

IDENTITIES WITH MULTIPLICATIVE GENERALIZED (α, α) -DERIVATIONS OF SEMIPRIME RINGS

GURNINDER SINGH SANDHU¹, AYŞE AYRAN², AND NEŞET AYDIN²

ABSTRACT. Let R be a semiprime ring and α be an automorphism of R . A mapping $F : R \rightarrow R$ (not necessarily additive) is called multiplicative generalized (α, α) -derivation if there exists a unique (α, α) -derivation d of R such that $F(xy) = F(x)\alpha(y) + \alpha(x)d(y)$ for all $x, y \in R$. In the present paper, we intend to study several algebraic identities involving multiplicative generalized (α, α) -derivations on appropriate subsets of semiprime rings and collect the information about the commutative structure of these rings.

1. INTRODUCTION

Troughout this paper, R denotes an associative semiprime ring with center $Z(R)$. A ring R is said to be prime if for any $a, b \in R$, $aRb = (0)$ implies either $a = 0$ or $b = 0$ and is called semiprime if $aRa = (0)$ implies $a = 0$. It is straight forward to observe that every prime ring is semiprime but the converse is not true in general, e.g., $\mathbb{Z} \times \mathbb{Z}$, which is a semiprime ring but not prime. For a fixed integer $n \geq 1$, a ring is said to be n -torsion free if $nx = 0$ for all $x \in R$ implies $x = 0$. For any $x, y \in R$, we denote the commutator $xy - yx$ and the anti-commutator $xy + yx$ by the symbols $[x, y]$ and $(x \circ y)$, respectively. An additive mapping $d : R \rightarrow R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. The very first example of a derivation is the differential operator Δ on $C[0, 1]$, the ring of the real valued differentiable functions on $[0, 1]$. The notion of derivation has been generalized in many directions. Brešar [6] introduced the notion of generalized derivation, which is an additive mapping

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$F : R \rightarrow R$ satisfying the relation $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, where d is the associated derivation of R . An additive mapping $H : R \rightarrow R$ such that $H(xy) = H(x)y$ for all $x, y \in R$ is called the left multiplier of R . Clearly, with $d = 0$, every left multiplier is a generalized derivation and with $F = d$, every derivation is a generalized derivation. Let $\alpha : R \rightarrow R$ be an automorphism of R . Then an additive mapping $\delta : R \rightarrow R$ is said to be an (α, α) -derivation if $\delta(xy) = \delta(x)\alpha(y) + \alpha(x)\delta(y)$ for all $x, y \in R$. Note that every $(1_R, 1_R)$ -derivation is the ordinary derivation of R , where 1_R stands for the identity mapping of R . Thus one can now think of the notion of generalized (α, α) -derivation, which is a unified notion of both generalized derivation and (α, α) -derivation. Accordingly, an additive mapping $\xi : R \rightarrow R$ is said to be a generalized (α, α) -derivation if there exists a unique (α, α) -derivation δ of R such that $\xi(xy) = \xi(x)\alpha(y) + \alpha(x)\delta(y)$ for all $x, y \in R$. If we drop the assumption of additivity of ξ , then it is called multiplicative generalized (α, α) -derivation associated with (α, α) -derivation δ , e.g. let S be any ring and $R = \{ae_{12} + be_{13} + ce_{23} : \forall a, b, c \in S\}$ be a subring of $M_3(S)$, the ring of 3×3 matrices over S . Define a mapping $F : R \rightarrow R$ by $ae_{12} + be_{13} + ce_{23} \mapsto (bc)e_{13}$, $d : R \rightarrow R$ by $ae_{12} + be_{13} + ce_{23} \mapsto -ae_{12} + be_{13}$ and $\alpha : R \rightarrow R$ by $ae_{12} + be_{13} + ce_{23} \mapsto -ae_{12} + be_{13} - ce_{23}$. Then α is an automorphism of R and F is a multiplicative generalized (α, α) -derivation associated with (α, α) -derivation δ .

The study of commutative structure of prime rings with derivations has been initiated long back by Posner [15]. Precisely, Posner proved that: *If R is a prime ring and d is a derivation of R such that $[d(x), x] \in Z(R)$ for all $x \in R$, then either R is commutative or $d = 0$.* Since then this result has been extended in many directions by a number of algebraists. In 1987, Bell and Martindale [4] extended this result to the class of semiprime rings and proved that: if a semiprime ring R admits a derivation d which is nonzero and centralizing on a left ideal U of R , then R contains a nonzero central ideal. In the same line of investigation, Daif [7] examined the commutativity of semiprime rings admitting derivations that satisfy the identities: (i) $d([x, y]) - [xy] = 0$, (ii) $d([x, y]) + [x, y] = 0$. Further, Fošner et al. [12] studied a more general situation and proved that: if R is a 2-torsion free semiprime ring and d is a derivation of R such that $[[d(x), x], d(x)] = 0$ for all $x \in R$, then D maps R into $Z(R)$. Dhara [8] discussed the generalized derivations of semiprime rings that act as homomorphisms and anti-homomorphism on appropriate subsets of the ring. More specifically, he proved the following.

1. Let R be a semiprime ring, I a nonzero ideal of R and F a nonzero generalized derivation of R associated with a derivation d . If $F(xy) = F(x)F(y)$ for all $x, y \in I$, then $d(I) = (0)$, F is commuting left multiplier mapping on I .
2. Let R be a semiprime ring, I a nonzero ideal of R and F a nonzero generalized derivation of R associated with a derivation d . If $F(xy) = F(y)F(x)$ for all $x, y \in I$, then $d(I) = (0)$, R contains a nonzero central ideal.

