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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR SIXTH-ORDER ORDINARY NONLINEAR DIFFERENTIAL PROBLEMS

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Abstract. This manuscript is devoted to the investigation of existence and multiplicity criteria for non-zero weak solutions to a class of sixth-order ordinary differential equations that emerge in the context of spatial pattern formation, with particular relevance to thin elastic structures, phase-transition phenomena, and materials science. Adopting a variational framework, we scrutinize the associated energy functional and invoke critical point theory to prove the existence of at least one, two, and, under suitable conditions, three distinct weak solutions. The results are derived under fairly general hypotheses that cover a broad spectrum of nonlinearities and boundary effects pertinent to physical applications.

1. Introduction

The present article addresses the following nonlinear sixth-order boundary value problem, which depends on a positive real parameter λ:

$$\begin{cases} -u^{(vi)} + Au^{(iv)} - Bu'' + Cu = \lambda f(x,u), \quad x \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases} \tag{1.1}$$

where A, B and C are given real constants and $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, i.e.,

(f_1) $x \mapsto f(x,s)$ is measurable for every $s \in \mathbb{R}$,
(f_2) $s \mapsto f(x,s)$ is continuous for almost every $x \in [0,1]$,
(f_3) for all $\rho > 0$ the function $x \mapsto \sup_{|s|\le\rho} |f(x,s)|$ belongs to $L^1(0,1)$.

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The interest in problem (1.1) is largely motivated by the study of stationary solutions to sixth-order parabolic differential equations of the form

$$\frac{\partial u}{\partial t} = -\frac{\partial^6 u}{\partial x^6} + A\frac{\partial^4 u}{\partial x^4} - B\frac{\partial^2 u}{\partial x^2} + Cu - \lambda h(t, x, u(t, x)).$$

Such equations become particularly relevant when examining interfaces between two phases, as they allow for a more refined description of interface structures and capture the behavior of phase fronts during liquid–solid transitions [1, 9, 17].

Over recent years, the existence and multiplicity of solutions for sixth-order equations have been extensively investigated through variational techniques. For instance, in [8, 19], the authors employed a minimization theorem and Clark's theorem to discuss periodic solutions for semilinear fourth- and sixth-order equations arising in phase-transition models. In a related direction, [12] utilized two Brezis–Nirenberg linking theorems to assure the existence of at least two or three periodic solutions under super-quadratic growth conditions, namely

$$\lim_{t\to+\infty} \frac{F(x,t)}{t^2} = +\infty, \quad \lim_{t\to 0} \frac{F(x,t)}{t^2} = 0,$$

uniformly in x, with $F(x,t) = \int_0^t f(x,s)\,ds$. Further contributions include the works [15, 21], where the theory of fixed point index in cones was applied to establish the existence of one or two positive solutions for (1.1) under appropriate assumptions on the coefficients A, B, C.

A distinct line of research has focused on infinitely many solutions. In [6], an unbounded sequence of classical solutions was obtained for (1.1) by imposing an oscillatory behavior of $f(x,\cdot)$ at infinity and requiring λ to lie in a well-specified interval (see [6, Theorem 3.2]). More recently, in [4], the authors applied a non-zero local minimum theorem to establish the existence of a nontrivial solution under growth conditions that are only required on a possibly bounded set; in the autonomous case, they showed that sublinearity at zero suffices. The multiplicity issue was further addressed in [16], where three solutions (including three nonnegative ones) were obtained by imposing suitable sign conditions and algebraic inequalities on the nonlinearity. Additionally, [18] employed critical point theorems to derive general existence results for weak solutions.

Within this broad context, the goal of the present paper is threefold. Firstly, we provide a concrete estimate of the positive interval of λ for which (1.1) admits at least one nontrivial weak solution, assuming that $f(x,t)$ has a uniform sublinear growth on a bounded interval $[0,1]$ that may be away from zero or infinity. Secondly, we establish the existence of two distinct solutions along with uniform parameter estimates, by invoking [3, Theorem 3.2]. In contrast to our first main result, which requires the $(P.S.)^{[r]}$ condition, this second theorem relies on the (AR)-condition to guarantee the usual (PS)-condition, a strategy successfully employed in [5, 10, 11, 14]. Finally, and most notably, we demonstrate the existence of three solutions for (1.1) a topic that

has received comparatively less attention under subcritical growth of $f(x,\cdot)$. The key tool here is the critical point theorem stated in [7, Theorem 3.6].

