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SOME APPLICATION ON HURWITZ LERCH ZETA FUNCTION DEFINED BY A GENERALIZATION OF THE SRIVASTAVA ATTIYA OPERATOR

K. A. CHALLAB[1], M. DARUS[1], AND F. GHANIM[2]

Abstract. In this article, we study certain properties of Hurwitz-Lerch zeta function involving the generalized Srivastava-Attiya operator. The authors also study the differential subordination, differential superordination and sandwich-type properties for the new operator which is defined on the space of normalized analytic function in the open unit disc.

1. Introduction

Let $A(\mathbb{U})$ denote a class of all analytic functions defined in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. For $a\in\mathbb{C}$, $j\in\mathbb{N}=\{1,2,\ldots\}$, let

$$A[a,j]=\left\{f\in A(\mathbb{U}):f(z)=a+a_jz^j+a_{j+1}z^{j+1}+\cdots\right\}.$$

We denote a subclass of $A[a,1]$ by A whose members are of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k\quad(z\in\mathbb{U}).\tag{1.1}$$

We denote by C the class of convex (univalent) functions in $\mathbb{U}$ and satisfying

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0\quad(z\in\mathbb{U}).$$

For two functions $f,g\in A(\mathbb{U})$, we say f is subordinate to g, or g is superordinate to f in $\mathbb{U}$ and write $f(z)\prec g(z)$, $z\in\mathbb{U}$, if there exists a Schwarz function ω, analytic in $\mathbb{U}$

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with $\omega(0) = 0$, and $|\omega(z)| < 1$, $z \in \mathbb{U}$ such that $f(z) = g(\omega(z))$, $z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 1.1. [7]. Let

$$\phi : \mathbb{C}^2 \to \mathbb{C}$$

and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the following differential subordination:

$$\phi(p(z), zp'(z)) \prec h(z) \quad (z \in \mathbb{U}), \tag{1.2}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination or, more simply, a dominant if

$$p \prec q \quad (z \in \mathbb{U}),$$

for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies the following condition:

$$\tilde{q} \prec q \quad (z \in \mathbb{U}),$$

for all dominants q of (1.2) is said to be the best dominant.

Miller and Mocanu [8] introduced the following notion of differential superordinations, as the dual concept of differential subordinations.

Definition 1.2. [8] Let

$$\phi : \mathbb{C}^2 \longrightarrow \mathbb{C}$$

and let h be analytic in $\mathbb{U}$. If p and $\phi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the following differential superordination:

$$h(z) \prec \phi(p(z), zp'(z)) \quad (z \in \mathbb{U}), \tag{1.3}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solution of the differential superordination or, more simply, a subordinant if

$$q \prec p \quad (z \in \mathbb{U}),$$

for all p satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies the following condition:

$$q \prec \tilde{q} \quad (z \in \mathbb{U}),$$

for all subordinants q of (1.3) is said to be the best subordinant.

Definition 1.3. [8] We denote by ϱ the class of functions f that are analytic and injective on $\mathbb{U}\backslash E(f)$, where

$$E(f) = \{\xi : \ \xi \in \partial\,\mathbb{U} \text{ and } \lim_{z \longrightarrow \xi} f(z) = \infty\},$$

and are such that

$$f'(\xi) \neq 0 \quad (\xi \in \partial\mathbb{U}\backslash E(f))\,.$$

We also consider the function classes H and D defined by

$$H := \{f : f \ \in \ A\,[a, 1]\,, f(0) = 0 \text{ and } f'(0) = 1\}$$

and

$$D := \{\varphi : \varphi \ \in \ A, \ \varphi(0) = 1 \text{ and } \varphi(z) \ \neq \ 0 \ (z \ \in \ \mathbb{U})\},$$

respectively.

Let S^* and κ be the subclasses of A consisting of all functions which are, respectively, starlike in $\mathbb{U}$ and convex in $\mathbb{U}$ (see, for details, [7]).

The Srivastava-Attiya operator is defined as [12] (see also [1, 15, 20]):

$$J_{s,a}\,(f)\,(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+a}{k+a}\right)^s a_k z^k, \tag{1.4}$$

where $z \in \mathbb{U}$, $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $s \in \mathbb{C}$ and $f \in A\,[a, 1]$.

In fact, (1.4) can be written as

$$J_{s,a}\,(f)\,(z) := G_{s,a}\,(z) * f\,(z)\,.$$

In terms of Hadamard product (or convolution) where $G_{s,a}\,(z)$ is given by

$$G_{s,a}\,(z) := (1+a)^s \left[\Phi\,(z, s, a) - a^{-s}\right] \quad (z \in \mathbb{U}). \tag{1.5}$$

The function $\Phi\,(z, s, a)$ involved in the right-hand side of (1.5) is the well known Hurwitz-Lerch zeta function defined by (see, for example [19], p. 121 et seq.)

$$\Phi\,(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s}, \tag{1.6}$$

where $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $s \in \mathbb{C}$, when $|z| < 1$, $\mathrm{Re}\,(s) > 1$ when $|z| = 1$.

