

GENERALIZED DONOHO-STARK UNCERTAINTY INEQUALITIES IN n -DIFFERENT LINEAR CANONICAL DUNKL DOMAINS

AHMED SAOUDI

ABSTRACT. This paper studies Donoho-Stark type uncertainty principles for the linear canonical Dunkl transform. We derive new concentration inequalities across three linear canonical Dunkl domains, linking time and band-limiting parameters with the measures of the associated measurable sets. These results extend and unify existing uncertainty principles for several classical and fractional transforms, including the Dunkl transform, fractional Dunkl transform, fractional Hankel transform, and the one-dimensional linear canonical transform. We further generalize the framework to n linear canonical Dunkl domains for $n \geq 3$, establishing a comprehensive multi-domain uncertainty principle.

1. INTRODUCTION

The classical uncertainty principle states that a nonzero function and its Fourier transform cannot both be highly localized at the same time. This fundamental result, arising from mathematics and physics, has far-reaching consequences, especially in signal processing. A significant generalization is the Donoho-Stark uncertainty principle, which recasts the Heisenberg inequality in terms of measurable sets. This approach has demonstrated strong effectiveness in applications, particularly for designing methods of signal recovery and reconstruction (see [7, 11]).

The linear canonical transform was initially developed by Collins [5] in the field of paraxial optics during the early 1970s and, independently, by Quesne et al. [22] in the context of quantum mechanics. Wolf later extended the framework to Hermitian operators in [29], investigating the behavior of information and uncertainty under

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linear phase-space transformations. Since the 1990s, the combined contributions of Collins and Quesne have been increasingly referenced in a growing body of research, evolving into a substantial field with applications in radar and optical system analysis, mathematical physics, pattern recognition, phase retrieval, filter design, and both analog and digital signal processing, among other areas.

In recent developments, the linear canonical transform has been generalized across multiple mathematical frameworks. Notable extensions include its adaptation to the Fourier-Bessel context, as explored in [1, 2], and the Dunkl framework addressed in [25]. Further expansions have been investigated within the Jacobi-Dunkl setting [15], as well as within Stockwell-based formulations [6, 17]. Additional contributions have extended the transform to quadratic-phase systems [26], quaternion algebras [27], and octonion structures, including both octonion linear canonical domains [16] and offset octonion settings [3]. Moreover, advancements in signal processing have led to an adaptive non-uniform partition algorithm utilizing the linear canonical transform for enhanced signal representation [31], among other innovative adaptations.

Our principal goal in this article is to investigate and analyze uncertainty principles pertaining to measurable sets associated with the linear canonical Dunkl transform $\mathcal{B}_\mu^A$, following the approach of Pan Lian [20] and Kou et al. [19]. This perspective unifies and extends several integral transforms previously introduced by Ghazouani et al. [14]. For $M \in SL(2, \mathbb{R})$ with $b \neq 0$, the linear canonical Dunkl transform of a function f belonging to $L_\nu^1(\mathbb{R})$ is defined as

$$\mathcal{D}_\nu^M f(t) = \begin{cases} \frac{\sigma_\nu}{(ib)^{\nu+1}} \int_{\mathbb{R}} E_\nu^M(t, w) f(w) |w|^{2\nu+1} dw, & b \neq 0, \\ \frac{e^{\frac{i}{2}at^2}}{|a|^{\nu+1}} f(t/a), & b = 0, \end{cases}$$

where $E_\nu^M(t, w)$ represent the linear canonical Dunkl kernel given by (2.2).

In this work, we establish new uncertainty principles of Donoho-Stark type in the framework of the linear canonical Dunkl transform $\mathcal{D}_\nu^M$. First, we derive concentration inequalities for a function and its transforms in three distinct linear canonical Dunkl domains, yielding quantitative bounds that generalize earlier results for the classical Dunkl, fractional Dunkl, and linear canonical transforms. Specifically, under appropriate time- and band-limiting conditions on measurable sets, we prove two main uncertainty inequalities (Theorems 3.2 and 3.5) that relate the concentration parameters κ_{E_j} and the measures of the sets involved. Notably, by specializing to the one-dimensional linear canonical transform (i.e., taking $\nu = -1/2$), we recover the corresponding uncertainty principles in three linear canonical domains established by Pan Lian [20, Theorem 3.2 and 3.5]. These results unify and extend existing uncertainty relations for special cases such as the fractional Hankel transform, the Dunkl-Fresnel transform, and the fractional Fourier transform. Furthermore, we generalize the analysis to n linear canonical Dunkl domains ($n \geq 3$) and obtain a comprehensive uncertainty principle (Theorem 4.1) that accounts for multiple transform domains simultaneously. Our approach leverages the algebraic properties of the transform,

notably the additivity property and makes use of projection operators associated with measurable sets. The results not only broaden the theoretical landscape of uncertainty principles in harmonic analysis but also provide a flexible framework applicable to signal processing, phase-space analysis, and other areas where generalized integral transforms are employed.

The remainder of this paper is organized as follows. In Section 2, we recall the definition and essential properties of the linear canonical Dunkl transform, including its inversion formula, Plancherel theorem, and additivity property. Section 3 is devoted to the study of uncertainty principles in three linear canonical Dunkl domains. We first present a Donoho-Stark type uncertainty inequality and then establish two main concentration inequalities, along with several corollaries for special cases such as the fractional Dunkl transform and the classical Dunkl transform. In Section 4, we extend the analysis to the general setting of n linear canonical Dunkl domains and prove a comprehensive uncertainty principle.

2. LINEAR CANONICAL DUNKL TRANSFORM

We will employ the following function spaces throughout this work.

- $\mathcal{C}_0(\mathbb{R})$: the space of continuous functions on $\mathbb{R}$ that vanish at infinity.
- $S(\mathbb{R})$: the Schwartz space of smooth functions on $\mathbb{R}$ that, together with all their derivatives, are rapidly decreasing.
- $L^p_\nu(\mathbb{R})$: the weighted L^p -space of measurable functions on $\mathbb{R}$ satisfying

$$\|f\|_{\nu,p} = \left(\int_0^{+\infty} |f(w)|^p |w|^{2\nu+1} dw \right)^{\frac{1}{p}} < +\infty, \quad \text{if } 1 \leq p < +\infty,$$

$$\|f\|_\infty = \operatorname{ess\,sup}_{w \in \mathbb{R}} |f(w)| < +\infty, \quad \text{if } p = +\infty,$$

equipped with the norm $\|\cdot\|_{p,\nu}$. Moreover, $\mathcal{L}^2_\nu$ denotes the Hilbert space endowed with the inner product

$$\langle f, g \rangle_\nu = \int_{\mathbb{R}} f(w) \overline{g(w)} |w|^{2\nu+1} dw.$$