It is worth emphasizing that our variational framework is built upon that developed in [4], where problem (1.1) was considered on a general interval $[a,b]$ and the existence of at least one nontrivial solution was proved without imposing assumption (E_1) (see Theorem 3.1).

The remainder of the paper is structured as follows. Section 2 recalls some preliminary results. Section 3 presents our main findings: under suitable conditions on f, we establish the existence of at least one, two, and three distinct weak solutions for (1.1), with the parameter λ belonging to a positive interval in each case.

2. Preliminaries

Consider the Sobolev subspace X of $H^3(0,1)\cap H_0^1(0,1)$ defined as

$$X=\{u\in H^3(0,1)\cap H_0^1(0,1) : u''(0)=u''(1)=0\},$$

endowed with the norm

$$\|u\|=\left(\|u'''\|_2^2+\|u''\|_2^2+\|u'\|_2^2+\|u\|_2^2\right)^{\frac{1}{2}}, \quad \text{for all } u\in X, \tag{2.1}$$

where $\|\cdot\|_2$ stands for the standard $L^2(0,1)$-norm. The norm $\|\cdot\|$ is naturally induced by the inner product

$$\langle u,v\rangle=\int_0^1 (u'''(x)v'''(x)+u''(x)v''(x)+u'(x)v'(x)+u(x)v(x))dx, \quad \text{for all } u,v\in X.$$

Clearly $(X,\|\cdot\|)\hookrightarrow(C^0(0,1),\|\cdot\|_\infty)$ and this embedding is compact. We next recall some useful Poincaré-type inequalities.

Proposition 2.1. ([6, Proposition 2.1]). *For every $u\in X$, setting $k=\frac{1}{\pi^2}$, one has*

$$\|u^i\|_2^2\le k^{j-i}\|u^j\|_2^2, \quad i=0,1,2, j=1,2,3, \ \textit{with } i<j. \tag{2.2}$$

Remark 2.1. The constants in (2.2) are sharp; this can be verified by testing with the function $\sin(px)$, for which equalities are attained. In this connection, we recall the general inequality

$$\|v\|_2^2\le 4k\|v'\|_2^2, \tag{2.3}$$

which holds for every $v\in H^1([0,1])$ admitting some $c\in[0,1]$ such that $v(c)=0$, with equality for suitable functions v (see, for instance, [13, page 182]). Applying (2.3) to $v=u'$ yields $\|u'\|_2^2\le 4k\|u''\|_2^2$, which, as indicated by (2.2), does not provide the best constant, in contrast to (2.3). This discrepancy is due to the enhanced regularity of u' arising from the fact that $u\in X$.

For later purposes, we introduce an equivalent norm on X that preserves the Hilbert space structure. To this end, given $A,B,C\in\mathbb{R}$, define the functional $N:X\to\mathbb{R}$ by

$$N(u)=\|u'''\|_2^2+A\|u''\|_2^2+B\|u'\|_2^2+C\|u\|_2^2,$$

for every $u\in X$.

Proposition 2.2. ([6, Proposition 2.3]). *Let $k = \frac{1}{\pi^2}$. The condition*

(H) $\max\{-Ak, -Ak - Bk^2, -Ak - Bk^2 - Ck^3\} < 1$,

holds if and only if one of the following alternatives is satisfied:

$(H)_1$ $A \geq 0$, $B \geq 0$, $C \geq 0$,
$(H)_2$ $A \geq 0$, $B \geq 0$, $C < 0$ *and* $-Ak - Bk^2 - Ck^3 < 1$,
$(H)_3$ $A \geq 0$, $B < 0$, $C \geq 0$ *and* $-Ak - Bk^2 < 1$,
$(H)_4$ $A \geq 0$, $B < 0$, $C < 0$ *and* $-Ak - Bk^2 - Ck^3 < 1$,
$(H)_5$ $A < 0$, $B \geq 0$, $C \geq 0$ *and* $-Ak < 1$,
$(H)_6$ $A < 0$, $B \geq 0$, $C < 0$ *and* $\max\{-Ak, -Ak - Bk^2 - Ck^3\} < 1$,
$(H)_7$ $A < 0$, $B < 0$, $C \geq 0$ *and* $-Ak - Bk^2 < 1$,
$(H)_8$ $A < 0$, $B < 0$, $C < 0$ *and* $-Ak - Bk^2 - Ck^3 < 1$.