Moreover, Dhara and Mozumder [10] generalized these results by studying the identities $F(xy) \pm F(x)F(y) \in Z(R)$ and $F(xy) \pm F(y)F(x) \in Z(R)$ on (semi)prime

ring R , where F is multiplicative generalized derivation of R . Very recently, Tiwari and Sharma [17] studied the following identities: (i) $G(xy) \pm F(x)F(y) \in Z(R)$; (ii) $G(xy) \pm F(x)F(y) \pm \alpha(yx) = 0$; (iii) $G(xy) \pm F(x)F(y) \pm \alpha(xy) \in Z(R)$; (iv) $G(xy) \pm F(x)F(y) \pm \alpha([x, y]) = 0$; (v) $G(xy) \pm F(x)F(y) \pm \alpha(x \circ y) = 0$ on Lie ideals of prime rings, where F and G are generalized (α, α) -derivations and α is an automorphism.

On the other hand, Atteya [2] proved that a semiprime ring R that admits a generalized derivation F , contains a nonzero central ideal if any one of the following identity holds true: (i) $F(xy) \pm xy \in Z(R)$; (ii) $F(xy) \pm yx \in Z(R)$; (iii) $F(x)F(y) - xy \in Z(R)$; (iv) $F(x)F(y) + yx \in Z(R)$ for all $x, y \in I$, a nonzero ideal of R . In 2013, Dhara et al. [9] studied the following identities: (i) $[d(x), F(y)] = \pm[x, y]$; (ii) $[d(x), F(y)] = \pm(x \circ y)$; (iii) $d(x) \circ F(y) = \pm(x \circ y)$; (iv) $d(x) \circ F(y) = \pm[x, y]$ on ideals of semiprime rings, where F is generalized derivation with associated derivation d . Then it seems more interesting to consider such identities with multiplicative derivations. In this direction, Kumar and Sandhu [14] investigated the following identities: (i) $F([x, y]) \pm xy = 0$; (ii) $F([x, y]) \pm yx = 0$; (iii) $F(x \circ y) \pm xy = 0$; (iv) $F(x \circ y) \pm yx = 0$; (v) $d(x)F(y) \pm xy = 0$; (vi) $d(x)F(y) \pm yx = 0$; (vii) $[F(x), y] \pm x \circ G(y) = 0$; (viii) $F(x) \circ y \pm x \circ G(y) = 0$ on semiprime R , where F and G are multiplicative (generalized)-derivations with the associated mappings d and g respectively. In 2017, Tiwari et al. [16] examined the structure of semiprime rings involving two multiplicative (generalized)-derivations that satisfy a list of algebraic identities on appropriate subsets of the ring.

In light of the above discussion, in this paper, our aim is to study certain identities with multiplicative generalized (α, α) -derivations of semiprime rings. More precisely, we characterize the following situations: (i) $G(xy) \pm F(x)F(y) = 0$; (ii) $G(xy) \pm F(x)F(y) \pm xy = 0$; (iii) $F(x)y + yG(x) = 0$; (iv) $F(xy) \pm G(xy) = 0$; (v) $F(xy) \pm G(yx) = 0$; (vi) $\alpha(x) \circ F(y) \pm G(yx) = 0$; (vii) $[\alpha(x), F(y)] \pm G(yx) = 0$; (viii) $\alpha(x) \circ F(y) \pm \alpha([x, y]) = 0$; (ix) $[\alpha(x), F(y)] \pm \alpha(x \circ y) = 0$; (x) $[F(x), d(y)] \pm x \circ \alpha(y) = 0$; (xi) $d(x) \circ d(y) \pm F(xy) = 0$; (xii) $[d(x), d(y)] \pm F(xy) = 0$; (xiii) $[d(x), F(y)] \pm F([x, y]) = 0$; (xiv) $d(x) \circ F(y) \pm F(x \circ y) = 0$, for all x, y in an appropriate subset of R , where F and G are multiplicative generalized (α, α) -derivations with associated (α, α) -derivations d and g , respectively.

2. PRELIMINARY RESULTS

Lemma 2.1 (Brauer's trick [11]). *A group G cannot be the union of two of its proper subgroups.*

Lemma 2.2. ([1, Lemma 2.1]). *If R is a semiprime ring and I is an ideal of R , then I is a semiprime ring.*

Lemma 2.3. ([4, Theorem 3]). *Let R be a semiprime ring and U be a nonzero left ideal of R . If R admits a derivation D which is nonzero on U and centralizing on U , then R contains a nonzero central ideal.*

Lemma 2.4. ([13, Lemma 1.1.5]). *Let R be a semiprime ring and ρ be a right ideal of R . Then $Z(\rho) \subset Z(R)$.*

Lemma 2.5. ([3, Lemma 3.1]). *Let R be a 2-torsion free semiprime ring and U be a nonzero left ideal of R . If $a, b \in R$ such that $axb + bxa = 0$ for all $x \in U$, then $axb = 0 = bxa$ for all $x \in U$.*

Lemma 2.6. ([12, Theorem 1]). *Let R be a 2-torsion free semiprime ring and $D : R \rightarrow R$ be a derivation satisfying the relation $[[D(x), x], D(x)] = 0$ for all $x \in R$. In this case D maps R into $Z(R)$.*

Lemma 2.7. ([5, Lemma 1]). *Let R be a semiprime ring, I be a nonzero ideal of R and $a \in I$ and $b \in R$. If $aIb = (0)$, then $ab = ba = 0$.*

Lemma 2.8. *Let R be a 2-torsion free semiprime ring and I be a nonzero ideal of R . If R admits an (α, α) -derivation d such that $d(I) \subseteq Z(R)$, then $d(R) \subseteq Z(R)$.*

Proof. By hypothesis, we have

$$(2.1) \quad [r, d(x)] = 0, \quad \text{for all } x \in I, r \in R.$$

Replacing x by xs in (2.1), where $s \in R$, we get

$$(2.2) \quad d(x)[r, \alpha(s)] + [r, \alpha(x)]d(s) + \alpha(x)[r, d(s)] = 0.$$

Replacing x by sx in (2.2), we find

$$(2.3) \quad d(s)\alpha(x)[r, \alpha(s)] + \alpha(s)d(x)[r, \alpha(s)] + \alpha(s)[r, \alpha(x)]d(s) \\ + [r, \alpha(s)]\alpha(x)d(s) + \alpha(s)\alpha(x)[r, d(s)] = 0.$$