Definition 2.1. Let $M \in SL(2, \mathbb{R})$ such that $b \neq 0$. The linear canonical Dunkl transform of $f \in \mathcal{L}^1_\mu$ is given by

$$(2.1) \quad \mathcal{D}^M_\nu f(t) = \begin{cases} \frac{\sigma_\nu}{(ib)^{\nu+1}} \int_{\mathbb{R}} E^M_\nu(t, w) f(w) |w|^{2\nu+1} dw, & b \neq 0, \\ \frac{e^{\frac{i}{2} \frac{a}{b} t^2}}{|a|^{\nu+1}} f(t/a), & b = 0, \end{cases}$$

where the kernel $E^M_\nu(t, w)$ is defined as (see [14, (3.30)])

$$(2.2) \quad E^M_\nu(t, w) = e^{\frac{i}{2} \left(\frac{d}{b} t^2 + \frac{a}{b} w^2 \right)} E_\nu(-it/b, w),$$

where $E_\nu(t, w)$ denotes the Dunkl kernel of type A_2 (see [24])

$$(2.3) \quad E_\nu(it, w) = j_\nu(tw) + \frac{itw}{2(\nu+1)} j_{\nu+1}(tw)$$

and the constant σ_ν is given by

$$\sigma_\nu = \frac{1}{2^{\nu+1}\Gamma(\nu+1)}.$$

In the above, j_ν is the normalized spherical Bessel function given by

$$(2.4) \quad j_\nu(t) := 2^\nu \Gamma(\nu+1) \frac{J_\nu(t)}{t^\nu} = \Gamma(\nu+1) \sum_{n=0}^{+\infty} \frac{(-1)^n (t/2)^{2n}}{n! \Gamma(n+\nu+1)},$$

and J_ν represents the classical Bessel function (see [28]).

The transform $\mathcal{D}_\nu^M$ generalizes numerous integral transforms through specific choices of the matrix M , the multiplicity parameter ν , and the function spaces involved. Examples include the following.

- For $M = (0, -1, 1, 0)$, $\mathcal{D}_\nu^M$ reduces to the Dunkl transform introduced by Dunkl [8–10]:

$$\mathcal{D}_\nu f(t) = \sigma_\nu \int_{\mathbb{R}} f(t) E_\nu(-it, w) |w|^{2\nu+1} dw, \quad \nu \geq -\frac{1}{2}.$$

- For $M = (\cos \theta, \sin \theta, -\sin \theta, \cos \theta)$, $\mathcal{D}_\nu^M$ becomes the fractional Dunkl transform (up to a constant unimodular factor $(e^{i\theta})^{\nu+1}$) defined by Ghazouani et al. [12, 13]

$$\mathcal{D}_\nu^\theta f(t) = \begin{cases} A_{\nu,\theta} \int_{\mathbb{R}} e^{\frac{i}{2}(t^2+w^2) \cot \alpha} E_\nu\left(\frac{-it}{\sin(\theta)}, w\right) f(w) |w|^{2\nu+1} dw, & \theta \in \frac{\mathbb{R}}{n\pi\mathbb{Z}}, \\ f(t), & \theta = 2n\pi, \\ f(-t), & \theta = (2n+1)\pi, \end{cases}$$

where $A_{\nu,\alpha} = \frac{e^{i(\nu+1)(\hat{\theta}\pi/2 - (\theta-2n\pi))}}{\Gamma(\nu+1)(2|\sin(\theta)|)^{\nu+1}}$, $n \in \mathbb{Z}$ and $\hat{\theta} = \text{sgn}(\sin \theta)$.

- For $M(\tau) = (1, \tau, 0, 1)$, $\mathcal{D}_\nu^M$ reduces to the one-dimensional Dunkl-type Fresnel transform defined by Rösler [23]

$$\mathcal{D}_\nu^\tau(f)(t) = \begin{cases} \frac{\sigma_\nu}{(i\tau)^{\nu+1}} \int_{\mathbb{R}} e^{\frac{i}{2\tau}(t^2+w^2)} E_\nu(-it/\tau, w) f(w) |w|^{2\nu+1} dw, & \tau \neq 0, \\ f(t), & \tau = 0. \end{cases}$$

- The even part of $\mathcal{D}_\nu^M$ reduces to the canonical Hankel transform defined by [30]

$$\mathcal{H}_\nu^M f(t) = \begin{cases} \frac{2\sigma_\nu}{(ib)^{\nu+1}} \int_0^{+\infty} e^{\frac{i}{2}(\frac{d}{b}t^2 + \frac{a}{b}w^2)} j_\nu\left(\frac{tw}{b}\right) f(w) w^{2\nu+1} dw, & b \neq 0, \\ \frac{e^{\frac{i}{2}\frac{a}{b}t^2}}{|a|^{\nu+1}} f(t/a), & b = 0. \end{cases}$$

- For $M = (\cos \theta, \sin \theta, -\sin \theta, \cos \theta)$, the even part of $\mathcal{D}_\nu^M$ yields the fractional Hankel-type transform, up to a constant phase factor $(e^{i\theta})^{\nu+1}$ [18]

$$\mathcal{H}_\nu^\theta f(t) = \begin{cases} c_\theta \int_0^{+\infty} e^{-\frac{i}{2} \cot(\theta)(t^2+w^2)} j_\nu\left(\frac{tw}{\sin(\theta)}\right) f(w) w^{2\nu+1} dw, & \theta \in \frac{\mathbb{R}}{n\pi\mathbb{Z}}, \\ f(t), & \theta = 2n\pi, \\ f(-t), & \alpha = (2n+1)\pi, \end{cases}$$

where $c_\theta = \frac{2e^{i(\nu+1)((\theta-2n\pi)-\hat{\theta}\pi/2)}}{\Gamma(\nu+1)(2|\sin(\theta)|)^{\nu+1}}$.

- For $\nu = -\frac{1}{2}$, $\mathcal{D}_\nu^M$ reduces to the one-dimensional linear canonical transform described by Bultheel et al. [4]

$$\mathcal{F}^M f(t) = \begin{cases} \frac{1}{(2i\pi b)^{1/2}} \int_{\mathbb{R}} e^{\frac{i}{2}(\frac{d}{b}t^2 + \frac{a}{b}w^2) - \frac{i}{b}tw} f(w) dw, & b \neq 0, \\ \frac{e^{\frac{i}{2}\frac{c}{a}|t|^2}}{\sqrt{|a|}} f(t/a), & b = 0. \end{cases}$$

- For $\nu = -\frac{1}{2}$ and $M = (\cos \theta, \sin \theta, -\sin \theta, \cos \theta)$, $\mathcal{D}_\nu^M$ becomes the fractional Fourier transform defined by [21]

$$\mathcal{F}^\theta f(t) = \begin{cases} \frac{e^{i(\theta\pi/4 - (\theta - 2n\pi)/2)}}{\sqrt{2\pi|\sin(\theta)|}} \int_{\mathbb{R}} K_\theta(t, w) f(w) dw, & \theta \in \frac{\mathbb{R}}{n\pi\mathbb{Z}}, \\ f(t), & \theta = 2n\pi, \\ f(-t), & \theta = (2n+1)\pi, \end{cases}$$

with $K_\theta(t, w) = e^{-\frac{i}{2}(t^2+w^2)\cot(\theta) + \frac{itw}{\sin(\theta)}}$ for all $\theta \in \mathbb{R} \setminus \pi\mathbb{Z}$.

- For $\nu = -\frac{1}{2}$ and $M = (0, -1, 1, 0)$, $\mathcal{D}_\nu^M$ reduces to the classical Fourier transform:

$$\mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-it\xi} dt.$$

The following proposition gathers several fundamental properties of the linear canonical Dunkl transform. These results form essential tools and will be repeatedly used in the subsequent analysis and developments of this paper.