Moreover, with the notation

$$\delta = \begin{cases} 1, & \text{if } (H)_1 \text{ holds,} \\ \min\{1, 1 + Ak + Bk^2 + Ck^3\}, & \text{if } (H)_2 \text{ or } (H)_4 \text{ holds,} \\ \min\{1, 1 + Ak + Bk^2\}, & \text{if } (H)_3 \text{ holds,} \\ 1 + Ak, & \text{if } (H)_5 \text{ holds,} \\ \min\{1 + Ak, 1 + Ak + Bk^2 + Ck^3\}, & \text{if } (H)_6 \text{ holds,} \\ 1 + Ak + Bk^2, & \text{if } (H)_7 \text{ holds,} \\ 1 + Ak + Bk^2 + Ck^3, & \text{if } (H)_8 \text{ holds,} \end{cases} \tag{2.4}$$

the following propositions hold.

Proposition 2.3. ([6, Proposition 2.4]). *Under assumption (H), there exists a constant $m > 0$ such that*

$$N(u) \geq m\|u\|^2, \quad \text{for all } u \in X. \tag{2.5}$$

Proposition 2.4. ([6, Proposition 2.5]). *Assume (H) and define*

$$\|u\|_X = \sqrt{N(u)}, \quad \text{for all } u \in X. \tag{2.6}$$

Then, $\|\cdot\|_X$ is a norm equivalent to $\|\cdot\|$ from (2.1)*, and $(X, \|\cdot\|_X)$ is a Hilbert space.*

Proposition 2.5. ([6, Proposition 2.6]). *If (H) is satisfied, then*

$$\|u\|_\infty \leq \frac{k}{2\sqrt{\delta}}\|u\|_X, \quad \text{for all } u \in X, \tag{2.7}$$

where δ is given in (2.4).

Definition 2.1. Let Φ and Ψ be two continuously Gâteaux differentiable functionals on a real Banach space X, and fix $r \in \mathbb{R}$. The functional $I = \Phi - \Psi$ is said to satisfy the Palais-Smale condition cut off above at r (denoted by $(P.S.)^{[r]}$) if every sequence $\{u_n\}_{n\in\mathbb{N}}$ in X fulfilling

(α_1) $\{I(u_n)\}_{n\in\mathbb{N}}$ is bounded,
(α_2) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$,
(α_3) $\Phi(u_n) < r$ for each $n \in \mathbb{N}$,

has a convergent subsequence.

The subsequent theorems constitute the primary analytical tools for establishing our main results in the next section.

Theorem 2.1. ([3, Theorem 2.3], [2, Theorem 5.1]). *Let X be a real Banach space, $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\overline{v} \in X$, with $0 < \Phi(\overline{v}) < r$ such that*

(i_1) $\frac{\sup_{\Phi(u)\leq r} \Psi(u)}{r} < \frac{\Psi(\overline{v})}{\Phi(\overline{v})}$,

(i_2) for all

$$\lambda \in \Lambda := \left(\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{\Phi(u)\leq r} \Psi(u)} \right)$$

the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the $(P.S.)^{[r]}$ condition.

Then, for each $\lambda \in \Lambda$ there is $u_{0,\lambda} \in \Phi^{-1}(0, r)$ such that $I'_\lambda(u_{0,\lambda}) = \vartheta_{X^*}$ and $I_\lambda(u_{0,\lambda}) < I_\lambda(u)$ for all $u \in \Phi^{-1}(0, r)$.