Recently, a new family of γ-generalized Hurwitz-Lerch zeta function was investigated by Srivastava (see, for example, [14,17,18] and [4]). Srivastava considered the following function:

$$\begin{aligned} &\Phi^{\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q}_{\gamma_1,\ldots,\gamma_p;\mu_1,\ldots,\mu_q}\,(z, s, a; b, \gamma) \\ =&\frac{1}{\gamma\,\Gamma\,(s)} \cdot \sum_{k=0}^{\infty} \frac{\Pi_{j=1}^{p}(\gamma_j)_{k\rho_j}}{(a+k)^s \cdot \Pi_{j=1}^{q}(\mu_j)_{k\sigma_j}} \cdot H^{2,0}_{0,2}\left[(a+k)\,b^{\frac{1}{\gamma}}\Big|\overline{(s,1)\,,\left(0, \frac{1}{\gamma}\right)}\right] \frac{z^k}{k!} \\ &(\min\{\mathrm{Re}\,(a)\,, \mathrm{Re}\,(s)\} > 0; \mathrm{Re}\,(b) > 0; \gamma > 0)\,, \end{aligned} \tag{1.7}$$

where $\gamma_j \in \mathbb{C}$, $j = 1, \dots, p$, and $\mu_j \in \mathbb{C} \backslash \mathbb{Z}_0^-$, $j = 1, \dots, q$, $\rho_j > 0$, $j = 1, \dots, p$, $\sigma_j > 0$, $j = 1, \dots, q$, $1 + \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j \geqslant 0$, and the equality in the convergence condition holds true for suitably bounded values of $|z|$ given by

$$|z| < \nabla := \left(\prod_{j=1}^{p} {\rho_j}^{-\rho_j} \right) \cdot \left(\prod_{j=1}^{q} \sigma_j^{\sigma_j} \right).$$

Here, and for the remainder of this paper, $(\gamma)_k$ denotes the Pochhammer symbol defined in terms of Gamma function, by

$$(\gamma)_k := \frac{\Gamma(\gamma + k)}{\Gamma(\gamma)} = \begin{cases} \gamma(\gamma+1)\cdots(\gamma+n-1) & (k = n \in \mathbb{N},\ \gamma \in \mathbb{C}), \\ 1 & (k = 0,\ \gamma \in \mathbb{C} \backslash \{0\}). \end{cases}$$

Definition 1.4. The H-function involved in the right-hand side of (1.7) is the well-known Fox's H-Function [5, Definition 1.1] (see also [16]) defined by

$$\begin{aligned} H_{p,q}^{m,n}(z) &= H_{p,q}^{m,n}\left[z \middle|_{(b_1,B_1),\dots,(b_q,B_q)}^{(a_1,A_1),\dots,(a_p,A_p)} \right] \\ &= \frac{1}{2\pi i} \int_{\ell} \Xi(s) z^{-s} ds \quad (z \in \mathbb{C} \backslash \{0\}, |\arg(z)| < \pi), \end{aligned}$$

where

$$\Xi(s) = \frac{\Pi_{j=1}^{m} \Gamma(b_j + B_j s) \cdot \Pi_{j=1}^{n} (1 - a_j - A_j s)}{\Pi_{j=n+1}^{p} \Gamma(a_j + A_j s) \cdot \Pi_{j=m+1}^{q} \Gamma(1 - b_j - B_j s)},$$

an empty product is interpreted as 1, m, n, p and q are integers such that $1 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$, $A_j > 0$, $j = 1, \dots, p$, $B_j > 0$, $j = 1, \dots, q$, $a_j \in \mathbb{C}$, $j = 1, \dots, p$, $b_j \in \mathbb{C}$, $j = 1, \dots, q$ and ℓ is a suitable Mellin-Barnes type contour separating the poles of the gamma functions

$$\{\Gamma(b_j + B_j s)\}_{j=1}^{m}$$

from the poles of gamma functions

$$\{\Gamma(1 - a_j + A_j s)\}_{j=1}^{n}.$$

It is worthy to mention that using the fact that [11, p. 1496, Remark 7]

$$\lim_{b \to 0} \left\{ H_{0,2}^{2,0} \left[(a+k)\, b^{\frac{1}{\gamma}} \middle| \overline{(s,1), \left(0, \frac{1}{\gamma}\right)} \right] \right\} = \gamma\, \Gamma(s) \quad (\lambda > 0).$$

Equation (1.6) reduces to

$$\begin{aligned} &\Phi_{\gamma_1,\dots,\gamma_p;\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p;\sigma_1,\dots,\sigma_q)}(z, s, a; 0, \gamma) := \Phi_{\gamma_1,\dots,\gamma_p;\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p;\sigma_1,\dots,\sigma_q)}(z, s, a) \\ &= \sum_{k=0}^{\infty} \frac{\Pi_{j=1}^{p} (\gamma_j)_{k\rho_j}}{(a+k)^s \cdot \Pi_{j=1}^{q} (\mu_j)_{k\sigma_j}} \cdot \frac{z^k}{k!}. \end{aligned} \tag{1.8}$$

Definition 1.5. The function $\Phi^{(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}_{\gamma_1,\ldots,\gamma_p;\mu_1,\ldots,\mu_q}(z,s,a)$ involved in (1.8) is the multi-parameter extension and generalization of the Hurwtiz- Lerch zeta function $\Phi(z,s,a)$ introduced by Srivastava et al. [18, p. 503, Eq. (6.2)], defined by

$$\Phi^{(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}_{\gamma_1,\ldots,\gamma_p;\mu_1,\ldots,\mu_q}(z,s,a) := \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}(\gamma_j)_{k\rho_j}}{(a+k)^s \cdot \prod_{j=1}^{q}(\mu_j)_{k\sigma_j}} \cdot \frac{z^k}{k!},$$

where $p,q \in \mathbb{N}_0,\ \gamma_j \in \mathbb{C},\ j=1,\ldots,p,\ a,\mu_j \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ j=1,\ldots,q,\ \rho_j,\sigma_k \in \mathbb{R}^+$, $j=1,\ldots,p,\ k=1,\ldots,q,\ \Delta > -1$ when $s,z \in \mathbb{C}$, $\Delta=-1$ and $s\in\mathbb{C}$ when $|z|<\nabla^*$, $\Delta=-1$ and $\operatorname{Re}(\Xi) > \frac{1}{2}$ when $|z| = \nabla^*$ with

$$\nabla^* := \left(\prod_{j=1}^{p} \rho_j^{-\rho_j}\right) \cdot \left(\prod_{j=1}^{q} {\sigma_j}^{\sigma_j}\right)$$

and

$$\Delta := \sum_{j=1}^{q}\sigma_j - \sum_{j=1}^{p}\rho_j \ \text{ and } \ \Xi := s + \sum_{j=1}^{q}\mu_j - \sum_{j=1}^{p}\gamma_j + \frac{p-q}{2}.$$