Proposition 2.1 (see [14]). *Let M be a matrix belonging to $SL(2, \mathbb{R})$.*

- (a) *Suppose that $b \neq 0$. Then, for all $f \in L_\nu^1(\mathbb{R})$, $\mathcal{D}_\nu^M f$ belongs to $\mathcal{C}_0(\mathbb{R})$ and satisfies*

$$\|\mathcal{D}_\nu^M f\|_{\nu, \infty} \leq \frac{1}{\Gamma(\nu+1)(2|b|)^{\nu+1}} \|f\|_{\nu, 1}.$$

- (b) *For every f in $L_\nu^1(\mathbb{R})$ such that $\mathcal{D}_\nu^M f \in L_\nu^1(\mathbb{R})$,*

$$(2.5) \quad (\mathcal{D}_\nu^{M^{-1}} \circ \mathcal{D}_\nu^M) f = f \quad \text{a.e.} \quad \text{and} \quad (\mathcal{D}_\nu^M \circ \mathcal{D}_\nu^{M^{-1}}) f = f \quad \text{a.e.}$$

- (c) *The linear canonical Dunkl transform $\mathcal{D}_\nu^M$ defines a bijective mapping from $\mathcal{S}(\mathbb{R})$ onto itself. Furthermore, for any $f \in \mathcal{S}(\mathbb{R})$,*

$$(\mathcal{D}_\nu^M)^{-1} f = \mathcal{D}_\nu^{M^{-1}} f.$$

- (d) (Plancherel theorem). *Let f and g be functions in $L_\nu^1(\mathbb{R})$, and let $M \in SL(2, \mathbb{R})$ with $b \neq 0$. Then,*

$$\int_{\mathbb{R}} \mathcal{D}_\nu^M f(t) \overline{g(t)} |t|^{2\nu+1} dt = \int_{\mathbb{R}} f(t) \overline{\mathcal{D}_\nu^{M^{-1}} g(t)} |t|^{2\nu+1} dt.$$

- (e) *For every $f \in \mathcal{S}(\mathbb{R})$ and $M \in SL(2, \mathbb{R})$*

$$\|\mathcal{D}_\nu^M f\|_{\nu, 2} = \|f\|_{\nu, 2}.$$

(f) Let $M \in SL(2, \mathbb{R})$. If $f \in L_\nu^1(\mathbb{R}) \cap L_\nu^2(\mathbb{R})$, then $\mathcal{D}_\nu^M f \in L_\nu^2(\mathbb{R})$ and $\|\mathcal{D}_\nu^M f\|_{\nu,2} = \|f\|_{\nu,2}$.

(g) Let $M \in SL(2, \mathbb{R})$. There exists a unique unitary operator on $L_\nu^2(\mathbb{R})$ that coincides with $\mathcal{D}_\nu^M$ on $L_\nu^1(\mathbb{R}) \cap L_\nu^2(\mathbb{R})$; this extension is denoted by $f \mapsto \mathcal{D}_\nu^M f$.

The following theorem state the additivity property for linear canonical Dunkl transform.

Theorem 2.1 (see [14]). Let $M_1, M_2 \in SL(2, \mathbb{R})$ and $f \in L_\nu^1(\mathbb{R})$ with $D_\nu^{M_2} f \in L_\nu^1(\mathbb{R})$. Then, we have

$$(2.6) \quad D_\nu^{M_1} D_\nu^{M_2} f = e^{i\psi} D_\nu^{M_1 M_2} f,$$

where ψ is constant phase see [14, Theorem 4.2].

3. UNCERTAINTY RELATIONS IN THREE LINEAR CANONICAL DUNKL DOMAINS

The Donoho-Stark uncertainty principle is a powerful generalization of the classical Heisenberg inequality, shifting emphasis from pointwise behavior to the behavior of functions on measurable sets. To present this uncertainty principle, we need to recall the following notations and definitions.

Definition 3.1. Let E_1 and E_2 be two measurable subsets of $\mathbb{R}$ and f a non-zero function in $L_\nu^2(\mathbb{R})$. We say that

(a) f is κ_{E_1} time-limited in the $L_\nu^2(\mathbb{R})$ -norm if

$$\|f - \chi_{E_1} f\|_{\nu,2} \leq \kappa_{E_1},$$

where

$$\chi_{E_1}(t) = \begin{cases} 1, & t \in E_1, \\ 0, & \text{otherwise,} \end{cases}$$

(b) f is κ_{E_2} linear canonical Dunkl band-limited in the $L_\nu^2(\mathbb{R})$ -norm if its linear canonical Dunkl transform $\mathcal{D}_\nu^{M_1} f$ satisfies

$$\|\mathcal{D}_\nu^{M_1} f - \chi_{E_2} \mathcal{D}_\nu^{M_1} f\|_{\nu,2} \leq \kappa_{E_2} \|\mathcal{D}_\nu^{M_1} f\|_{\nu,2}.$$

Define the time-limiting and frequency-limiting operators on the Hilbert space $L_\nu^2(\mathbb{R})$, denoted by π_{E_1} and Π_{E_2} , respectively, as

$$\pi_{E_1} f(t) = \chi_{E_1}(t) f(t), \quad t \in \mathbb{R},$$

and

$$\Pi_{E_2}(f)(t) = \mathcal{D}_\nu^{M_1^{-1}} \left[\chi_{E_2}(\cdot) \mathcal{D}_\nu^{M_1}(f)(\cdot) \right](t), \quad t \in \mathbb{R}.$$

Remark 3.1. Let E_1 and E_2 be two measurable subsets of $\mathbb{R}$ and f a non-zero function in $L_\nu^2(\mathbb{R})$.

(a) f is κ_{E_1} time-limited in the $L_\nu^2(\mathbb{R})$ -norm if and only if

$$\|\pi_{E_1^c} f\|_{\nu,2} \leq \kappa_{E_1} \|f\|_{\nu,2}.$$

(b) f is κ_{E_2} linear canonical Dunkl band-limited in the $L^2_\nu(\mathbb{R})$ -norm if and only if

$$\begin{aligned} \left\| \mathcal{D}_\nu^{M_1} f - \mathcal{D}_\nu^{M_1} (\Pi_{E_2} f) \right\|_{\nu,2} &= \left\| \mathcal{D}_\nu^{M_1} f - \chi_{E_2} \mathcal{D}_\nu^{M_1} f \right\|_{\nu,2} = \left\| \Pi_{E_2^c} \mathcal{D}_\nu^{M_1} f \right\|_{\nu,2} \\ &\leq \kappa_{E_2} \left\| \mathcal{D}_\nu^{M_1} f \right\|_{\nu,2}. \end{aligned}$$

Define the norms of the time-limiting and frequency-limiting operators respectively by

$$\|\pi_{E_1}\| = \sup_{f \in L^2_\nu(\mathbb{R})} \frac{\|\pi_{E_1} f\|_{\nu,2}}{\|f\|_{\nu,2}}$$

and

$$\|\Pi_{E_2}\| = \sup_{f \in L^2_\nu(\mathbb{R})} \frac{\|\Pi_{E_2} f\|_{\nu,2}}{\|f\|_{\nu,2}}.$$

Using Plancherel's theorem, the norm of Π_{E_2} can also be expressed as

$$\|\Pi_{E_2}\| = \sup_{\mathcal{D}_\nu^{M_1} f \in L^2_\nu(\mathbb{R})} \frac{\|\chi_{E_2} \mathcal{D}_\nu^{M_1} f\|_{\nu,2}}{\|\mathcal{D}_\nu^{M_1} f\|_{\nu,2}}.$$

Consequently, π_{E_1} and Π_{E_2} are projection operators with norm 1.