Theorem 2.2. ([3, Theorem 3.2]). *Let X be a real Banach space, $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ and assume that, for each*

$$\lambda \in \left(0, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)} \right)$$

the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the $(P.S.)$ condition and is unbounded from below. Then, for each

$$\lambda \in \left(0, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)} \right)$$

the functional I_λ admits two distinct critical points.

Theorem 2.3. ([7, Theorem 3.6]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive and continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\overline{v} \in X$, with $r < \Phi(\overline{v})$ such that

(j_1) $\frac{\sup_{\Phi(u)\leq r} \Psi(u)}{r} < \frac{\Psi(\overline{v})}{\Phi(\overline{v})}$,

(j_2) for all $\lambda \in \Lambda_r := \left(\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{\Phi(u)\leq r} \Psi(u)} \right)$ the functional $\Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

To make the variational structure of the problem (1.1) explicit, we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ by setting

$$\Phi(u) := \frac{1}{2}\|u\|_X^2, \quad \Psi(u) := \int_0^1 F(x, u(x))dx, \quad \text{for all } u \in X, \tag{2.8}$$

where $F(x,t) = \int_0^t f(x,s)ds$ for $(x,t) \in \Omega \times \mathbb{R}$.

A standard computation confirms that Φ and Ψ are continuously Gâteaux differentiable, with derivatives given by

$$\Phi'(u)(v) = \int_0^1 \Big(u'''(x)v'''(x) + Au''(x)v''(x) + Bu'(x)v'(x) + Cu(x)v(x)\Big)dx$$

and

$$\Psi'(u)(v) = \int_0^1 f(x, u(x))v(x)dx,$$

for every $u, v \in X$.

Definition 2.2. A function $u \in X$ is termed a weak solution of problem (1.1) if it satisfies

$$\begin{aligned}&\int_0^1 \Big(u'''(x)v'''(x) + Au''(x)v''(x) + Bu'(x)v'(x) + Cu(x)v(x)\Big)dx\\ =&\lambda \int_0^1 f(x, u(x))v(x)dx, \quad \text{for all } v \in X.\end{aligned}$$

Consequently, the weak solutions of (1.1) are precisely the critical points of the functional $\Phi - \lambda\Psi$.

3. Main Results

The first result of this section establishes the existence of at least one weak solution for problem (1.1). For this purpose, we define the following auxiliary constant, which will play a key role in our subsequent analysis:

$$\mu := 4\delta\pi^4 \left(90\left(\frac{12}{5}\right)^5 + 4\left(\frac{12}{5}\right)^4 A + \frac{1248}{175}B + \frac{493}{756}C\right)^{-1}. \tag{3.1}$$

Theorem 3.1. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathèodory function. Condition (H) is assumed throughout. In addition, we postulate the existence of two positive parameters, τ and η, satisfying the following condition:*

(E_1) $\eta < \sqrt{\mu}\tau$,

(E_2) $\frac{\int_0^1 \max_{|s|\le\tau} F(x,s)dx}{\tau^2} < \mu \frac{\int_{\frac{5}{12}}^{\frac{7}{12}} F(x,\eta)dx}{\eta^2}$,

(E_3) $F(x,t) \ge 0$ *for each* $(x,t) \in ([0, \frac{5}{12}] \cup [\frac{7}{12}, 1]) \times \mathbb{R}$.

Then, for each

$$\lambda \in \Lambda_w := 2\delta\pi^4 \left(\frac{\eta^2}{\mu \int_{\frac{5}{12}}^{\frac{7}{12}} F(x,\eta)dx}, \frac{\tau^2}{\int_0^1 \max_{|s|\le\tau} F(x,s)dx}\right), \tag{3.2}$$

the problem (1.1) *admits at least one nontrivial solution* $u_\lambda \in X$ *such that* $\|u_\lambda\|_\infty \leq \tau$ *and* $\|u_\lambda\|_X < 4\pi^2\tau$.