The following linear operator was introduced by Srivastava and Gaboury [13]:

$$J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}(f) :\ A(\mathbb{U}) \to A(\mathbb{U}),$$

defined by

$$J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}(f)(z) = G^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}(z) * f(z), \tag{1.9}$$

where $*$ denotes the Hadamard product (or convolution) of analytic functions and function $G^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}(z)$ is given by

$$\begin{aligned} G^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}(z) :=& \frac{\gamma \prod_{j=1}^{q}(\mu_j)\,\Gamma(s)\,(a+1)^s}{\prod_{j=1}^{p}(\gamma_j)} \Lambda(a+1,b,s,\gamma)^{-1} \\ & \times \left[\Phi^{(1,\ldots,1,1,\ldots,1)}_{\gamma_1,\ldots,\gamma_p;\mu_1,\ldots,\mu_q}(z,s,a;b,\gamma) - \frac{a^{-s}}{\gamma\,\Gamma(s)}\Lambda(a,b,s,\gamma)\right] \\ =& z + \sum_{k=2}^{\infty} \frac{\prod_{j=1}^{p}(\gamma_j+1)_{k-1}}{\prod_{j=1}^{q}(\mu_j+1)_{k-1}} \left(\frac{a+1}{a+k}\right)^s \left(\frac{\Lambda(a+k,b,s,\gamma)}{\Lambda(a+1,b,s,\gamma)}\right) \frac{z^k}{k!}, \end{aligned} \tag{1.10}$$

with

$$\Lambda(a,b,s,\gamma) := H^{2,0}_{0,2}\left[ab^{\frac{1}{\gamma}}\middle|\overline{(s,1),\left(0,\frac{1}{\gamma}\right)}\right].$$

Combining (1.9) and (1.10), we obtain

$$J^{s,\,a,\,\gamma}_{(\gamma_p),(\mu_q),b}(f)(z) = z + \sum_{k=2}^{\infty} \frac{\prod_{j=1}^{p}(\gamma_j+1)_{k-1}}{\prod_{j=1}^{q}(\mu_j+1)_{k-1}} \left(\frac{a+1}{a+k}\right)^s \left(\frac{\Lambda(a+k,b,s,\gamma)}{\Lambda(a+1,b,s,\gamma)}\right) a_k \frac{z^k}{k!},$$

where $\gamma_j \in \mathbb{C},\ j=1,\ldots,p$ and $\mu_j \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ j=1,\ldots,q,\ p \leqslant q+1,\ z\in\mathbb{U}$, with

$$\min\{\operatorname{Re}(a),\operatorname{Re}(s)\} > 0, \gamma > 0 \text{ if } \operatorname{Re}(b) > 0$$

and

$$s \in \mathbb{C}, a \in \mathbb{C}\backslash\mathbb{Z}_0^- \text{ if } b = 0.$$

The following lemmas will be required in our present investigation.

Lemma 1.1. [9, Miller and Mocanu] *Suppose that the function*

$$G : \mathbb{C}^2 \longrightarrow \mathbb{C}$$

satisfies the following condition:

$$\mathrm{Re}\{G(is, t)\} \leq 0,$$

for all real s and for all

$$t \leq -\frac{n(1+s^2)}{2} \quad (n \in \mathbb{N}).$$

If the function

$$p(z) = 1 + p_n z^n + \cdots$$

is analytic in $\mathbb{U}$ and

$$\mathrm{Re}\{G(p(z), zp'(z))\} > 0 \quad (z \in \mathbb{U}),$$

then

$$\mathrm{Re}\{p(z)\} > 0 \quad (z \in \mathbb{U}).$$

Lemma 1.2. [6, Miller and Mocanu] *Let*

$$\beta, \gamma \in \mathbb{C} \quad (\beta \neq 0) \text{ and } h \in A(\mathbb{U}) \quad (h(0) = c).$$

If

$$\mathrm{Re}\{\beta h(z) + \gamma)\} > 0 \quad (z \in \mathbb{U}),$$

then the solution of the following differential equation:

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \quad (z \in \mathbb{U}, q(0) = c)$$

is analytic in $\mathbb{U}$ and satisfies the following inequality:

$$\mathrm{Re}\{\beta q(z) + \gamma\} > 0 \quad (z \in \mathbb{U}).$$

Lemma 1.3. [7, Miller and Mocanu] *Let*

$$p \in \varrho \quad (p(0) = a)$$

and let

$$q(z) = a + a_n z^n + \cdots$$

be analytic in $\mathbb{U}$ with

$$q(z) \neq a \quad (n \in \mathbb{N}).$$

If the function q is not subordinate to p, then there exist points

$$z_0 = r_0 e^{i\theta} \in \mathbb{U} \text{ and } \zeta_0 \in \partial\mathbb{U}\backslash E(f)$$

for which

$$q(\mathbb{U}_{r_0}) \subset p(\mathbb{U}), q(z_0) = p(\zeta_0) \text{ and } z_0 q'(z_0) = m\zeta_0 p'(\zeta_0) \quad (m \geq n).$$

Let

$$N := N(c) = \frac{|c|\sqrt{1+2\mathrm{Re}(c)} + \mathrm{Im}(c)}{\mathrm{Re}(c)} \quad (c \in \mathbb{C}, \mathrm{Re}(c) > 0).$$

If R is the univalent function defined in $\mathbb{U}$ by

$$R(z) = \frac{2Nz}{1-z^2} \quad (z \in \mathbb{U}),$$

then the open door function (see [7]) defined by

$$R_c(z) := R\left(\frac{z+b}{1+\bar{b}z}\right) \quad (b = R^{-1}(c), z \in \mathbb{U}). \tag{1.11}$$

Remark 1.1. The function R_c defined by (1.11) is univalent in $\mathbb{U}$, with $R_c(0) = c$, $R_c(\mathbb{U}) = R(\mathbb{U})$ is the complex plane with slits along the half-lines given by $\mathrm{Re}(w) = 0$ and $|\mathrm{Im}(w)| \geq N$.