We now present the Donoho-Stark uncertainty principle for the linear canonical Dunkl transform established in our preprint work with T. Moumni and I. Zayed.

Theorem 3.1 (Donoho-Stark theorem). *Let E_1 and E_2 be two measurable subsets of $\mathbb{R}$ with $0 < |E_1|, |E_2| < +\infty$. Assume that $\kappa_{E_1} + \kappa_{E_2} < 1$. If f is κ_{E_1} time-limited and κ_{E_2} linear canonical Dunkl band-limited in the $L^2_\nu(\mathbb{R})$ -norm, then*

$$(1 - \kappa_{E_1} - \kappa_{E_2})^2 |b|^{2(\nu+1)} \leq \sigma_\nu^2 |E_1| \cdot |E_2|,$$

where $|E_1|$ and $|E_2|$ denote the measures of E_1 and E_2 , respectively.

To develop uncertainty inequalities involving three distinct linear canonical Dunkl domains, namely for f , $\mathcal{D}_\nu^{M_1}(f)$, and $\mathcal{D}_\nu^{M_2}(f)$, we use Π_{E_1} to denote the projection defined via $\mathcal{D}_\nu^{M_1}$ and the set E_1 , and Π_{E_2} for the projection defined via $\mathcal{D}_\nu^{M_2}$ and E_2 . Specifically,

$$\begin{aligned} \Pi_{E_1} f &= \mathcal{D}_\nu^{M_1^{-1}} \left[\chi_{E_1} \mathcal{D}_\nu^{M_1}(f) \right], \\ \Pi_{E_2} f &= \mathcal{D}_\nu^{M_2^{-1}} \left[\chi_{E_2} \mathcal{D}_\nu^{M_2}(f) \right]. \end{aligned}$$

For simplicity, we omit explicit reference to M_1 and M_2 in the notation for Π_{E_1} and Π_{E_2} .

Proposition 3.1. *Let $M_1 = (a_{M_1}, b_{M_1}, c_{M_1}, d_{M_1})$ and $M_2 = (a_{M_2}, b_{M_2}, c_{M_2}, d_{M_2})$ be two matrices with $\det(M_1) = \det(M_2) = 1$ and $b_{M_1} b_{M_2} \neq 0$. Then, for every $f \in L^2_\nu(\mathbb{R})$,*

$$\left\| \mathcal{D}_\nu^{M_1} (\Pi_{E_1} \pi_{E_3} \Pi_{E_2} f) \right\|_{\nu,2} \leq \sigma_\nu^2 |b_{M_1} b_{M_2}|^{-(\nu+1)} |E_1|^{\frac{1}{2}} |E_3| \cdot |E_2|^{\frac{1}{2}} \|f\|_{\nu,2},$$

where $|E_1|$, $|E_3|$ and $|E_2|$ represent the measures of the sets E_1 , E_3 and E_2 .

Proof. Assume that the measures of E_1 , E_3 , and E_2 are finite. By definition,

$$\mathcal{D}_\nu^{M_1} (\Pi_{E_1} \pi_{E_3} \Pi_{E_2} f) = \chi_{E_1} \left(\mathcal{D}_\nu^{M_1} (\pi_{E_3} \Pi_{E_2} f) \right).$$

Hence,

$$(3.1) \quad \left\| \mathcal{D}_\nu^{M_1} (\Pi_{E_1} \pi_{E_3} \Pi_{E_2} f) \right\|_{\nu,2} = \left(\int_{E_1} \left| \chi_{E_1} \mathcal{D}_\nu^{M_1} (\pi_{E_3} \Pi_{E_2} f) \right|^2 w^{2\nu+1} dw \right)^{\frac{1}{2}}.$$

Moreover,

$$\mathcal{D}_\nu^{M_1} (\pi_{E_3} \Pi_{E_2} f) (w) = \frac{\sigma_\nu}{(ib)^{\nu+1}} \int_{E_3} E_\nu^{M_1}(w, x) (\Pi_{E_2} f)(x) x^{2\nu+1} dx.$$

Applying Hölder's inequality and the bound for the kernel, we obtain

$$\begin{aligned} \left| \mathcal{D}_\nu^{M_1} (\pi_{E_3} \Pi_{E_2} f) (w) \right| &= \left| \frac{\sigma_\nu}{(ib)^{\nu+1}} \int_{E_3} E_\nu^{M_1}(w, x) (\Pi_{E_2} f)(x) x^{2\nu+1} dx \right| \\ &\leq \frac{\sigma_\nu}{|b|^{\nu+1}} \left(\int_{E_3} \left| E_\nu^{M_1}(w, x) \right|^2 x^{2\nu+1} dx \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{E_3} |(\Pi_{E_2} f)(x)|^2 x^{2\nu+1} dx \right)^{\frac{1}{2}} \\ (3.2) \quad &= \frac{\sigma_\nu}{|b|^{\nu+1}} |E_3|^{\frac{1}{2}} \left(\int_{E_3} |(\Pi_{E_2} f)(x)|^2 x^{2\nu+1} dx \right)^{\frac{1}{2}}. \end{aligned}$$

Furthermore, using Hölder's inequality and the Riemann-Lebesgue lemma yields

$$\begin{aligned} \left(\int_{E_3} |(\Pi_{E_2} f)(x)|^2 x^{2\nu+1} dx \right)^{\frac{1}{2}} &= \left(\int_{E_3} \left| \left(\mathcal{D}_\nu^{M_2^{-1}} \chi_{E_2} \mathcal{D}_\nu^{M_2} f \right) (x) \right|^2 x^{2\nu+1} dx \right)^{\frac{1}{2}} \\ &\leq \left(\int_{E_3} x^{2\nu+1} dx \right)^{\frac{1}{2}} \left\| \mathcal{D}_\nu^{M_2^{-1}} \left(\chi_{E_2} \mathcal{D}_\nu^{M_2} f \right) \right\|_{\infty, \nu} \\ &= |E_3|^{\frac{1}{2}} \left\| \mathcal{D}_\nu^{M_2^{-1}} \left(\chi_{E_2} \mathcal{D}_\nu^{M_2} f \right) \right\|_{\infty, \nu} \\ &= |E_3|^{\frac{1}{2}} \sigma_\nu |b_{M_2^{-1}}|^{-(\nu+1)} \left\| \chi_{E_2} \mathcal{D}_\nu^{M_2} f \right\|_{1, \nu} \\ (3.3) \quad &\leq \frac{\sigma_\nu}{|b_{M_2^{-1}}|^{\nu+1}} |E_3|^{\frac{1}{2}} |E_2|^{\frac{1}{2}} \left\| \mathcal{D}_\nu^{M_2}(f) \right\|_{\nu,2}. \end{aligned}$$