Proof. The principal objective of this segment is to implement Theorem 2.1 in the context of problem (1.1). To this end, we use the space X defined in Section 2, endowed with the norm $\|\cdot\|_X$ from (2.6), and we choose the functionals Φ and Ψ precisely as defined in (2.8). It was already established in the preceding section that Φ and Ψ belong to $C^1(X)$. A straightforward verification reveals that Φ is coercive and bounded from below, and that its derivative is a homeomorphism. In order to prove that Φ' possesses a continuous inverse, we rely on [20, Theorem 26.A(d)]; according to this result, it is enough to demonstrate the coercivity, hemicontinuity, and uniform monotonicity of Φ'. Indeed, for an arbitrary $u \in X$, we have

$$\frac{\langle \Phi'(u), u\rangle}{\|u\|_X} = \|u\|_X,$$

i.e., Φ' is coercive. We show that Φ' is uniformly monotone. Thus, for every $u, v \in X$, we deduce that

$$\langle \Phi'(u) - \Phi'(v), u - v\rangle \geq c_1\|u - v\|_X^2,$$

for some positive constant c_1, from which the uniform monotonicity of Φ' follows directly. Furthermore, the compactness inherent in the embedding $X \hookrightarrow C^0(0,1)$ guarantees that Ψ' is compact as an operator. Consequently, for any $\lambda > 0$ (in fact, for every $\lambda \in \mathbb{R}$), the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$ condition for arbitrary $r > 0$ (see [2, Proposition 2.1]). We proceed by applying Theorem 2.1 to the aforementioned functional I_λ. Notice that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$ is already satisfied. Accordingly, it remains solely to verify condition (i_1). For this purpose, we set $r := 2\delta\pi^4\tau^2$, and select $w(x)$ in the following manner:

$$w(x) = \begin{cases} \eta v(x), & \text{if } x \in [0, \frac{5}{12}), \\ \eta, & \text{if } x \in [\frac{5}{12}, \frac{7}{12}], \\ \eta v(1-x), & \text{if } x \in (\frac{7}{12}, 1], \end{cases} \tag{3.3}$$

where $v(x) = v(x) = \frac{3}{8}\left(\frac{12}{5}\right)^5 x^5 - \frac{5}{4}\left(\frac{12}{5}\right)^3 x^3 + \frac{9}{2}x$, a direct computation shows that $w \in X$ and, in particular

$$\Phi(w) = \frac{2\delta\pi^4}{\mu}\eta^2.$$

Hence, it follows from (E_1) that $0 < \Phi(w) < r$. Let us begin by observing that, in view of (2.2) one has

$$\begin{aligned} \{v \in X : \Phi(v) \leq r\} &\subset \left\{v \in C^0(0,1) : \|v\|_\infty \leq \frac{k}{\sqrt{\delta}}\sqrt{\frac{r}{2}}\right\} \\ &= \{v \in C^0(0,1) : \|v\|_\infty \leq \tau\}. \end{aligned}$$

Hence, the condition (E_2), implies that, for each $u \in \Phi^{-1}(-\infty, r]$

$$\begin{aligned} \frac{\sup_{u\in\Phi^{-1}(-\infty,r]}\Psi(u)}{r} &\leq \frac{1}{2\delta\pi^4}\cdot\frac{\int_0^1 \max_{|s|\leq\tau} F(x,s)dx}{\tau^2} \\ &\leq \frac{\mu}{2\delta\pi^4}\cdot\frac{\int_{\frac{5}{12}}^{\frac{7}{12}} F(x,\eta)dx}{\eta^2} \leq \frac{\Psi(w)}{\Phi(w)}, \end{aligned} \tag{3.4}$$

which means that, the condition (i_1) of Theorem 2.1 is verified. Hence, applying Theorem 2.1, for each $\lambda \in \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(u)\leq r}\Psi(u)}\right[$ the functional I_λ admits at least one critical point u_λ such that $0 < \Phi(u_\lambda) < r$, which is a nontrivial solution of the problem (1.1). □

The theoretical findings of Theorem 3.1 are substantiated through a concrete illustrative example presented below.