Lemma 1.4. [7, Miller and Mocanu] *Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$, $\alpha + \delta = \beta + \gamma$, $\mathrm{Re}(\alpha + \delta) > 0$ and $\phi, \varphi \in D$. If $f \in A_{\varphi,\alpha,\delta}$, where*

$$A_{\varphi,\alpha,\delta} := \left\{ f : f \in H \text{ and } \alpha\frac{zf'(z)}{f(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec R_{\alpha+\delta}(z), z \in \mathbb{U} \right\} \tag{1.12}$$

and $R_{\alpha+\delta}$ is defined by (1.11) with $c = \alpha + \delta$, then

$$J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}(f) \in H, \frac{J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}(f)(z)}{z} \neq 0 \quad (z \in \mathbb{U})$$

and

$$\mathrm{Re}\left(\beta\frac{z(J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}(f)(z))'}{J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}(f)(z)} + \frac{z\phi'(z)}{\phi(z)} + \gamma\right) > 0 \quad (z \in \mathbb{U}),$$

where $J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}(f)(z)$ is given in (1.9).

A function $L(z,t)$ defined on $\mathbb{U} \times [0,\infty)$ is the subordination chain (or lowner chain) if $L(\cdot,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \in [0,\infty)$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$ and $L(z,s) \prec L(z,t)$, $z \in \mathbb{U}$, $0 \leq s \leq t$.

Lemma 1.5. [8, Miller and Mocanu] *Let $q \in A[a,1]$ and $\phi : \mathbb{C}^2 \to \mathbb{C}$. Also set*

$$\mu(q(z), zq'(z)) =: h(z) \quad (z \in \mathbb{U}).$$

If

$$L(z,t) = \mu(q(z), tzq'(z))$$

is a subordination chain and

$$p \in A[a, 1] \cap \varrho,$$

then the following subordination condition:

$$h(z) \prec \mu(p(z), zp'(z)) \quad (z \in \mathbb{U}),$$

implies that

$$q(z) \prec p(z) \quad (z \in \mathbb{U}).$$

Furthermore, if

$$\mu(q(z), tzp'(z)) = h(z)$$

has a univalent solution $q \in \varrho$ then q is the best subordinant.

Lemma 1.6. [10, Pommerenke] *Let the function $L(z, t)$ be given by*

$$L(z, t) = a_1(t)z + \cdots$$

with

$$a_1(t) \neq 0 \text{ and } \lim_{t \to \infty} |a_1(t)| = \infty.$$

Suppose also that $L(\cdot, t)$ is analytic in $\mathbb{U}$ for all $t \geq 0$ and that $L(z, \cdot)$ is continuously differentiable on $[0, \infty)$ for all $z \in \mathbb{U}$. If the function $L(z, t)$ satisfies the following inequalities:

$$\operatorname{Re}\left(\frac{z\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right) > 0 \quad (z \in \mathbb{U}; 0 \leq t < \infty)$$

and

$$|L(z, t)| \leq k_0 |a_1(t)| \quad (|z| < r_0 < 1, 0 \leq t < \infty),$$

for some positive constants k_0 and r_0, then $L(z, t)$ is a subordination chain.

2. A Set of Main Results

An overall subordination property involving (1.9) is included in Theorem 2.1 below.

Theorem 2.1. *Let $f, g \in A_{\varphi,\alpha,\delta}$ where $A_{\varphi,\alpha,\delta}$ is defined by* (1.12). *Suppose also that*

$$\operatorname{Re}\left(1 + \frac{zv''(z)}{v'(z)}\right) > -\rho \quad \left(z \in \mathbb{U}, v(z) := z\left(\frac{g(z)}{z}\right)^{\alpha} \varphi(z)\right), \tag{2.1}$$

where

$$\rho = \frac{1 + |\beta + \gamma - 1|^2 - |1 - (\beta + \gamma - 1)^2|}{4\operatorname{Re}(\beta + \gamma - 1)} \quad (\operatorname{Re}(\beta + \gamma - 1) > 0). \tag{2.2}$$

Then the next subordination relation:

$$z\left(\frac{f(z)}{z}\right)^{\alpha} \varphi(z) \prec z\left(\frac{g(z)}{z}\right)^{\alpha} \varphi(z) \quad (z \in \mathbb{U}) \tag{2.3}$$

indicates that

$$z\left(\frac{J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}\,(f)\,(z)}{z}\right)^{\beta}\phi(z)\prec z\left(\frac{J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}\,(g)\,(z)}{z}\right)^{\beta}\phi(z)\quad(z\in\mathbb{U}).$$

Moreover, the function $z\left(\frac{J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}(g)(z)}{z}\right)^{\beta}\phi(z)$ *is the best dominant.*