Substituting (3.3) into (3.2) and then into (3.1), we obtain

$$\left\| \mathcal{D}_\nu^{M_1} (\Pi_{E_1} \pi_{E_3} \Pi_{E_2} f) \right\|_{\nu,2} \leq \frac{\sigma_\nu^2}{|b_{M_1} b_{M_2^{-1}}|^{\nu+1}} |E_1|^{\frac{1}{2}} |E_3| \cdot |E_2|^{\frac{1}{2}} \|f\|_{\nu,2}.$$

Since $b_{M_2^{-1}} = -b_{M_2}$, it follows that

$$\left\| \mathcal{D}_\nu^{M_1} (\Pi_{E_1} \pi_{E_3} \Pi_{E_2} f) \right\|_{\nu,2} \leq \frac{\sigma_\nu^2}{|b_{M_1} b_{M_2}|^{\nu+1}} |E_1|^{\frac{1}{2}} |E_3| \cdot |E_2|^{\frac{1}{2}} \|f\|_{\nu,2}.$$

□

3.1. First uncertainty inequality. We now present the first uncertainty principle governing three distinct linear canonical Dunkl transform domains. The matrices M_1 and M_2 are those introduced in Proposition 3.1, and we assume $b_{M_1}b_{M_2} \neq 0$.

Theorem 3.2. *Let f be a non-zero function in $L^2_\nu(\mathbb{R})$. Assume that f is κ_{E_3} time-limited on E_3 , $\mathcal{D}_\nu^{M_1}(f)$ is κ_{E_1} -limited on E_1 , and $\mathcal{D}_\nu^{M_2}(f)$ is κ_{E_2} -limited on E_2 . Then,*

$$(1 - \kappa_{E_1} - \kappa_{E_3} - \kappa_{E_2}) \leq \sigma_\nu^2 |b_{M_1}b_{M_2}|^{-(\nu+1)} |E_1|^{\frac{1}{2}} |E_3| \cdot |E_2|^{\frac{1}{2}}.$$

Proof. Without loss of generality, assume that E_1 , E_3 and E_2 have finite measure. By the given concentration properties

$$\begin{aligned} \|f - \pi_{E_3}f\|_{\nu,2} &= \left(\int_{\mathbb{R} \setminus E_3} |f(x)|^2 dx \right)^{\frac{1}{2}} \leq \kappa_{E_3} \|f\|_{\nu,2}, \\ \|\mathcal{D}_\nu^{M_1}(f) - \mathcal{D}_\nu^{M_1}(\Pi_{E_1}f)\|_{\nu,2} &\leq \kappa_{E_1} \|\mathcal{D}_\nu^{M_1}(f)\|_{\nu,2}, \\ \|\mathcal{D}_\nu^{M_2}(f) - \mathcal{D}_\nu^{M_2}(\Pi_{E_2}f)\|_{\nu,2} &\leq \kappa_{E_2} \|\mathcal{D}_\nu^{M_2}(f)\|_{\nu,2}. \end{aligned}$$

From these we obtain the two approximations

$$\begin{aligned} \|\mathcal{D}_\nu^{M_1}(\Pi_{E_1}f) - \mathcal{D}_\nu^{M_1}(\Pi_{E_1}\pi_{E_3}f)\|_{\nu,2} &= \|\chi_{E_1}\mathcal{D}_\nu^{M_1}(f) - \chi_{E_1}\mathcal{D}_\nu^{M_1}(\pi_{E_3}f)\|_{\nu,2} \\ &\leq \|\mathcal{D}_\nu^{M_1}(f - \pi_{E_3}f)\|_{\nu,2} \\ &= \|f - \pi_{E_3}f\|_{\nu,2} \end{aligned}$$

and

$$\begin{aligned} \|\mathcal{D}_\nu^{M_1}(\Pi_{E_1}\pi_{E_3}f) - \mathcal{D}_\nu^{M_1}(\Pi_{E_1}\pi_{E_3}\Pi_{E_2}f)\|_{\nu,2} &= \|\chi_{E_1}\mathcal{D}_\nu^{M_1}(\pi_{E_3}f) - \chi_{E_1}\mathcal{D}_\nu^{M_1}(\pi_{E_3}\Pi_{E_2}f)\|_{\nu,2} \\ &\leq \|\mathcal{D}_\nu^{M_1}(\pi_{E_3}f - \pi_{E_3}\Pi_{E_2}f)\|_{\nu,2} \\ &= \|\pi_{E_3}f - \pi_{E_3}\Pi_{E_2}f\|_{\nu,2} \\ &\leq \|f - \Pi_{E_2}f\|_{\nu,2} \\ &= \|\mathcal{D}_\nu^{M_2}(f) - \mathcal{D}_\nu^{M_2}(\Pi_{E_2}f)\|_{\nu,2} \\ &\leq \kappa_{E_2} \|f\|_{\nu,2}. \end{aligned}$$

Now, using the triangle inequality and Proposition 3.1,

$$\begin{aligned} \|\mathcal{D}_\nu^{M_1}(f)\|_{\nu,2} &\leq \|\mathcal{D}_\nu^{M_1}(f) - \mathcal{D}_\nu^{M_1}(\Pi_{E_1}f)\|_{\nu,2} + \|\mathcal{D}_\nu^{M_1}(\Pi_{E_1}f) - \mathcal{D}_\nu^{M_1}(\Pi_{E_1}\pi_{E_3}f)\|_{\nu,2} \\ &\quad + \|\mathcal{D}_\nu^{M_1}(\Pi_{E_1}\pi_{E_3}f) - \mathcal{D}_\nu^{M_1}(\Pi_{E_1}\pi_{E_3}\Pi_{E_2}f)\|_{\nu,2} \\ &\quad + \|\mathcal{D}_\nu^{M_1}(\Pi_{E_1}\pi_{E_3}\Pi_{E_2}f)\|_{\nu,2} \\ &\leq \kappa_{E_1} \|\mathcal{D}_\nu^{M_1}(f)\|_{\nu,2} + \kappa_{E_3} \|f\|_{\nu,2} + \kappa_{E_2} \|\mathcal{D}_\nu^{M_2}(f)\|_{\nu,2} \\ &\quad + \sigma_\nu^2 |b_{M_1}b_{M_2}|^{-(\nu+1)} |E_1|^{\frac{1}{2}} |E_3| \cdot |E_2|^{\frac{1}{2}} \|f\|_{\nu,2}. \end{aligned}$$

This implies

$$(1 - \kappa_{E_1} - \kappa_{E_3} - \kappa_{E_2}) \left\| \mathcal{D}_\nu^{M_1}(f) \right\|_{\nu,2} \leq \sigma_\nu^2 |b_{M_1} b_{M_2}|^{-(\nu+1)} |E_1|^{\frac{1}{2}} |E_3| \cdot |E_2|^{\frac{1}{2}} \|f\|_{\nu,2}.$$

Applying the Plancherel theorem for the linear canonical Dunkl transform gives the desired result. $\square$

Setting $M_1 = M_2$ and $E_1 = E_2$, Theorem 3.2 reduces to the following.