Utilization of equation (2.2) in (2.3) gives that

$$d(s)\alpha(x)[r, \alpha(s)] + [r, \alpha(s)]\alpha(x)d(s) = 0, \quad \text{for all } x \in I, r, s \in R.$$

Applying Lemma 2.5, we have

$$(2.4) \quad d(s)\alpha(x)[r, \alpha(s)] = 0.$$

Taking $x\alpha^{-1}(p)$ instead of x in (2.4), we get

$$d(s)\alpha(x)p[r, \alpha(s)] = 0, \quad \text{for all } x \in I, r, s, p \in R.$$

We now can easily arrive at $d(s)[\alpha(x), \alpha(s)]pd(s)[\alpha(x), \alpha(s)] = 0$ for all $x \in I$ and $s, p \in R$. It forces that $d(s)[\alpha(x), \alpha(s)] = 0$ for all $x \in I$ and $s \in R$. Linearizing on s , we get

$$(2.5) \quad d(r)[\alpha(x), \alpha(s)] + d(s)[\alpha(x), \alpha(r)] = 0, \quad \text{for all } x \in I, r, s \in R.$$

Substituting sx in place of s in (2.5), we get

$$(2.6) \quad d(r)[\alpha(x), \alpha(s)]\alpha(x) + d(s)\alpha(x)[\alpha(x), \alpha(r)] + \alpha(s)d(x)[\alpha(x), \alpha(r)] = 0.$$

Right multiply (2.5) by $\alpha(x)$ and then subtract from (2.6) to obtain

$$d(s)[\alpha(x), [\alpha(x), \alpha(r)]] + \alpha(s)d(x)[\alpha(x), \alpha(r)] = 0, \quad \text{for all } x \in I, r, s \in R.$$

In particular, taking $r = x$ in (2.5), we get $d(x)[\alpha(x), \alpha(s)] = 0$ for all $x \in I$ and $s \in R$. Using this in the above relation, we get

$$(2.7) \quad d(s)[\alpha(x), [\alpha(x), \alpha(r)]] = 0, \quad \text{for all } x \in I, r, s \in R.$$

Replacing r by rp in (2.7), where $p \in R$, we find

$$\begin{aligned} d(s)\alpha(r)[\alpha(x), [\alpha(x), \alpha(p)]] + 2d(s)[\alpha(x), \alpha(r)][\alpha(x), \alpha(p)] \\ + d(s)[\alpha(x), [\alpha(x), \alpha(r)]]\alpha(p) = 0. \end{aligned}$$

Using (2.7), we get

$$d(s)\alpha(r)[\alpha(x), [\alpha(x), \alpha(p)]] + 2d(s)[\alpha(x), \alpha(r)][\alpha(x), \alpha(p)] = 0.$$

In view of (2.7), it can be re-written as

$$-\alpha(s)d(r)[\alpha(x), [\alpha(x), \alpha(p)]] + 2d(s)[\alpha(x), \alpha(r)][\alpha(x), \alpha(p)] = 0.$$

Again using (2.7), we get $2d(s)[\alpha(x), \alpha(r)][\alpha(x), \alpha(p)] = 0$ for all $x \in I$ and $r, s, p \in R$. Since R is 2-torsion free, we obtain

$$d(s)[\alpha(x), \alpha(r)][\alpha(x), \alpha(p)] = 0.$$

Replacing p by qp , where $q \in R$, we get

$$d(s)[\alpha(x), \alpha(r)]\alpha(q)[\alpha(x), \alpha(p)] = 0.$$

Replacing q by $q\alpha^{-1}(d(s))$ and put $p = r$ in the above relation, we get

$$d(s)[\alpha(x), \alpha(r)]Rd(s)[\alpha(x), \alpha(r)] = (0), \quad \text{for all } x \in I, r, s \in R.$$

Since R is semiprime, we get

$$d(s)[\alpha(x), \alpha(r)] = 0, \quad \text{for all } x \in I, r, s \in R.$$

Replacing s by su in the above relation

$$(2.8) \quad d(s)\alpha(u)[\alpha(x), \alpha(r)] = 0, \quad \text{for all } x \in I, r, s, u \in R.$$

Replacing u by xu in (2.8), we get

$$(2.9) \quad d(s)\alpha(x)\alpha(u)[\alpha(x), \alpha(r)] = 0, \quad \text{for all } x \in I, r, s, u \in R.$$

Left multiply (2.8) by $\alpha(x)$ and then subtract from (2.9), we obtain

$$[d(s), \alpha(x)]\alpha(u)[\alpha(x), \alpha(r)] = 0.$$

In particular, put $r = \alpha^{-1}(d(s))$, we get

$$[\alpha(x), d(s)]\alpha(u)[\alpha(x), d(s)] = 0, \quad \text{for all } x \in I, u, s \in R.$$

It implies that $[\alpha(x), d(s)] = 0$ for all $x \in I$ and $s \in R$. That means $d(R) \subseteq Z(I')$, where $I' = \alpha(I)$. And hence by Lemma 2.4, we are done. \square

3. MAIN RESULTS

Theorem 3.1. *Let R be a semiprime ring, I be a nonzero ideal of R and α be an automorphism of R . Let F and G be multiplicative generalized (α, α) -derivations of R associated with nonzero (α, α) -derivations d and g respectively. If $G(xy) \pm F(x)F(y) = 0$ for all $x, y \in I$, then R contains a nonzero central ideal and d and g maps R into $Z(R)$.*

Proof. Assume that

$$(3.1) \quad G(xy) + F(x)F(y) = 0, \quad \text{for all } x, y \in I.$$

Replacing y by yr in (3.1), where $r \in R$, we find

$$\begin{aligned} 0 &= G(xyr) + F(x)F(yr) \\ &= G(xy)\alpha(r) + \alpha(xy)g(r) + F(x)F(y)\alpha(r) + F(x)\alpha(y)d(r). \end{aligned}$$