Proof. Let us define the functions F and G by

$$F(z):=z\left(\frac{J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}\,(f)\,(z)}{z}\right)^{\beta}\phi(z)\text{ and }G(z):=z\left(\frac{J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}\,(g)\,(z)}{z}\right)^{\beta}\phi(z),\tag{2.4}$$

respectively. We note that the functions F and G are clearly defined by Lemma 1.4. Without loss of generality, suppose that G is analytic and univalent on $\mathbb{U}$ and

$$g'(\zeta)\neq 0\quad(|\zeta|=1).$$

First we show that, if the function q is defined as

$$q(z):=1+\frac{zG''(z)}{G'(z)}\quad(z\in\mathbb{U}),\tag{2.5}$$

then

$$\operatorname{Re}\{q(z)\}>0\quad(z\in\mathbb{U}).$$

Indeed, from the definition of (1.9), we achieve

$$\left(\frac{J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}\,(g)\,(z)}{z}\right)^{\beta}\left(\beta\frac{z(J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}\,(g)\,(z))'}{J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}\,(g)\,(z)}+\gamma\right)\frac{1}{\beta+\gamma}=\left(\frac{g(z)}{z}\right)^{\alpha}\varphi(z).\tag{2.6}$$

Also we have

$$\beta\frac{z(J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}\,(g)\,(z))'}{J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}\,(g)\,(z)}=\beta-1+\frac{zG'(z)}{G(z)}.\tag{2.7}$$

It follows from (2.6) and (2.7) that

$$(\beta+\gamma)v(z)=(\beta+\gamma-1)G(z)+zG'(z).\tag{2.8}$$

Now, by using simple calculation with (2.8), we obtain

$$\begin{aligned}1+\frac{zv''(z)}{v'(z)}&=1+\frac{zG''(z)}{G'(z)}+\frac{zq'(z)}{q(z)+\beta+\gamma-1}\\&=q(z)+\frac{zq'(z)}{q(z)+\beta+\gamma-1}=:h(z).\end{aligned}\tag{2.9}$$

Therefore, from (2.1), we have

$$\operatorname{Re}\{h(z)+\beta+\gamma-1\}>0\quad(z\in\mathbb{U}).$$

Hence, using Lemma 1.2, we can conclude that the differential equation (2.9) includes a solution $q \in A(\mathbb{U})$ with

$$q(0) = h(0) = 1.$$

Now we place

$$H(u, v) = u + \frac{v}{u + \beta + \gamma - 1} + \rho, \tag{2.10}$$

where ρ is given by (2.2). From (2.1), (2.9) and (2.10), we get

$$\mathrm{Re}\{H(q(z), zq'(z))\} > 0 \quad (z \in \mathbb{U}).$$

We now continue to exhibit that

$$\mathrm{Re}\{H(is, t)\} \leq 0 \quad \left(s \in \mathbb{R}, t \leq -\frac{(1+s^2)}{2}\right). \tag{2.11}$$

Indeed, from (2.10), we have the following

$$\begin{aligned} \mathrm{Re}\{H(is, t)\} &= \mathrm{Re}(is + \frac{t}{is + \beta + \gamma - 1} + \rho) \\ &= \frac{t\mathrm{Re}(\beta + \gamma - 1)}{|\beta + \gamma - 1 + is|^2} + \rho \leq -\frac{E_\rho(s)}{2|\beta + \gamma - 1 + is|^2}, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} E_\rho(s) :=& [\mathrm{Re}(\beta + \gamma - 1) - 2\rho]s^2 - 4\rho[\mathrm{Im}(\beta + \gamma - 1)]s - 2\rho|\beta + \gamma - 1|^2 \\ &+ \mathrm{Re}(\beta + \gamma - 1). \end{aligned}$$

For ρ given by (2.2), we noticeably that the coefficient of s^2 in the quadratic expression $E_\rho(s)$ given above is positive or equal to zero and also $E_\rho(s)$ is a perfectly square. Hence, from (2.12), we note that (2.11) holds true. Thus, with the use of Lemma 1.1, we can conclude that

$$\mathrm{Re}\{q(z)\} > 0 \quad (z \in \mathbb{U}),$$

that is, that the function G which defined by (2.4) is convex in $\mathbb{U}$.

Continually we will prove that the subordination condition (2.3) denotes that

$$F(z) \prec G(z) \quad (z \in \mathbb{U}) \tag{2.13}$$

for the functions F and G defined by (2.4). For this aim, we will consider the function $L(z, t)$ given by

$$L(z, t) := \left(\frac{\beta + \gamma - 1}{\beta + \gamma}\right) G(z) + \left(\frac{1 + t}{\beta + \gamma}\right) zG'(z) \quad (z \in \mathbb{U}, 0 \leq t < \infty).$$

We know that

$$\left.\frac{\partial L(z, t)}{\partial z}\right|_{z=0} = G'(0)\left(1 + \frac{t}{\beta + \gamma}\right) \neq 0 \quad (0 \leq t < \infty, \mathrm{Re}(\beta + \gamma - 1) > 0).$$

This indicates that the function $L(z,t)$ given by

$$L(z,t) = a_1(t)z + \cdots \quad (z \in \mathbb{U}, 0 \le t < \infty),$$

which satisfy the following condition:

$$a_1(t) \neq 0 \quad (0 \le t < \infty).$$

Since G is convex in $\mathbb{U}$ and $\operatorname{Re}(\beta+\gamma-1) > 0$, we have

$$\operatorname{Re}\left(\frac{z\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right) = \operatorname{Re}\left\{\beta+\gamma-1+(1+t)\left(1+\frac{zG''(z)}{G'(z)}\right)\right\} > 0 \quad (z \in \mathbb{U}).$$

Moreover, since the function G is convex in $\mathbb{U}$, the following distortion inequalities (see [2]) clearly hold true:

$$\frac{r}{1+r} \le |G(z)| \le \frac{r}{1-r} \quad (|z| \le r < 1) \tag{2.14}$$

and

$$\frac{1}{(1+r)^2} \le |G'(z)| \le \frac{1}{(1-r)^2} \quad (|z| \le r < 1). \tag{2.15}$$