Theorem 3.3. *Let f be a non-zero function in $L_\nu^2(\mathbb{R})$, such that f is κ_{E_3} time-limited on E_3 and $\mathcal{D}_\nu^{M_1}(f)$ is κ_{E_1} -limited on E_1 . Then,*

$$(1 - 2\kappa_{E_1} - \kappa_{E_3}) \leq \sigma_\nu^2 |b_{M_1}|^{-2(\nu+1)} |E_1| \cdot |E_3|.$$

Remark 3.2. When $\kappa_{E_1} = 0$, we obtain a sharper bound than the result established in our preprint work with T. Moumni and I. Zayed

$$(1 - \kappa_{E_3}) \leq \sigma_\nu^2 |b_{M_1}|^{-2(\nu+1)} |E_1| \cdot |E_3|.$$

For $\nu = -\frac{1}{2}$, the linear canonical Dunkl transform is reduced to the one-dimensional linear canonical transform and we recover the uncertainty inequality in three linear canonical domains introduced by Pan Lian in [20, Theorem 3.2].

Corollary 3.1. *Let f be a non-zero function in $L^2(\mathbb{R})$. Assume that f is κ_{E_3} time-limited on E_3 , $\mathcal{F}^{M_1}(f)$ is κ_{E_1} -limited on E_1 , and $\mathcal{F}^{M_2}(f)$ is κ_{E_2} -limited on E_2 . Then,*

$$(1 - \kappa_{E_1} - \kappa_{E_3} - \kappa_{E_2}) \leq (2\pi)^{-1} |b_{M_1} b_{M_2}|^{-(\nu+1)} |E_1|^{\frac{1}{2}} |E_3| \cdot |E_2|^{\frac{1}{2}}.$$

When $M_i = (\cos \alpha_i, \sin \alpha_i, -\sin \alpha_i, \cos \alpha_i)$ for $\alpha_i \in \mathbb{R}$ and $i = 1, 2$, the linear canonical Dunkl transform reduces to the fractional Dunkl transform $\mathcal{D}_\nu^\theta$. For $M_1 = (\cos \alpha_1, \sin \alpha_1, -\sin \alpha_1, \cos \alpha_1)$ and $M_2 = (\cos \alpha_2, \sin \alpha_2, -\sin \alpha_2, \cos \alpha_2)$ such that $\sin \alpha_1 \sin \alpha_2 \neq 0$ and $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \pi\mathbb{Z}$, Theorem 3.2 yields the following uncertainty inequality in three fractional Dunkl domains.

Theorem 3.4. *Let f be a non-zero function in $L_\nu^2(\mathbb{R})$. Assume that f is κ_{E_3} time-limited on E_3 , $\mathcal{D}_\nu^{\alpha_1}(f)$ is κ_{E_1} -limited on E_1 , and $\mathcal{D}_\nu^{\alpha_2}(f)$ is κ_{E_2} -limited on E_2 . Then,*

$$(1 - \kappa_{E_1} - \kappa_{E_3} - \kappa_{E_2}) \leq \sigma_\nu^2 |\sin \alpha_1 \sin \alpha_2|^{-(\nu+1)} |E_1|^{\frac{1}{2}} |E_3| \cdot |E_2|^{\frac{1}{2}}.$$

Setting $\alpha_1 = \alpha_2$ and $E_1 = E_2$ gives the following corollary.

Corollary 3.2. *Let f be a non-zero function in $L_\nu^2(\mathbb{R})$ such that f is κ_{E_3} time-limited on E_3 and $\mathcal{D}_\nu^{\alpha_1}(f)$ is κ_{E_1} -limited on E_1 . Then,*

$$(1 - 2\kappa_{E_1} - \kappa_{E_3}) \leq \sigma_\nu^2 |\sin \alpha_1|^{-2(\nu+1)} |E_1| \cdot |E_3|.$$

For $M_1 = (0, 1, -1, 0)$, the linear canonical Dunkl transform reduces to the classical Dunkl transform $\mathcal{D}_\nu$. Then Theorem 3.3 yields the following uncertainty relation.

Corollary 3.3. *Let f be a non-zero function in $L_\nu^2(\mathbb{R})$ such that f is κ_{E_3} time-limited on E_3 and $\mathcal{D}_\nu(f)$ is κ_{E_1} -limited on E_1 . Then,*

$$(1 - 2\kappa_{E_1} - \kappa_{E_3}) \leq \sigma_\nu^2 |E_1| \cdot |E_3|.$$

Remark 3.3. For the special case $\kappa_{E_1} = 0$, we obtain a sharper bound for the Donoho-Strack uncertainty principle for the Dunkl transform

$$(1 - \kappa_{E_3}) \leq \sigma_\nu^2 |E_1| \cdot |E_3|.$$

3.2. Second uncertainty inequality. We now present the second uncertainty inequality in three linear canonical Dunkl domains.

Theorem 3.5. *Let f be a non-zero function in $L_\nu^2(\mathbb{R})$, and let E_1 , E_3 and E_2 be measurable subsets of $\mathbb{R}$ satisfying*

$$|E_1|^{\frac{1}{2}} |E_3| \cdot |E_2|^{\frac{1}{2}} < \sigma_\nu^2 |b_{M_1} b_{M_2}|^{-(\nu+1)} |E_1|^{\frac{1}{2}}, \quad \kappa_{E_1}^2 + \kappa_{E_3}^2 + \kappa_{E_2}^2 < 1.$$

If f is κ_{E_3} time-limited on E_3 , $\mathcal{D}_\nu^{M_1}(f)$ is κ_{E_1} -limited on E_1 , and $\mathcal{D}_\nu^{M_2}(f)$ is κ_{E_2} -limited on E_2 , then

$$1 - (\kappa_{E_1}^2 + \kappa_{E_3}^2 + \kappa_{E_2}^2)^{\frac{1}{2}} \leq \sigma_\nu^2 |b_{M_1} b_{M_2}|^{-(\nu+1)} |E_1|^{\frac{1}{2}} |E_3| \cdot |E_2|^{\frac{1}{2}}.$$

Proof. Denote by $\mathcal{I}$ the identity operator. Then,

$$\begin{aligned} \mathcal{I} &= \Pi_{E_1} + \Pi_{E_1^c} = \Pi_{E_1} \pi_{E_3} + \Pi_{E_1} \pi_{E_3^c} + \Pi_{E_1^c} \\ &= \Pi_{E_1} \pi_{E_3} \Pi_{E_2} + \Pi_{E_1} \pi_{E_3} \Pi_{E_2^c} + \Pi_{E_1} \pi_{E_3^c} + \Pi_{E_1^c}. \end{aligned}$$

By orthogonality of the operators Π_{E_1} and π_{E_3} ,

$$\begin{aligned} \|f - \Pi_{E_1} \pi_{E_3} \Pi_{E_2} f\|_{\nu,2}^2 &= \|\Pi_{E_1} \pi_{E_3} \Pi_{E_2^c} f + \Pi_{E_1} \pi_{E_3^c} f + \Pi_{E_1^c} f\|_{\nu,2}^2 \\ &= \|\Pi_{E_1} \pi_{E_3} \Pi_{E_2^c} f\|_{\nu,2}^2 + \|\Pi_{E_1} \pi_{E_3^c} f\|_{\nu,2}^2 + \|\Pi_{E_1^c} f\|_{\nu,2}^2. \end{aligned}$$

Since $\|\pi_{E_3}\| = \|\Pi_{E_1}\| = \|\Pi_{E_2}\| = 1$, we obtain

$$(3.4) \quad \|f - \Pi_{E_1} \pi_{E_3} \Pi_{E_2} f\|_{\nu,2}^2 \leq \|\Pi_{E_2^c} f\|_{\nu,2}^2 + \|\pi_{E_3^c} f\|_{\nu,2}^2 + \|\Pi_{E_1^c} f\|_{\nu,2}^2.$$