By using (3.1), we get

$$(3.2) \quad 0 = \alpha(xy)g(r) + F(x)\alpha(y)d(r), \quad \text{for all } x, y \in I, r \in R.$$

Replacing x by xs in (3.2), where $s \in R$, we get

$$(3.3) \quad 0 = \alpha(xsy)g(r) + F(x)\alpha(s)\alpha(y)d(r) + \alpha(x)d(s)\alpha(y)d(r).$$

Putting sy instead of y in (3.2), where $s \in R$, we get

$$(3.4) \quad 0 = \alpha(xsy)g(r) + F(x)\alpha(sy)d(r).$$

Combining (3.3) and (3.4), we obtain

$$0 = \alpha(x)d(s)\alpha(y)d(r), \quad \text{for all } x, y \in I, r, s \in R.$$

Substituting wy for y and then putting $y = x$ and $s = r$, where $w \in R$, we have

$$0 = \alpha(x)d(r)\alpha(w)\alpha(x)d(r), \quad \text{for all } x \in I, r, w \in R.$$

Since R is semiprime ring, we find that

$$(3.5) \quad 0 = \alpha(x)d(r), \quad \text{for all } x \in I, r \in R.$$

Replacing x by xy in (3.5), where $y \in I$, we have

$$(3.6) \quad 0 = \alpha(x)\alpha(y)d(r), \quad \text{for all } x, y \in I, r \in R.$$

Because I is ideal of R , $\alpha(I)$ is ideal of R . In view of Lemma 2.7, we have $\alpha(x)d(r) = 0$ and $d(r)\alpha(x) = 0$ for all $x \in I, r \in R$. That is

$$(3.7) \quad 0 = [\alpha(x), d(r)], \quad \text{for all } x \in I, r \in R.$$

Moreover, a particular case of (3.7) implies that $[x, \varphi(x)] = 0$ for all $x \in I$, where $\varphi = \alpha^{-1}d$ is an ordinary derivation of R . By Lemma 2.3, R contains a nonzero central ideal of R . Further, from equation (3.7), we have $d(r) \in Z(I)$ for all $r \in R$. In view of Lemma 2.4, we conclude that d maps R into $Z(R)$.

Using (3.4) in (3.5), we obtain $0 = \alpha(x)\alpha(s)\alpha(y)g(r)$ for all $x, y \in I$ and $r, s \in R$. Taking $\alpha^{-1}(g(r))s$ instead of s in the last expression, we get in particular

$$(0) = \alpha(x)g(r)R\alpha(x)g(r), \quad \text{for all } x \in I, r \in R.$$

It yields that

$$0 = \alpha(x)g(r), \quad \text{for all } x \in I, r \in R.$$

Since this expression is same as (3.5) but with g instead of d . Therefore, the same technique implies g maps R into $Z(R)$, as desired.

Using similar approach, we conclude that the same result holds for $G(xy) - F(x)F(y) = 0$ for all $x, y \in I$. \square

Theorem 3.2. *Let R be a semiprime ring, I be a nonzero ideal of R and α be an automorphism of R . Let F and G be multiplicative generalized (α, α) -derivations of R associated with nonzero (α, α) -derivations d and g respectively. If $G(xy) \pm F(x)F(y) \pm xy = 0$ for all $x, y \in I$ then R contains a nonzero central ideal and, d and g maps R into $Z(R)$.*

Proof. Assume that

$$(3.8) \quad G(xy) + F(x)F(y) + xy = 0, \quad \text{for all } x, y \in I.$$

Replacing y by yr in (3.8), where $r \in R$, we have

$$(3.9) \quad 0 = G(xy)\alpha(r) + \alpha(xy)g(r) + F(x)F(y)\alpha(r) + F(x)\alpha(y)d(r) + xyr.$$

Right multiplying (3.8) by $\alpha(r)$, we get

$$(3.10) \quad 0 = G(xy)\alpha(r) + F(x)F(y)\alpha(r) + xy\alpha(r), \quad \text{for all } x, y \in I, r \in R.$$

Subtracting (3.10) from (3.9), we obtain

$$(3.11) \quad 0 = \alpha(xy)g(r) + F(x)\alpha(y)d(r) + xyr - xy\alpha(r), \quad \text{for all } x, y \in I, r \in R.$$

Replacing x by xs in (3.11), where $s \in R$, we have

$$(3.12) \quad 0 = \alpha(xsy)g(r) + F(x)\alpha(s)\alpha(y)d(r) + \alpha(x)d(s)\alpha(y)d(r) + xsyr - xsy\alpha(r).$$

Again replacing y by sy in (3.11), where $s \in R$, we get

$$(3.13) \quad 0 = \alpha(xsy)g(r) + F(x)\alpha(sy)d(r) + xsyr - xsy\alpha(r).$$

Combining (3.12) and (3.13), we obtain

$$(3.14) \quad 0 = \alpha(x)d(s)\alpha(y)d(r), \quad \text{for all } x, y \in I, r, s \in R.$$

Substituting wx for y and r for s in (3.14), where $w \in R$, we have

$$0 = \alpha(x)d(r)\alpha(w)\alpha(x)d(r), \quad \text{for all } x \in I, r, w \in R.$$

It follows that

$$0 = \alpha(x)d(r), \quad \text{for all } x \in I, r \in R.$$

This expression also appeared as equation (3.5) in Theorem 3.1, so the result is followed in the same way.