Example 3.1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as follows:

$$f(t) = 2te^{-2|t|}(1 - |t|).$$

Owing to Theorem 3.1, for each $\lambda \in (240\pi^4e^4, 18\pi^4e^2 \times 10^2)$ the problem

$$\begin{cases} -u^{(vi)} + u^{(iv)} = \lambda f(u), \quad x \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases}$$

admits at least one non-zero solution u_λ such that $\|u_\lambda\|_\infty < 30$ and $\|u_\lambda\|_X < 120\pi^2$. Indeed, in this case, we have $A = 1, B = C = 0$. For which a simple computation shows that $\mu \simeq 5 \times 10^{-2}$ and from the definition of f,

$$f(t) = 2te^{-2|t|}(1 - |t|),$$

so that by picking $\eta = 2$ and $\tau = 30$ the assumptions (E_1) and (E_2) are verified.

The second principal outcome of this section concerns the existence of two distinct weak solutions for problem (1.1). This result is derived as a direct application of Theorem 2.2.

Theorem 3.2. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function satisfying condition (H). Furthermore, assume that*

(E_4) *there exist $\nu > 2$ and $R > 0$ such that*

$$0 < \nu F(x,t) < tf(x,t),$$

for all $x \in \Omega$ and $|t| \geq R$.

Assume also that $f(\cdot, 0) \not\equiv 0$ almost everywhere in $(0,1)$. Then, for any positive constant τ, and for every

$$\lambda \in \Lambda_r := \left(0, \frac{2\delta\pi^4\tau^2}{\int_0^1 \max_{|s|\leq\tau} F(x,s)dx}\right),$$

the problem (1.1) admits at least two non-zero weak solutions.

Proof. Consider the functionals Φ and Ψ as previously defined in Theorem 3.1; they satisfy all regularity hypotheses required by Theorem 2.2. Following the same line of reasoning as in the proof of Theorem 3.1, and choosing $r = 2\delta\pi^4\tau^2$, for each $\lambda \in \Lambda_r$ we obtain

$$\frac{\sup_{u\in\Phi^{-1}(-\infty,r]} \Psi(u)}{r} \leq \frac{1}{2\delta\pi^4} \frac{\int_0^1 \max_{|s|\leq\tau} F(x,s)dx}{\tau^2} < \frac{1}{\lambda}.$$

Next, by (E_4), a standard argument yields the existence of a positive constant m such that

$$F(x,t) \geq m|t|^\nu, \quad \text{for all } x \in \Omega.$$

Consequently, for every $\lambda \in \Lambda_r$, any $u \in X \setminus \{0\}$, and $t > 1$, we have

$$\begin{aligned} I_\lambda(tu(x)) &= \Phi(tu(x)) - \lambda \int_0^1 F(x,tu)dx \\ &\leq \frac{1}{2}t^2\|u\|_X^2 - m\lambda t^\nu \int_0^1 |u|^\nu dx. \end{aligned}$$

Since $\nu > 2$, it follows that $I_\lambda(tu) \to -\infty$ as $t \to +\infty$. Hence, I_λ is unbounded from below.

It remains to verify that I_λ satisfies the (PS)-condition. To this end, let $\{u_n\}_{n\in\mathbb{N}} \subset X$ be a sequence such that $\{I_\lambda(u_n)\}_{n\in\mathbb{N}}$ is bounded and $I'_\lambda(u_n) \to 0$ in X^* as $n \to +\infty$. Then there exists a positive constant s_0 satisfying

$$|I_\lambda(u_n)| \leq s_0, \quad \|I'_\lambda(u_n)\| \leq s_0, \quad \text{for all } n \in \mathbb{N}.$$

Employing condition (E_4) together with the definition of I'_λ, we deduce that, for all $n \in \mathbb{N}$,

$$\begin{aligned} \nu s_0 + s_0\|u_n\|_X &\geq \nu I_\lambda(u_n) - I'_\lambda(u_n)u_n \\ &\geq \frac{\nu}{2}\|u_n\|_X^2 - \|u_n\|_X^2 + \lambda \int_0^1 (f(x,u_n)u_n - \nu F(x,u_n))dx \\ &\geq \left(\frac{\nu}{2} - 1\right)\|u_n\|_X^2 - \lambda(\nu+1)R\|M_R\|_{L^1(0,1)}. \end{aligned}$$