Hence, by using (2.14) and (2.15). It can be easily seen the second supposition of Lemma 1.6 is satisfied. Thus, the function $L(z,t)$ is a subordination chain. From the definition of a subordination chain we note that

$$v(z) = \left(\frac{\beta+\gamma-1}{\beta+\gamma}\right)G(z) + \left(\frac{1}{\beta+\gamma}\right)zG'(z) = L(z,0)$$

and

$$L(z,0) \prec L(z,t) \quad (z \in \mathbb{U}, 0 \le t < \infty).$$

This implies that

$$L(\zeta,t) \notin L(\mathbb{U},0) = v(\mathbb{U}) \quad (\zeta \in \partial\mathbb{U}, 0 \le t < \infty).$$

We now assume that the function F is not subordinate to G, then, by Lemma 1.3, $\exists z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ where

$$F(z_0) = G(\zeta_0) \text{ and } z_0F(z_0) = (1+t)\zeta_0G'(\zeta_0) \quad (0 \le t < \infty).$$

Hence, we get

$$\begin{aligned} L(\zeta_0,t) &= \left(\frac{\beta+\gamma-1}{\beta+\gamma}\right)G(\zeta_0) + \left(\frac{1+t}{\beta+\gamma}\right)\zeta_0G'(\zeta_0) \\ &= \left(\frac{\beta+\gamma-1}{\beta+\gamma}\right)F(z_0) + \left(\frac{1}{\beta+\gamma}\right)z_0F'(z_0) = z\left(\frac{f(z_0}{z_0}\right)^{\alpha}\varphi(z_0) \in v(\mathbb{U}), \end{aligned}$$

by the subordination condition (2.3). This conflicts the previous observation that

$$L(\zeta_0,t) \notin v(\mathbb{U}).$$

Thus, the subordination condition (2.3) should imply the subordination specified by (2.13). By consider $F(z) = G(z)$, we note that the function G is the best dominant. This clearly completes the proof of Theorem 2.1. $\square$

Remark 2.1. We see that ρ specified by (2.2) in Theorem 2.1 satisfies the next inequality: $0 < \rho \leq \frac{1}{2}$.

We will prove a solution to a dual problem of Theorem 2.1, by substitute the subordinations by superordinations.

Theorem 2.2. *let $f, g \in A_{\varphi,\alpha,\delta}$ where $A_{\varphi,\alpha,\delta}$ is defined by* (1.12). *Suppose also that*

$$\operatorname{Re}\left(1+\frac{zv''(z)}{v'(z)}\right) > -\rho \quad \left(z \in \mathbb{U}, v(z) := z\left(\frac{g(z)}{z}\right)^{\alpha}\varphi(z)\right),$$

where ρ is given by (2.2). *If the function*

$$z\left(\frac{f(z)}{z}\right)^{\alpha}\varphi(z)$$

is univalent in $\mathbb{U}$ and

$$z\left(\frac{J^{s,\,a,\,\gamma}_{(\gamma_p),(\mu_q),b}\,(f)\,(z)}{z}\right)^{\beta}\phi(z) \in \varrho,$$

then the following superordination relation:

$$z\left(\frac{g(z)}{z}\right)^{\alpha}\varphi(z) \prec z\left(\frac{f(z)}{z}\right)^{\alpha}\varphi(z) \quad (z \in \mathbb{U}) \tag{2.16}$$

implies that

$$z\left(\frac{J^{s,\,a,\,\gamma}_{(\gamma_p),(\mu_q),b}\,(g)\,(z)}{z}\right)^{\beta}\phi(z) \prec z\left(\frac{J^{s,\,a,\,\gamma}_{(\gamma_p),(\mu_q),b}\,(f)\,(z)}{z}\right)^{\beta}\phi(z) \quad (z \in \mathbb{U}).$$

Furthermore, the function

$$z\left(\frac{J^{s,\,a,\,\gamma}_{(\gamma_p),(\mu_q),b}\,(g)\,(z)}{z}\right)^{\beta}\phi(z)$$

is the best subordinant.

Proof. By defining the functions F and G by (2.4), we first note from (2.6) and (2.7) that

$$v(z) = \left(\frac{\beta+\gamma-1}{\beta+\gamma}\right)G(z) + \left(\frac{1}{\beta+\gamma}\right)zG'(z) =: \mu(G(z), zG'(z)). \tag{2.17}$$

Using a simple calculation, the last Equation (2.17) we can obtian the following relationship:

$$1+\frac{zv''(z)}{v'(z)} = q(z) + \frac{zq'(z)}{q(z)+\beta+\gamma-1},$$

noticeably the function q is determined by (2.5). Then, with the use of the same method as in proof of Theorem 2.1, we obtain that

$$\mathrm{Re}\{q(z)\} > 0 \quad (z \in \mathbb{U}).$$

The function G which defined by (2.4) is convex (univalent) in $\mathbb{U}$.

We prove the following superordination condition (2.16) which implies that

$$F(z) \prec G(z) \quad (z \in \mathbb{U}). \tag{2.18}$$

For this aim, we can consider the function $L(z,t)$ defined by

$$L(z,t) := \left(\frac{\beta+\gamma-1}{\beta+\gamma}\right) G(z) + \left(\frac{t}{\beta+\gamma-1}\right) zG'(z) \quad (z \in \mathbb{U}, 0 \leq t < \infty).$$

As proved previously, the function G is convex in $\mathbb{U}$ and $\mathrm{Re}(\beta+\gamma-1) > 0$, easily we can get that $L(z,t)$ is a subordination chain as we proved in the proof of Theorem 2.1. Thus, according to Lemma 1.5, we achieve that the superordination condition (2.16) should imply the superordination given by (2.18). Moreover, since the differential equation (2.17) includes the univalent solution G, which is the best subordinant of the specified differential superordination. Consequently we have achieve the proof of Theorem 2.2. $\square$

With combine Theorem 2.1 and 2.2, we can get the following sandwich-type theorem.