Now replacing x by ux in (3.9), where $u \in R$, we get

$$0 = \alpha(u)\alpha(xy)g(r) + uxyr - uxy\alpha(r).$$

Combining with the above relation, it implies that $(\alpha(u) - u)\alpha(xy)g(r) = 0$ for all $x, y \in I$ and $u, r \in R$. Using (3.11), it gives

$$(\alpha(u) - u)xy(\alpha(r) - r) = 0, \quad \text{for all } x, y \in I, r, u \in R.$$

It implies that $x\alpha(r) = xr$ for all $x \in I$ and $r \in R$. Now, using this in (3.11), we obtain $0 = \alpha(xy)g(r)$ for all $x, y \in I$ and $r \in R$. That is, $\alpha(x)\alpha(y)g(r) = 0$. Since α is an automorphism, replacing x by $x\alpha^{-1}(g(r))$ in the last relation to get $\alpha(x)g(r)\alpha(y)g(r) = 0$. Further it implies that $\alpha(x)g(r)R\alpha(y)g(r) = (0)$ for all $x, y \in I$ and $r \in R$. In particular, $\alpha(x)g(r)R\alpha(x)g(r) = (0)$ for all $x \in I$ and $r \in R$. Hence, we get $0 = \alpha(x)g(r)$ for all $x \in I$ and $r \in R$. By repeating the similar argument as above, we get our conclusion.

Using similar approach, we conclude that the same result holds for $G(xy) - F(x)F(y) - xy = 0$, $G(xy) - F(x)F(y) + xy = 0$ and $G(xy) + F(x)F(y) - xy = 0$ for all $x, y \in I$. \square

Theorem 3.3. *Let R be a semiprime ring, I be a nonzero ideal of R and α be an automorphism on R such that $\alpha(I) = I$. Let F and G be multiplicative generalized (α, α) -derivations of R associated with nonzero (α, α) -derivations d and g respectively. If $F(x)y + yG(x) = 0$ for all $x, y \in I$, then F maps I into $Z(R)$ and G maps I into $Z(R)$. Moreover, R contains a nonzero central ideal and $d + g$ maps R into $Z(R)$.*

Proof. Assume that

$$(3.15) \quad F(x)y + yG(x) = 0, \quad \text{for all } x, y \in I.$$

Substituting xz in place of x in (3.15), where $z \in I$, we obtain

$$(3.16) \quad 0 = F(x)\alpha(z)y + \alpha(x)d(z)y + yG(x)\alpha(z) + y\alpha(x)g(z).$$

Replacing y by $\alpha(z)y$ in (3.15), we have

$$(3.17) \quad 0 = F(x)\alpha(z)y + \alpha(z)yG(x), \quad \text{for all } x, y \in I, z \in R.$$

Subtracting (3.17) from (3.16) and using (3.15), we get

$$(3.18) \quad 0 = -F(x)y\alpha(z) + \alpha(z)F(x)y + \alpha(x)d(z)y + y\alpha(x)g(z).$$

Combining (3.16) and (3.18), we get

$$0 = [\alpha(z), F(x)]y, \quad \text{for all } x, y, z \in I.$$

Since I is semiprime, we have

$$0 = [\alpha(z), F(x)], \quad \text{for all } x, z \in I.$$

That is $F(I) \subset Z(I)$. In view of Lemma 2.4, we find $F(I) \subset Z(R)$. Now right multiplying (3.15) by $\alpha(z)$, where $z \in I$, we obtain

$$(3.19) \quad 0 = F(x)y\alpha(z) + yG(x)\alpha(z), \quad \text{for all } x, y, z \in I.$$

Replacing y with $y\alpha(z)$ in (3.15), we have

$$(3.20) \quad 0 = F(x)y\alpha(z) + y\alpha(z)G(x), \quad \text{for all } x, y, z \in I.$$

Application (3.19) and (3.20) yields that

$$0 = y[\alpha(z), G(x)], \quad \text{for all } x, y, z \in I.$$

By semiprimeness of I , we get

$$0 = [\alpha(z), G(x)], \quad \text{for all } x, y, z \in I.$$

That is $G(I) \subset Z(I)$. In view of Lemma 2.4, we find $G(I) \subset Z(R)$. In view of $F(I) \subset Z(I)$ and $G(I) \subset Z(I)$, our hypothesis yields

$$(3.21) \quad 0 = y(F + G)(x), \quad \text{for all } x, y \in I.$$

Replacing x by xr in (3.21), where $r \in R$, we find

$$(3.22) \quad 0 = y(F + G)(x)\alpha(r) + y\alpha(x)(d + g)(r), \quad \text{for all } x, y \in I, r \in R.$$

Application of (3.21) in (3.22), we have

$$(3.23) \quad 0 = y\alpha(x)(d + g)(r), \quad \text{for all } x, y \in I, r \in R.$$

In view of Lemma 2.7 and the fact that $\alpha(I) = I$, we have $y(d + g)(r) = 0$ and $(d + g)(r)y = 0$ for all $x \in I, r \in R$. That is

$$(3.24) \quad 0 = [y, (d + g)(r)], \quad \text{for all } y \in I, r \in R.$$

Moreover, a particular case of (3.24) implies that $[\alpha(y), (d + g)(y)] = 0$ for all $y \in I$. That is $[y, \varphi(y)] = 0$ for all $y \in I$, where $\varphi = \alpha^{-1}(d + g)$ is an ordinary derivation of R . By Lemma 2.3, R contains a nonzero central ideal of R . Applying Lemma 2.4 on relation (3.24), we find that $(d + g)(R) \subseteq Z(R)$. \square

Theorem 3.4. *Let R be a semiprime ring, I be a nonzero ideal of R and α be an automorphism of R . Let F and G be multiplicative generalized (α, α) -derivations associated with nonzero (α, α) -derivations d and g respectively. If $F(xy) \pm G(xy) = 0$ for all $x, y \in I$, then $d \pm g$ maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. By the given hypothesis, we have

$$F(xy) \pm G(xy) = 0 = (F \pm G)(xy), \quad \text{for all } x, y \in I.$$

Since sum of two multiplicative generalized (α, α) -derivations is a multiplicative generalized (α, α) -derivation, we take H in place of $F \pm G$, therefore our condition becomes $H(xy) = 0$ for all $x, y \in I$. Which is a particular case of our Theorem 3.1 (with $F = 0$). Hence, we are done. \square