Since $\nu > 2$, the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ follows immediately. Consequently, by the reflexivity of X, there exists a subsequence (still denoted by $\{u_n\}_{n\in\mathbb{N}}$) such that

$$u_n \rightharpoonup u, \quad \text{in } X.$$

The facts that $I'_\lambda(u_n) \to 0$ and $u_n \rightharpoonup u$ in X imply

$$(I'_\lambda(u_n) - I'_\lambda(u))(u_n - u) \to 0.$$

By the compact embedding $X \hookrightarrow C[0,1]$ and (f_3), passing to a subsequence if necessary, we also have

$$\int_0^1 (f(x,u_n) - f(x,u))(u_n - u)dx \to 0, \quad \text{as } n \to +\infty.$$

Furthermore, since Ψ' is compact and $u_n \rightharpoonup u$ in X,

$$\begin{aligned}\|u_n - u\|_X^2 =& \langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle \\ &+ \lambda\langle \Psi'(u_n) - \Psi'(u), u_n - u\rangle \to 0.\end{aligned}$$

Consequently, $u_n \to u$ strongly in X. Thus, I_λ fulfills the (PS)-condition, and consequently all assumptions of Theorem 2.2 are satisfied. Applying Theorem 2.2, for each $\lambda \in \Lambda_r$, the functional I_λ possesses at least two distinct critical points, which correspond precisely to the desired weak solutions of the problem (1.1). □

To illustrate the applicability of Theorem 3.2, we next present a concrete example.

Example 3.2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as $f(t) = 1 + 3t|t|$. Owing to Theorem 3.2, for each $\lambda \in (0, \frac{\pi^4}{5})$ the problem

$$\begin{cases} -u^{(vi)} - u'' = \lambda f(u), \quad x \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases}$$

admits at least two nontrivial solutions. Indeed, in this case, we have $A = C = 0$, $B = 1$ and from the definition of f,

$$F(t) = t + |t|^3.$$

Hence, by picking $\nu = \frac{5}{2}$, $R = 2$ and $\tau = 1$ the assumption (E_4) is verified.

The final part of this section is devoted to establishing the existence of at least three weak solutions for the problem (1.1).

Theorem 3.3. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Assume that condition (H) holds and suppose that there exist two positive constants τ and η, such that*

(E_5) $\eta > \sqrt{\mu}\tau$,

(E_6) $\limsup_{|t|\to+\infty} \frac{F(x,t)}{t^2} \leq 0$ *uniformly for almost every* $x \in (0,1)$,

and let (E_2) and (E_3) from Theorem 3.1 *hold. Then, for every $\lambda \in \Lambda_w$ as* (3.2) *in Theorem* 3.1*, the problem* (1.1) *admits at least three distinct weak solutions.*

Proof. The proof strategy consists of applying Theorem 2.3 to the functionals Φ and Ψ introduced earlier. These functionals satisfy all the regularity hypotheses required by Theorem 2.3.

Following the same argumentation as in the proof of Theorem 3.1, we define $w(x)$ by means of (3.3) and set $r = 2\delta\pi^4\tau^2$. In view of (E_5), we obtain $\Phi(w) > r > 0$. Consequently, by virtue of (E_2), inequality (3.4) holds, thereby fulfilling condition (j_1) of Theorem 2.3.

We next establish the coercivity of I_λ for every $\lambda \in \Lambda_w$. Condition (E_6) implies that, for any fixed $0 < \varepsilon < \min\left\{\frac{1}{2\lambda}, \frac{2\delta}{\lambda k^2}\right\}$, there exists $T > 0$ such that

$$F(x,t) \leq \varepsilon|t|^2, \quad \text{for all } x \in [0,1], |t| > T.$$

Moreover, since f is an L^1-Carathéodory function, condition (f_3) guarantees the existence of a function $w_1 \in L^1([0,1])$ satisfying

$$F(x,t) \leq w_1(x), \quad \text{for all } x \in [0,1], |t| \leq T.$$

For a fixed $x \in [0,1]$, we decompose $F(x,u(x)) = F_1(x,u(x)) + F_2(x,u(x))$, where F_1 corresponds to $|u(x)| > T$ and F_2 corresponds to $|u(x)| \leq T$. For every $u \in X$, we have

