiplexing block consist of $n+1$ inputs $z_0, \dots, z_n \in \{0,1\}$.

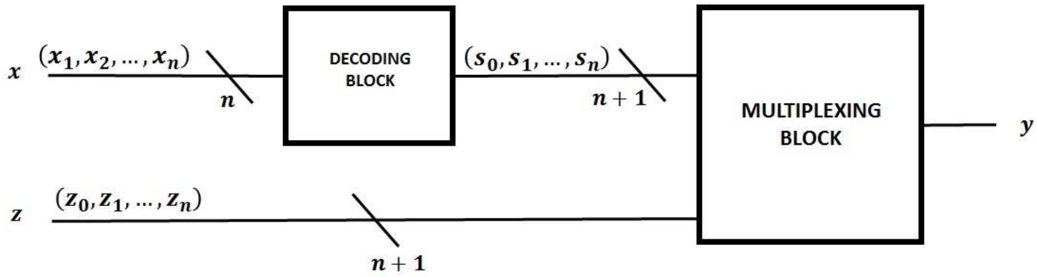

FIGURE 2. Combinational circuit associated to a degenerate Bernstein form with coefficients in $[0,1]$ (cf. [28,32,33]).

Since the circuit contains a multiplexing block, once it decodes the inputs $x_1, \dots, x_n$ then the boolean function $y = f(x_1, \dots, x_n)$ takes the form

$$y = \bigvee_{k=0}^{n} (z_k \wedge s_k), \tag{4.3}$$

which means that the output of the multiplexing block y is set to be the input z_k if $s_k = 1$.

Since $x_k \in \{0,1\}$, for $\lambda \le 0$ fixed each degenerate probability density could be given in terms of

$$P_\lambda\{X_k = x_k\} := (p_k)_{x_k,\lambda}(1-p_k)_{1-x_k,\lambda} = p_k^{x_k}(1-p_k)^{1-x_k}. \tag{4.4}$$

Using the association (4.4) for $(x_1, \dots, x_n)$, $(s_0, \dots, s_n)$ and $(z_1, \dots, z_n)$ we can choose discrete and independent random variables $(X_1, \dots, X_n)$, $(S_0, \dots, S_n)$ and $(Z_0, \dots, Z_n)$, such that $X_k \sim Be(p_k)$, $k = 1, \dots, n$, $S_j \sim Be(\hat{p}_j)$ and $Z_j \sim Be(\hat{\hat{p}}_j)$, $j = 0, \dots, n$. Similarly, we define

$$\begin{aligned} p_{X_k} &:= P_\lambda\{X_k = 1\} = p_k, \quad 1 - p_{X_k} := P_\lambda\{X_k = 0\} = 1 - p_k, \quad k = 1, 2, \dots, n, \\ p_{S_j} &:= P_\lambda\{S_j = 1\} = \hat{p}_j, \quad 1 - p_{S_j} := P_\lambda\{S_j = 0\} = 1 - \hat{p}_j, \\ p_{Z_j} &:= P_\lambda\{Z_j = 1\} = \hat{\hat{p}}_j, \quad 1 - p_{Z_j} := P_\lambda\{Z_j = 0\} = 1 - \hat{\hat{p}}_j, \quad j = 0, 1, \dots, n. \end{aligned}$$

Applying Theorem 4.1 to the function $y = f(x_1, \dots, x_n)$, we have that the stochastic logic yields a multivariate polynomial as in (4.1), such that $p_Y = \hat{F}(p_{X_1}, \dots, p_{X_n})$.

Let $g(t)$ be the polynomial associated to $\hat{F}$ for $a_k = t$, $k = 1, \dots, n$. Assume that $p_{X_1} = \cdots = p_{X_n} = t_0$, since s_j is set to 1 if and only if j out of n inputs of the decoding block are 1, the degnerate probability that S_j is 1 is

$$p_{S_j} = P_\lambda(S_j = 1) = \frac{1}{(1)_{n,\lambda}} \binom{n}{j} (t_0)_{\lambda,j} (1 - t_0)_{\lambda,n-j} = \hat{b}^n_{j,\lambda}(t_0), \quad j = 0, \dots, n, \ \lambda \le 0.$$

Now, assume that $p_{Z_j} = c_j^n$, $j = 0, \dots, n$. Then,

$$p_Y = P_\lambda\{Y = 1\} = \sum_{k=0}^{n} P_\lambda\{Y = 1|S_k = 1\}P_\lambda\{S_k = 1\}, \tag{4.5}$$

but from (4.3) is deduced that $S_j = 1$ implies $Y = Z_j$, so

$$P_\lambda\{Y = 1|S_j = 1\} = P_\lambda\{Z_j = 1\} = p_{Z_j} = c_j^n. \tag{4.6}$$

By (3.14), (4.2), (4.5) and (4.6) we obtain

$$g(t_0) = p_Y = P_\lambda\{Y = 1\} = \sum_{k=0}^{n} c_k^n \hat{b}_{k,\lambda}^n(t_0).$$

Hence, under the constrains imposed by us, the combinational circuit associated to the function $y = f(x_1, \dots, x_n)$ would require that $g(t)$ be a degenerate Bernstein form whose coefficients c_k^n belong to $[0, 1]$.

The next result summarizes the ideas above and provides an extension of [28, Theorem 2].

Theorem 4.2. *Let* $p(x) = \sum_{k=0}^{n} c_k^n \hat{b}_{k,\lambda}^n(x)$ *be any normalized degenerate Bernstein form with* $\lambda \leq 0$. *If the coefficients* $c_k^n \in [0, 1]$, $k = 0, \dots, n$, *then we can design a stochastic logic to compute the normalized degenerate Bernstein form. That is, there exists a multivariate polynomial* $\hat{F}$ *satisfying* (4.2) *such that its associated polynomial is* $p(x)$.

5. CONCLUSIONS

In this work, we analyzed the degenerate Bernstein basis functions (in the sense of T. Kim and D. S. Kim [15, 16]) and their associated forms, establishing several algebraic and analytic properties in parallel with their classical counterparts. We derived explicit formulas for degree elevation and reduction, bounds, and endpoint values, as well as comparative results on condition numbers that highlight the relative stability of different polynomial bases under perturbations. Furthermore, we showed how degenerate Bernstein forms can be naturally implemented within stochastic logic models of the Qian-Riedel-Rosenberg type. These results not only extend the theoretical framework of Bernstein-type polynomials but also reinforce their potential for applications in numerical analysis, computer-aided design, and stochastic computation.

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