On the other hand,

$$(3.5) \quad \|f - \Pi_{E_1} \pi_{E_3} \Pi_{E_2} f\|_{\nu,2} \geq \|f\|_{\nu,2} - \|\Pi_{E_1} \pi_{E_3} \Pi_{E_2} f\|_{\nu,2} \geq (1 - \|\Pi_{E_1} \pi_{E_3} \Pi_{E_2}\|) \|f\|_{\nu,2}.$$

From Proposition 3.1 we have

$$(3.6) \quad \left(1 - \sigma_\nu^2 |b_{M_1} b_{M_2}|^{-(\nu+1)} |E_1|^{\frac{1}{2}} |E_3| \cdot |E_2|^{\frac{1}{2}}\right) \|f\|_{\nu,2} \leq (1 - \|\Pi_{E_1} \pi_{E_3} \Pi_{E_2}\|) \|f\|_{\nu,2}.$$

Combining (3.4), (3.5) and (3.6) yields

$$\begin{aligned} \|f\|_{\nu,2}^2 &\leq \left(1 - \sigma_\nu^2 |b_{M_1} b_{M_2}|^{-(\nu+1)} |E_1|^{\frac{1}{2}} |E_3| |E_2|^{\frac{1}{2}}\right)^{-2} \\ &\quad \times \left(\|\Pi_{E_2^c} f\|_{\nu,2}^2 + \|\pi_{E_3^c} f\|_{\nu,2}^2 + \|\Pi_{E_1^c} f\|_{\nu,2}^2\right). \end{aligned}$$

Using the concentration hypotheses one has

$$\left(1 - \sigma_\nu^2 |b_{M_1} b_{M_2}|^{-(\nu+1)} |E_1|^{\frac{1}{2}} |E_3| \cdot |E_2|^{\frac{1}{2}}\right)^2 \leq \kappa_{E_1}^2 + \kappa_{E_3}^2 + \kappa_{E_2}^2.$$

Consequently,

$$1 - \left(\kappa_{E_1}^2 + \kappa_{E_3}^2 + \kappa_{E_2}^2 \right)^{\frac{1}{2}} \leq \sigma_\nu^2 |b_{M_1} b_{M_2}|^{-(\nu+1)} |E_1|^{\frac{1}{2}} |E_3| \cdot |E_2|^{\frac{1}{2}}.$$

□

For $\nu = -\frac{1}{2}$, the linear canonical Dunkl transform is reduced to the one-dimensional linear canonical transform and we recover the uncertainty inequality in three linear canonical domains introduced by Pan Lian in [20, Theorem 3.5].

Corollary 3.4. *Let f be a non-zero function in $L^2(\mathbb{R})$, and let E_1 , E_3 , and E_2 be measurable subsets of $\mathbb{R}$ satisfying*

$$|E_1|^{\frac{1}{2}} |E_3| |E_2|^{\frac{1}{2}} < 2\pi |b_{M_1} b_{M_2}|^{-(\nu+1)} |E_1|^{\frac{1}{2}}, \quad \kappa_{E_1}^2 + \kappa_{E_3}^2 + \kappa_{E_2}^2 < 1.$$

If f is κ_{E_3} time-limited on E_3 , $\mathcal{F}^{M_1}(f)$ is κ_{E_1} -limited on E_1 , and $\mathcal{F}^{M_2}(f)$ is κ_{E_2} -limited on E_2 , then

$$1 - \left(\kappa_{E_1}^2 + \kappa_{E_3}^2 + \kappa_{E_2}^2 \right)^{\frac{1}{2}} \leq \sigma_\nu^2 |b_{M_1} b_{M_2}|^{-(\nu+1)} |E_1|^{\frac{1}{2}} |E_3| \cdot |E_2|^{\frac{1}{2}}.$$

Setting $M_1 = M_2$ and $E_1 = E_2$ gives the following special case.

Theorem 3.6. *Let f be a non-zero function in $L_\nu^2(\mathbb{R})$ such that f is κ_{E_3} time-limited on E_3 and $\mathcal{D}_\nu^{M_1}(f)$ is κ_{E_1} -limited on E_1 . Then,*

$$1 - \left(2\kappa_{E_1}^2 + \kappa_{E_2}^2 \right)^{\frac{1}{2}} \leq \sigma_\nu^2 |b_{M_1}|^{-2(\nu+1)} |E_1| \cdot |E_3|.$$

4. UNCERTAINTY RELATIONS IN n LINEAR CANONICAL DUNKL DOMAINS

We extend the study of uncertainty principles from the case of three domains, discussed in the preceding section, to the general setting of n domains with $n \geq 3$. Consider n distinct matrices M_j , $j \in [0, n-1]$, such that $b_j \neq 0$ for each. The frequency-limiting operators Π_{E_j} associated with M_j are defined by

$$\Pi_{E_j} f = \mathcal{D}_\nu^{M_j^{-1}} \left[\chi_{E_j} \mathcal{D}_\nu^{M_j}(f) \right], \quad j \in [0, n-1].$$

Proposition 4.1. *Let f be a non-zero function in $L_\nu^2(\mathbb{R})$. Then,*

$$\left\| \mathcal{D}_\nu^{M_0} \left(\Pi_{E_0} \Pi_{E_1} \dots \Pi_{E_{n-1}} f \right) \right\|_{\nu,2} \leq \lambda |E_0|^{\frac{1}{2}} |E_1| \dots |E_{n-2}| \cdot |E_{n-1}|^{\frac{1}{2}} \|f\|_{\nu,2},$$

where $|E_j|$ denotes the measure of the set E_j for each $j \in [0, n-1]$ and the constant λ is given by

$$\lambda = \left| E_\nu^{M_0 M_1^{-1}} \right| \cdot \left| E_\nu^{M_1 M_2^{-1}} \right| \dots \left| E_\nu^{M_{n-2} M_{n-1}^{-1}} \right|,$$

with $|E_\nu^M(x, y)|$ representing the modulus of the kernel.