Theorem 2.3. *Let $f, g_k \in A_{\varphi,\alpha,\delta}$, $k = 1, 2$, where $A_{\varphi,\alpha,\delta}$ is defined by* (1.12). *Suppose also that*

$$\mathrm{Re}\left(1 + \frac{zv_k''(z)}{v_k'(z)}\right) > -\rho \quad \left(z \in \mathbb{U}, v_k(z) := z\left(\frac{g_k(z)}{z}\right)^{\alpha} \varphi(z),\ k = 1, 2\right), \tag{2.19}$$

where ρ is given by (2.2). *If the function $z(\frac{f(z)}{z})^{\alpha}\varphi(z)$ is univalent in $\mathbb{U}$ and*

$$z\left(\frac{J^{s,\,a,\,\gamma}_{(\gamma_p),(\mu_q),b}\,(f)\,(z)}{z}\right)^{\beta} \phi(z) \in \varrho,$$

then the following subordination relations:

$$z\left(\frac{g_1(z)}{z}\right)^{\alpha} \varphi(z) \prec z\left(\frac{f(z)}{z}\right)^{\alpha} \varphi(z) \prec z\left(\frac{g_2(z)}{z}\right)^{\alpha} \varphi(z) \quad (z \in \mathbb{U})$$

imply that

$$z\left(\frac{J^{s,\,a,\,\gamma}_{(\gamma_p),(\mu_q),b}\,(g_1)\,(z)}{z}\right)^{\beta} \phi(z) \prec z\left(\frac{J^{s,\,a,\,\gamma}_{(\gamma_p),(\mu_q),b}\,(f)\,(z)}{z}\right)^{\beta} \phi(z)$$

$$\prec z\left(\frac{J^{s,\,a,\,\gamma}_{(\gamma_p),(\mu_q),b}\,(g_2)\,(z)}{z}\right)^{\beta} \phi(z) \quad (z \in \mathbb{U}).$$

Moreover, the functions

$$z\left(\frac{J^{s,\,a,\,\gamma}_{(\gamma_p),(\mu_q),b}\left(g_1\right)(z)}{z}\right)^{\beta}\phi(z) \text{ and } z\left(\frac{J^{s,\,a,\,\gamma}_{(\gamma_p),(\mu_q),b}\left(g_2\right)(z)}{z}\right)^{\beta}\phi(z)$$

are the best subordinant and the best dominant, respectively.

3. Corollaries and Consequences

The supposition of Theorem 2.3 of the previous section that the functions

$$z\left(\frac{f(z)}{z}\right)^{\alpha} \text{ and } z\left(\frac{J^{s,\,a,\,\gamma}_{(\gamma_p),(\mu_q),b}\left(f\right)(z)}{z}\right)^{\beta}\phi(z)$$

require to be univalent in $\mathbb{U}$ will now be changed by different set of conditions in the next result.

Corollary 3.1. *Let $f, g_k \in A_{\varphi,\alpha,\delta}$, $k = 1, 2$, where $A_{\varphi,\alpha,\delta}$ is defined by* (1.12)*. Suppose also that the condition* (2.19) *is satisfied and*

$$\operatorname{Re}\left(1+\frac{z\psi''(z)}{\psi'(z)}\right) > -\rho \quad \left(z\in\mathbb{U}, \psi(z) := z\left(\frac{f(z)}{z}\right)^{\alpha}\varphi(z), f\in\varrho\right), \tag{3.1}$$

where ρ is given by (2.2)*. Then the following subordination relations:*

$$z\left(\frac{g_1(z)}{z}\right)^{\alpha}\varphi(z) \prec z\left(\frac{f(z)}{z}\right)^{\alpha}\varphi(z) \prec z\left(\frac{g_2(z)}{z}\right)^{\alpha}\varphi(z) \quad (z\in\mathbb{U})$$

imply that

$$\begin{aligned} z\left(\frac{J^{s,\,a,\,\gamma}_{(\gamma_p),(\mu_q),b}\left(g_1\right)(z)}{z}\right)^{\beta}\phi(z) &\prec z\left(\frac{J^{s,\,a,\,\gamma}_{(\gamma_p),(\mu_q),b}\left(f\right)(z)}{z}\right)^{\beta}\phi(z) \\ &\prec z\left(\frac{J^{s,\,a,\,\gamma}_{(\gamma_p),(\mu_q),b}\left(g_2\right)(z)}{z}\right)^{\beta}\phi(z) \quad (z\in\mathbb{U}). \end{aligned}$$

Moreover, the functions

$$z\left(\frac{J^{s,\,a,\,\gamma}_{(\gamma_p),(\mu_q),b}\left(g_1\right)(z)}{z}\right)^{\beta}\phi(z) \text{ and } z\left(\frac{J^{s,\,a,\,\gamma}_{(\gamma_p),(\mu_q),b}\left(g_2\right)(z)}{z}\right)^{\beta}\phi(z)$$

are the best subordinant and the best dominant, respectively.

Proof. So as to prove Corollary 3.1, we must show that the condition (2.19) indicates the univalence of ψ and

$$F(z) := z\left(\frac{J^{s,\,a,\,\gamma}_{(\gamma_p),(\mu_q),b}\left(f\right)(z)}{z}\right)^{\beta}\phi(z). \tag{3.2}$$

□

Since $0 < \rho \leq \frac{1}{2}$, we note from Remark 2.1 and the condition (3.1) that ψ is a close-to-convex function in $\mathbb{U}$ (see [3]) and hence ψ is univalent in $\mathbb{U}$. Moreover, with the use of the same techniques as in the proof of Theorem 2.3, we can show the convexity (univalence) of F defined by (3.2) in $\mathbb{U}$ and so the details may be neglected. Therefore, by using Theorem 2.3. we achieve Corollary 3.1.