Theorem 3.5. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R . Let F and G be multiplicative generalized (α, α) -derivations of R associated with nonzero (α, α) -derivations d and g respectively. If $F(xy) \pm G(yx) = 0$ for all $x, y \in I$, then d and g maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. Assume that

$$(3.25) \quad F(xy) + G(yx) = 0, \quad \text{for all } x, y \in I.$$

Replacing x by xr in (3.25), where $r \in R$, we have

$$(3.26) \quad 0 = F(xry) + G(yxr), \quad \text{for all } x, y \in I, r \in R.$$

Again replace y with ry in (3.25), we find

$$(3.27) \quad 0 = F(xry) + G(ryx), \quad \text{for all } x, y \in I, r \in R.$$

Combining (3.26) and (3.27), we get

$$G(yxr) = G(ryx), \quad \text{for all } x, y \in I, r \in R.$$

Putting yz instead of y in (3.25), where $z \in I$ and using (3.25), we get

$$(3.28) \quad 0 = -G(yx)\alpha(z) + G(yz)\alpha(x) + \alpha(xy)d(z) + \alpha(yz)g(x), \quad \text{for all } x, y, z \in I.$$

Substituting xs in place of x in (3.28), where $s \in R$, we obtain

$$(3.29) \quad -G(yxs)\alpha(z) + G(yz)\alpha(xs) + \alpha(xsy)d(z) + \alpha(yz)g(xs) = 0.$$

Taking sy instead of y in (3.28), where $s \in R$, we have

$$(3.30) \quad -G(syx)\alpha(z) + G(syz)\alpha(x) + \alpha(xsy)d(z) + \alpha(syz)g(x) = 0.$$

Combining (3.29) and (3.30) and using $G(yxs) = G(syx)$, we get

$$0 = G(yz)\alpha(xs) + \alpha(yz)g(xs) - G(syz)\alpha(x) - \alpha(syz)g(x).$$

Using $G(syz) = G(yzs)$ in last expression, we find

$$G(yz)\alpha(xs) + \alpha(yz)g(xs) - G(yzs)\alpha(x) - \alpha(syz)g(x) = 0, \quad \text{for all } x, y, z \in I, s \in R.$$

That is

$$(3.31) \quad G(yz)\alpha([x, s]) + \alpha(yz)(g(x)\alpha(s) - g(s)\alpha(x)) + \alpha(yzx)g(s) - \alpha(syz)g(x) = 0,$$

for all $x, y, z \in I, s \in R$. Replacing s by x in (3.31), we find

$$(3.32) \quad 0 = \alpha([yz, x])g(x), \quad \text{for all } x, y, z \in I.$$

Putting wy instead of y in (3.32), where $w \in I$, we have $0 = \alpha([w, x])\alpha(y)\alpha(z)g(x)$ for all $x, y, z, w \in I$. It implies that $(0) = \alpha([w, x])\alpha(I)\alpha(z)g(x)$ for all $x, y, z \in I$. Because I is an ideal of R , $\alpha(I)$ is an ideal of R . In view of Lemma 2.7, we have

$$(3.33) \quad 0 = \alpha([w, x])\alpha(z)g(x), \quad \text{for all } x, y, z \in I.$$

That is, $[I, x]I((\alpha^{-1}g)(x)) = (0)$ for all $x \in I$. Since I is a semiprime ring in itself, it must contain a family P of prime ideals such that $\cap P_\lambda = (0)$. Let P_{λ_1} be a typical member of this family and $x \in I$; by (3.33), we have find

$$[I, x] \subset P_{\lambda_1} \quad \text{or} \quad ((\alpha^{-1}g)(x)) \subset P_{\lambda_1}.$$

Let $A = \{x \in I : [I, x] \subset P_{\lambda_1}\}$ and $B = \{x \in I : ((\alpha^{-1}g)(x)) \subset P_{\lambda_1}\}$. Note that A and B are additive subgroups of I such that $A \cup B = I$. By using Brauer's trick, we obtain

$$[I, I] \subset P_{\lambda_1} \quad \text{or} \quad ((\alpha^{-1}g)(I)) \subset P_{\lambda_1}.$$

Together with these both cases, we have $[I, I]((\alpha^{-1}g)(I)) = (0)$. That is

$$(3.34) \quad \alpha([x, y])g(z) = 0, \quad \text{for all } x, y, z \in I.$$

Replacing y by ry in (3.34), where $r \in R$, we have $\alpha([x, r])\alpha(y)g(z) = 0$. That is

$$(3.35) \quad [\alpha(x), r]\alpha(y)g(z) = 0, \quad \text{for all } x, y, z \in I, r \in R.$$

Right multiplying (3.35) by $\alpha(x)$, we get

$$(3.36) \quad [\alpha(x), r]\alpha(y)g(z)\alpha(x) = 0.$$

Substituting yx for y in (3.35), we have

$$(3.37) \quad [\alpha(x), r]\alpha(y)\alpha(x)g(z) = 0.$$

Combining (3.36) and (3.37), we obtain $[\alpha(x), r]\alpha(y)[\alpha(x), g(z)] = 0$ for all $x, y, z \in I, r \in R$. In particular, we have $[\alpha(x), g(z)]\alpha(I)[\alpha(x), g(z)] = (0)$ for all $x, z \in I$. Since $\alpha(I)$ is semiprime ring, we obtain

$$(3.38) \quad [\alpha(x), g(z)] = 0, \quad \text{for all } x, z \in I.$$

That is $g(I) \subset Z(I')$, where $I' = \alpha(I)$. By Lemma 2.4, $g(I) \subset Z(R)$. In view of Lemma 2.8, we get $g(R) \subset Z(R)$. Moreover, a particular case of (3.38) implies that $[x, \varphi(x)] = 0$ for all $x \in I$, where $\varphi = \alpha^{-1}g$ is an ordinary derivation of R . By Lemma 2.3, R contains a nonzero central ideal of R .