$$\begin{aligned} I_\lambda(u) &= \frac{\|u\|_X^2}{2} - \lambda \int_0^1 F_1(x,u)dx - \lambda \int_0^1 F_2(x,u)dx \\ &\geq \frac{\|u\|_X^2}{2} - \lambda\varepsilon\|u\|_2^2 - \lambda\|w_1\|_1 \\ &\geq \left(\frac{1}{2} - \frac{\lambda\varepsilon k^2}{4\delta}\right)\|u\|_X^2 - \lambda\|w_1\|_{L^1(0,1)}. \end{aligned}$$

Given that $0 < \varepsilon < \min\left\{\frac{1}{2\lambda}, \frac{2\delta}{\lambda k^2}\right\}$, it follows that $I_\lambda \to +\infty$ as $\|u\|_X \to +\infty$. This proves coercivity, thereby establishing condition (j_2). Therefore, for every $\lambda \in \Lambda_w$, Theorem 2.3 ensures that I_λ possesses at least three critical points in X, which directly correspond to the weak solutions of the problem (1.1). □

We conclude the paper with the following illustrative example in support of Theorem 3.3.

Example 3.3. Let $f(x,t) = \alpha\, a(x)\, te^{-t^2}$ for all $(x,t) \in [0,1] \times \mathbb{R}$ where $\alpha > 0$ is a constant and $a \in L^1([0,1])$ is given by

$$a(x) = \begin{cases} 1, & x \in \left[\frac{5}{12}, \frac{7}{12}\right], \\ 0, & \text{otherwise.} \end{cases}$$

We observe that

$$F(x,t) = \frac{\alpha}{2} a(x)\left[1 - (1+t^2)e^{-t^2}\right]$$

and $\limsup_{|t|\to+\infty} \frac{F(x,t)}{t^2} = 0$ for all $x \in [0,1]$, and thus condition (E_6) is satisfied. Furthermore, since $F(x,t) \geq 0$ for all $(x,t) \in [0,1] \times \mathbb{R}$, condition (E_3) holds. We now prove that the condition (E_2) is fulfilled. Since $F(x,t)$ is increasing with respect to $|t|$, we have $\max_{|t|\leq\tau} F(x,t) = \frac{\alpha}{2} a(x)\left[1 - (1+\tau^2)e^{-\tau^2}\right]$. Thus,

$$\frac{\displaystyle\int_0^1 \max_{|t|\leq\tau} F(x,t)\,dx}{\tau^2} = \frac{\alpha}{2}\frac{1-(1+\tau^2)e^{-\tau^2}}{\tau^2}\int_0^1 a(x)\,dx = \frac{\alpha}{2}\cdot\frac{1-e^{-\tau^2}}{\tau^2}\int_0^1 a(x)\,dx.$$

On the other hand, we see that

$$\frac{\int_{5/12}^{7/12} F(x,\eta)\,dx}{\eta^2} = \frac{\displaystyle\int_{5/12}^{7/12} F(x,\eta)\,dx}{\eta^2} = \frac{\alpha}{2}\frac{1-(1+\eta^2)e^{-\eta^2}}{\eta^2}\int_{5/12}^{7/12} a(x)\,dx.$$

We also have $\int_0^1 a(x)\,dx = \int_{5/12}^{7/12} a(x)\,dx = \frac{1}{6}$. Thus, (E_2) reduces to

$$\frac{1-(1+\tau^2)e^{-\tau^2}}{\tau^2} < \mu \frac{1-(1+\eta^2)e^{-\eta^2}}{\eta^2}.$$

Fix $\eta = 1$. Since

$$\lim_{\tau \to 0^+} \frac{1-(1+\tau^2)e^{-\tau^2}}{\tau^2} = 0,$$

conditions (E_2) and (E_5) hold for all sufficiently small $\tau > 0$. Hence, for every $\lambda \in \Lambda_w$, problem (1.1) admits at least three distinct weak solutions.

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