By setting $\beta + \gamma = 2$ in Theorem 2.3, the following result can be achieved.

Corollary 3.2. *Let$f, g_k \in A_{\varphi,\alpha,2-\alpha}$, $k = 1, 2$, where $A_{\varphi,\alpha,2-\alpha}$ is defined by* (1.12) *with $\delta = 2 - \alpha$. Suppose also that*

$$\operatorname{Re}\left(1 + \frac{zv_k''(z)}{v_k'(z)}\right) > -\frac{1}{2} \quad \left(z \in \mathbb{U}, v_k(z) := z\left(\frac{g_k(z)}{z}\right)^{\alpha} \varphi(z),\ k = 1, 2\right).$$

If the function

$$z\left(\frac{f(z)}{z}\right)^{\alpha} \varphi(z)$$

is univalent in $\mathbb{U}$ and

$$z\left(\frac{J^{s,\,a,\,(2-\beta)}_{((2-\beta)_p),(\mu_q),b} f(z)}{z}\right)^{\beta} \phi(z) \in \varrho,$$

with $\gamma = 1 - \beta$ and $\delta = 1 - \alpha$, then the following subordination relations:

$$z\left(\frac{g_1(z)}{z}\right)^{\alpha} \varphi(z) \prec z\left(\frac{f(z)}{z}\right)^{\alpha} \varphi(z) \prec z\left(\frac{g_2(z)}{z}\right)^{\alpha} \varphi(z) \quad (z \in \mathbb{U})$$

imply that

$$\begin{aligned} z\left(\frac{J^{s,\,a,\,(2-\beta)}_{((2-\beta)_p),(\mu_q),b} (g_1)(z)}{z}\right)^{\beta} \phi(z) &\prec z\left(\frac{J^{s,\,a,\,(2-\beta)}_{((2-\beta)_p),(\mu_q),b} (f)(z)}{z}\right)^{\beta} \phi(z) \\ &\prec z\left(\frac{J^{s,\,a,\,(2-\beta)}_{((2-\beta)_p),(\mu_q),b} (g_2)(z)}{z}\right)^{\beta} \phi(z) \quad (z \in \mathbb{U}). \end{aligned}$$

Moreover, the functions

$$z\left(\frac{J^{s,\,a,\,(1-\beta)}_{((1-\beta)_p),(\mu_q),b} (g_1)(z)}{z}\right)^{\beta} \phi(z) \text{ and } z\left(\frac{J^{s,\,a,\,(1-\beta)}_{((1-\beta)_p),(\mu_q),b} (g_2)(z)}{z}\right)^{\beta} \phi(z)$$

are the best subordinant and the best dominant, respectively.

By setting $\beta + \gamma = 2 + i$ in Theorem 2.3, we are easily led to the following result.

Corollary 3.3. *Let $f, g_k \in A_{\varphi,\alpha,2+i-\alpha}$, $k = 1, 2$, where $A_{\varphi,\alpha,2+i-\alpha}$ is defined by* (1.12) *with $\delta = 2 + i - \alpha$. Suppose also that*

$$\operatorname{Re}\left(1 + \frac{zv_k''(z)}{v_k'(z)}\right) > -\frac{3 - \sqrt{5}}{4} \quad \left(z \in \mathbb{U}, v_k(z) := z\left(\frac{g_k(z)}{z}\right)^{\alpha} \varphi(z),\ k = 1, 2\right).$$

If the function

$$z\left(\frac{f(z)}{z}\right)^{\alpha}\varphi(z)$$

is univalent in $\mathbb{U}$ *and*

$$z\left(\frac{J^{s,\,a,\,(2+i-\beta)}_{((2+i-\beta)_p),(\mu_q),b}\left(f\right)(z)}{z}\right)^{\beta}\phi(z)\in\varrho,$$

with

$$\gamma=2+i-\beta \text{ and } \delta=2+i-\alpha,$$

then the following subordination relations:

$$z\left(\frac{g_1(z)}{z}\right)^{\alpha}\varphi(z)\prec z\left(\frac{f(z)}{z}\right)^{\alpha}\varphi(z)\prec z\left(\frac{g_2(z)}{z}\right)^{\alpha}\varphi(z)\quad (z\in\mathbb{U})$$

imply that

$$\begin{aligned} z\left(\frac{J^{s,\,a,\,(2+i-\beta)}_{((2+i)-\beta)_p),(\mu_q),b}\left(g_1\right)(z)}{z}\right)^{\beta}\phi(z) &\prec z\left(\frac{J^{s,\,a,\,(2+i-\beta)}_{((2+i-\beta)_p),(\mu_q),b}\left(f\right)(z)}{z}\right)^{\beta}\phi(z)\\ &\prec z\left(\frac{J^{s,\,a,\,(2+i-\beta)}_{((2+i-\beta)_p),(\mu_q),b}\left(g_2\right)(z)}{z}\right)^{\beta}\phi(z)\quad (z\in\mathbb{U}). \end{aligned}$$

Moreover, the functions

$$z\left(\frac{J^{s,\,a,\,(2+i-\beta)}_{((2+i-\beta)_p),(\mu_q),b}\left(g_1\right)(z)}{z}\right)^{\beta}\phi(z) \text{ and } z\left(\frac{J^{s,\,a,\,(2+i-\beta)}_{((2+i-\beta)_p),(\mu_q),b}\left(g_2\right)(z)}{z}\right)^{\beta}\phi(z)$$

are the best subordinate and the best dominant, respectively.

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[1] School of Mathematical Sciences,
University Kebangsaan Malaysia,
Bangi 43600 Selangor D. Ehsan, Malaysia
Email address: khalid_math1363@yahoo.com
Email address: maslina@ukm.edu.my

[2]Department of Mathematics,
University of Sharjah,
Sharjah, United Arab Emirates
Email address: fgahmed@sharjah.ac.ae