Now from (3.25), by replacing y by yr , where $r \in R$, we get

$$(3.39) \quad F(xyr) + G(yrx) = 0, \quad \text{for all } x, y \in I, r \in R.$$

And replacing x by rx in (3.25), we obtain

$$(3.40) \quad F(rxy) + G(yrx), \quad \text{for all } x, y \in I, r \in R.$$

Combining (3.39) and (3.40), we get

$$F(xyr) = F(rxy), \quad \text{for all } x, y \in I, r \in R.$$

Interchanging the role of x and y in this expression, we obtain

$$F(yxr) = F(ryx), \quad \text{for all } x, y \in I, r \in R.$$

This relation has already existed in the above proof for G . Hence by repeating the same arguments, we get that d maps R into $Z(R)$. □

Theorem 3.6. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R . Let F and G be multiplicative generalized (α, α) -derivations of R associated with nonzero (α, α) -derivations d and g respectively. If $\alpha(x) \circ F(y) \pm G(yx) = 0$ for all $x, y \in I$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. Assume that $\alpha(x) \circ F(y) \pm G(yx) = 0$ for all $x, y \in I$. That is

$$(3.41) \quad \alpha(x)F(y) + F(y)\alpha(x) \pm G(yx) = 0.$$

Replacing y by yx in (3.41), we find

$$\begin{aligned} &\alpha(x)F(y)\alpha(x) + \alpha(x)\alpha(y)d(x) + F(y)\alpha(x)\alpha(x) \\ &+ \alpha(y)d(x)\alpha(x) \pm G(yx)\alpha(x) \pm \alpha(yx)g(x) = 0. \end{aligned}$$

Expression (3.41) reduces it to

$$(3.42) \quad \alpha(x)\alpha(y)d(x) + \alpha(y)d(x)\alpha(x) \pm \alpha(yx)g(x) = 0.$$

Taking zy in place of y in (3.42), where $z \in I$, we get

$$(3.43) \quad \alpha(x)\alpha(z)\alpha(y)d(x) + \alpha(z)\alpha(y)d(x)\alpha(x) \pm \alpha(z)\alpha(yx)g(x) = 0.$$

By using (3.42), we find $[\alpha(x), \alpha(z)]\alpha(y)d(x) = 0$ for all $x, y, z \in I$. By using Lemma 2.7, it yields that $[\alpha(x), \alpha(y)]d(x) = 0$ for all $x, y \in I$. Replacing y by $\alpha^{-1}(r)y$, where $r \in R$, we get

$$[\alpha(x), r]\alpha(y)d(x) = 0, \quad \text{for all } x, y \in I, r \in R.$$

It implies that

$$(3.44) \quad [\alpha(x), R]R\alpha(I)d(x) = (0).$$

Since R contains a family S of prime ideals such that $\cap P_\lambda = (0)$. Let P be a typical member of this family and $x \in I$, by (3.44), we find

$$[\alpha(x), R] \subset P \quad \text{or} \quad \alpha(I)d(x) \subset P.$$

Let $A = \{x \in I : [\alpha(x), R] \subset P\}$ and $B = \{x \in I : \alpha(I)d(x) \subset P\}$. Note that A and B are the additive subgroups of R such that $A \cup B = I$. By using Brauer's trick, we obtain

$$[\alpha(I), R] \subset P \quad \text{or} \quad \alpha(I)d(I) \subset P.$$

Together with these both cases, we have $[\alpha(I), R]d(I) = 0$. That is

$$(3.45) \quad [\alpha(x), r]d(y) = 0, \quad \text{for all } x, y \in I, r \in R.$$

Replacing y by yt , in (3.45), where $t \in I$, we find

$$(3.46) \quad [\alpha(x), r]\alpha(y)d(t) = 0.$$

This expression is same as (3.35) of Theorem 3.5 with d instead of g . With the similar arguments, we get our conclusion. \square

By the same implications as in Theorem 3.6 with necessary modifications, we can get the following result.

Theorem 3.7. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R . Let F and G be multiplicative generalized (α, α) -derivations of R associated with nonzero (α, α) -derivations d and g , respectively. If $[\alpha(x), F(y)] \pm G(yx) = 0$ for all $x, y \in I$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Theorem 3.8. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R . Let F be a multiplicative generalized (α, α) -derivation of R associated with a nonzero (α, α) -derivation d . If $\alpha(x) \circ F(y) \pm \alpha([x, y]) = 0$ for all $x, y \in I$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. Assume that

$$(3.47) \quad \alpha(x) \circ F(y) \pm \alpha([x, y]) = 0, \quad \text{for all } x, y \in I.$$

Replacing y by yx in (3.47), we find

$$(\alpha(x) \circ F(y))\alpha(x) + (\alpha(x) \circ \alpha(y)d(x)) \pm \alpha([x, y])\alpha(x) = 0, \quad \text{for all } x, y \in I.$$

Equation (3.47) reduces it to

$$(3.48) \quad (\alpha(x) \circ \alpha(y)d(x)) = 0.$$

Replacing y by zy in (3.48), where $z \in I$, and using it, we get $[\alpha(x), \alpha(z)]\alpha(y)d(x) = 0$ for all $x, y, z \in I$. This expression also appeared in Theorem 3.6, hence the conclusion follows in the similar manner. □

By the same implications as in Theorem 3.8 with necessary modifications, we can get the following result.