Proof. We demonstrate the cases $n = 2$ and $n = 3$ and the general result follows by induction. For $n = 2$, the definition directly gives

$$\mathcal{D}_\nu^{M_0} (\Pi_{E_0} \Pi_{E_1} f) = \chi_{E_0} \left(\mathcal{D}_\nu^{M_0} (\Pi_{E_1} f) \right).$$

Consequently,

$$(4.1) \quad \left\| \mathcal{D}_\nu^{M_0} (\Pi_{E_0} \Pi_{E_1} f) \right\|_{\nu,2} = \left(\int_{E_0} \left| \chi_{E_0} \left(\mathcal{D}_\nu^{M_0} (\Pi_{E_1} f) \right) \right|^2 w^{2\nu+1} dw \right)^{\frac{1}{2}}.$$

Using the additivity property of the linear canonical Dunkl transform (2.6), we obtain

$$\begin{aligned} \left| \mathcal{D}_\nu^{M_0} (\Pi_{E_1} f) (w) \right| &= \left| \int_{\mathbb{R}} E_\nu^{M_0}(t, w) (\Pi_{E_1} f) (t) t^{2\nu+1} dt \right| \\ &= \left| \int_{E_1} E_\nu^{M_0 M_1^{-1}}(t, w) \left(\mathcal{D}_\nu^{M_1} f \right) (t) t^{2\nu+1} dt \right|. \end{aligned}$$

Applying Hölder's inequality yields

$$\begin{aligned} \left| \mathcal{D}_\nu^{M_0} (\Pi_{E_1} f) (w) \right| &\leq \left(\int_{E_1} \left| E_\nu^{M_0 M_1^{-1}}(t, w) \right|^2 t^{2\nu+1} dt \right)^{\frac{1}{2}} \left(\int_{E_1} \left| \left(\mathcal{D}_\nu^{M_1} f \right) (t) \right|^2 t^{2\nu+1} dt \right)^{\frac{1}{2}} \\ (4.2) \quad &= \left| E_\nu^{M_0 M_1^{-1}} \right| \cdot |E_1|^{\frac{1}{2}} \left(\int_{E_1} \left| \left(\mathcal{D}_\nu^{M_1} f \right) (t) \right|^2 t^{2\nu+1} dt \right)^{\frac{1}{2}}. \end{aligned}$$

Substituting (4.2) into (4.1) gives

$$\begin{aligned} \left\| \mathcal{D}_\nu^{M_0} (\Pi_{E_0} \Pi_{E_1} f) \right\|_{\nu,2} &= \left(\int_{E_0} \left| \chi_{E_0} \left(\mathcal{D}_\nu^{M_0} (\Pi_{E_1} f) \right) \right|^2 w^{2\nu+1} dw \right)^{\frac{1}{2}} \\ &\leq \left| E_\nu^{M_0 M_1^{-1}} \right| \cdot |E_0|^{\frac{1}{2}} |E_1|^{\frac{1}{2}} \left(\int_{E_1} \left| \left(\mathcal{D}_\nu^{M_1} f \right) (t) \right|^2 t^{2\nu+1} dt \right)^{\frac{1}{2}} \\ (4.3) \quad &\leq \left| E_\nu^{M_0 M_1^{-1}} \right| \cdot |E_0|^{\frac{1}{2}} |E_1|^{\frac{1}{2}} \|f\|_{\nu,2}. \end{aligned}$$

For $n = 3$, we replace f with $\Pi_{E_2} f$ in (4.3) to obtain

$$\left\| \mathcal{D}_\nu^{M_0} (\Pi_{E_0} \Pi_{E_1} \Pi_{E_2} f) \right\|_{\nu,2} \leq \left| E_\nu^{M_0 M_1^{-1}} \right| \cdot |E_0|^{\frac{1}{2}} |E_1|^{\frac{1}{2}} \left(\int_{E_1} \left| \left(\mathcal{D}_\nu^{M_1} (\Pi_{E_2} f) \right) (x) \right|^2 dx \right)^{\frac{1}{2}}.$$

Applying the estimate (4.2) once more yields

$$\left\| \mathcal{D}_\nu^{M_0} (\Pi_{E_0} \Pi_{E_1} \Pi_{E_2} f) \right\|_{\nu,2} \leq \left| E_\nu^{M_0 M_1^{-1}} \right| \cdot \left| E_\nu^{M_1 M_2^{-1}} \right| \cdot |E_0|^{\frac{1}{2}} |E_1| \cdot |E_2|^{\frac{1}{2}} \|f\|_{\nu,2}.$$

This completes the proof for the cases presented and the general result follows by induction on n . $\square$

Adapting the argument used in the proof of Theorem 3.5 leads to the following general result.

Theorem 4.1. *Let f be a function in $L_\nu^2(\mathbb{R})$ and let E_j , $j \in [0, n-1]$, be n measurable sets satisfying*

$$|E_0|^{\frac{1}{2}} |E_1| \cdots |E_{n-2}| \cdot |E_{n-1}|^{\frac{1}{2}} < \lambda^{-1},$$

and let $\kappa_{E_j} > 0$ be such that $\sum_{j=0}^{n-1} \kappa_{E_j}^2 < 1$. If $\mathcal{D}_\nu^{M_j} f$ is κ_{E_j} -limited on E_j for each j , then

$$1 - \left(\sum_{j=0}^{n-1} \kappa_{E_j}^2 \right)^{\frac{1}{2}} \leq \lambda |E_0|^{\frac{1}{2}} |E_1| \cdots |E_{n-2}| \cdot |E_{n-1}|^{\frac{1}{2}},$$

where $|E_j|$ is the measure of E_j for all $j \in [0, n-1]$ and

$$\lambda = \left| E_\nu^{M_0 M_1^{-1}} \right| \cdot \left| E_\nu^{M_1 M_2^{-1}} \right| \cdots \left| E_\nu^{M_{n-2} M_{n-1}^{-1}} \right|.$$

A careful inspection of the proof reveals that the conclusion remains valid when $M_0 = M_{n-1}$ with $n \geq 3$. In particular, taking $M_0 = M_{n-1}$ and $E_0 = E_{n-1}$ gives the following corollary.

Corollary 4.1. *Let f be a function in $L_\nu^2(\mathbb{R})$ and let E_j , $j \in [0, n-2]$, be $n-1$ measurable sets satisfying*

$$|E_0| \cdot |E_1| \cdots |E_{n-2}| \leq \lambda^{-1},$$

and let $\kappa_{E_j} > 0$ be such that $\sum_{j=0}^{n-2} \kappa_{E_j}^2 < 1$. If $\mathcal{D}_\nu^{M_j} f$ is κ_{E_j} -limited on E_j for each j , then

$$1 - \left(2\kappa_{E_0}^2 + \sum_{j=1}^{n-2} \kappa_{E_j}^2 \right)^{\frac{1}{2}} \leq \lambda |E_0| \cdot |E_1| \cdots |E_{n-2}|,$$

where $|E_j|$ is the measure of E_j for $j \in [0, n-2]$ and

$$\lambda = \left| E_\nu^{M_0 M_1^{-1}} \right| \cdot \left| E_\nu^{M_1 M_2^{-1}} \right| \cdots \left| E_\nu^{M_{n-2} M_0^{-1}} \right|.$$

CONCLUSION

In this paper, we have established new Donoho-Stark type uncertainty principles within the framework of the linear canonical Dunkl transform. By deriving concentration inequalities in three distinct linear canonical Dunkl domains, we have related time- and band-limiting parameters to the measures of the corresponding measurable sets. These results generalize and unify existing uncertainty principles for several integral transforms, including the Dunkl transform, fractional Dunkl transform, fractional Hankel transform, and the one-dimensional linear canonical transform. Furthermore, we extended the analysis to n linear canonical Dunkl domains ($n \geq 3$), yielding a comprehensive multi-domain uncertainty principle. Our approach leverages fundamental properties of the linear canonical Dunkl transform, such as additivity and the Plancherel theorem, providing a unified framework that broadens the theoretical landscape of uncertainty relations in harmonic analysis and offers potential applications in signal processing and phase-space analysis.

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DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, NORTHERN BORDER UNIVERSITY,
ARAR, SAUDI ARABIA

Email address: ahmed.saoudi@ipeim.rnu.tn

ORCID iD: <https://orcid.org/0000-0003-4422-2054>