Theorem 3.9. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R . Let F be a multiplicative generalized (α, α) -derivation of R associated with a nonzero (α, α) -derivation d . If $[\alpha(x), F(y)] \pm \alpha(x \circ y) = 0$ for all $x, y \in I$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Theorem 3.10. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R . Let F be a multiplicative generalized (α, α) -derivation of R associated with a nonzero (α, α) -derivation d . If $[F(x), d(y)] \pm (x \circ \alpha(y)) = 0$ for all $x, y \in I$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. Assume that $[F(x), d(y)] - (x \circ \alpha(y)) = 0$ for all $x, y \in I$. That is

$$(3.49) \quad F(x)d(y) - d(y)F(x) - x\alpha(y) - \alpha(y)x = 0.$$

Replacing y by yz in (3.49), where $z \in I$, we find

$$\begin{aligned} &F(x)d(y)\alpha(z) + F(x)\alpha(y)d(z) - d(y)\alpha(z)F(x) \\ &- \alpha(y)d(z)F(x) - x\alpha(y)\alpha(z) - \alpha(y)\alpha(z)x = 0. \end{aligned}$$

Right multiplying (3.49) by $\alpha(z)$ and then comparing with the above expression to obtain

$$(3.50) \quad [F(x), \alpha(y)d(z)] + d(y)[F(x), \alpha(z)] - \alpha(y)[\alpha(z), x] = 0, \quad \text{for all } x, y, z \in I.$$

Taking zy instead of y in (3.50), we have

$$\begin{aligned} &\alpha(z)[F(x), \alpha(y)d(z)] + [F(x), \alpha(z)]\alpha(y)d(z) + d(z)\alpha(y)[F(x), \alpha(z)] \\ &+ \alpha(z)d(y)[F(x), \alpha(z)] - \alpha(z)\alpha(y)[\alpha(z), x] = 0, \quad \text{for all } x, y, z \in I. \end{aligned}$$

Application of (3.50) yields

$$[F(x), \alpha(z)]\alpha(y)d(z) + d(z)\alpha(y)[F(x), \alpha(z)] = 0, \quad \text{for all } x, y, z \in I.$$

In view of Lemma 2.5, it follows that

$$[F(x), \alpha(z)]\alpha(y)d(z) = 0 = d(z)\alpha(y)[F(x), \alpha(z)], \quad \text{for all } x, y, z \in I.$$

Let us consider the expression

$$(3.51) \quad [F(x), \alpha(z)]\alpha(y)d(z) = 0.$$

Replacing x by xz in (3.51), we get

$$(3.52) \quad [F(x), \alpha(z)]\alpha(z)\alpha(y)d(z) + [\alpha(x)d(z), \alpha(z)]\alpha(y)d(z) = 0.$$

Substituting zy in place of y in (3.51) and then subtracting it from (3.52) in order to find

$$(3.53) \quad [\alpha(x)d(z), \alpha(z)]\alpha(y)d(z) = 0, \quad \text{for all } x, y, z \in I.$$

Replacing x by wx in (3.53), where $w \in R$, and using it to obtain

$$[\alpha(w), \alpha(z)]\alpha(x)d(z)\alpha(y)d(z) = 0, \quad \text{for all } x, y, z, w \in I.$$

In particular, taking $y = r[w, z]x$ in above expression, where $r \in R$, we get

$$[\alpha(w), \alpha(z)]\alpha(x)d(z)R[\alpha(w), \alpha(z)]\alpha(x)d(z) = (0), \quad \text{for all } x, z, w \in I.$$

Since R is semiprime ring, it implies that $[\alpha(w), \alpha(z)]\alpha(x)d(z) = 0$ for all $x, z, w \in I$. This expression also appeared in Theorem 3.6, so the result is followed in the same way.

Using similar approach we conclude that the same result holds for $[F(x), d(y)] + (x \circ \alpha(y)) = 0$ for all $x, y \in I$. \square

Theorem 3.11. *Let R be a 2-torsion free semiprime ring and α be an automorphism of R . Let F be a multiplicative generalized (α, α) -derivation of R associated with a nonzero (α, α) -derivation d . If $d(x) \circ d(y) \pm F(xy) = 0$ for all $x, y \in R$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. We assume that $d(x) \circ d(y) \pm F(xy) = 0$ for all $x, y \in R$. That is

$$(3.54) \quad d(x)d(y) + d(y)d(x) \pm F(xy) = 0.$$

Replacing y by yz in (3.54) and using it, we get

$$(3.55) \quad d(x)\alpha(y)d(z) + \alpha(y)d(z)d(x) + d(y)[\alpha(z), d(x)] \pm \alpha(xy)d(z) = 0.$$

Taking xy in place of y in (3.55) to get

$$(3.56) \quad d(x)\alpha(x)\alpha(y)d(z) + \alpha(x)\alpha(y)d(z)d(x) + d(x)\alpha(y)[\alpha(z), d(x)] \\ + \alpha(x)d(y)[\alpha(z), d(x)] \pm \alpha(x)\alpha(xy)d(z) = 0, \quad \text{for all } x, y, z \in R.$$

Application of (3.55) in (3.56) yields

$$[d(x), \alpha(x)]\alpha(y)d(z) + d(x)\alpha(y)[\alpha(z), d(x)] = 0, \quad \text{for all } x, y, z \in R.$$

In particular for $x = z$, we get

$$(3.57) \quad [d(x), \alpha(x)]\alpha(y)d(x) = d(x)\alpha(y)[d(x), \alpha(x)], \quad \text{for all } x, y \in R.$$

Replacing y by $y\alpha^{-1}(d(x))$ in (3.57) and using it, we have

$$d(x)\alpha(y)[d(x), [d(x), \alpha(x)]] = 0, \quad \text{for all } x, y \in R.$$

It implies that

$$[d(x), [d(x), \alpha(x)]]\alpha(y)[d(x), [d(x), \alpha(x)]] = 0, \quad \text{for all } x, y \in R.$$

Using semiprimeness of R , we get $[d(x), [d(x), \alpha(x)]] = 0$ for all $x \in R$. That is equivalent to $[\varphi(x), [\varphi(x), x]] = 0$ for all $x \in R$, where $\varphi = \alpha^{-1}d$, which is an ordinary derivation of R . Invoking Lemma 2.6, we get φ maps R into $Z(R)$, i.e.,

$$[\varphi(x), y] = 0, \quad \text{for all } x, y \in R.$$

In particular, we have $[\varphi(x), x] = 0$ for all $x \in R$, and hence R contains a nonzero central ideal by Lemma 2.3. \square

Theorem 3.12. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R . Let F be a multiplicative generalized (α, α) -derivation of R associated with a nonzero (α, α) -derivation d . If $[d(x), d(y)] \pm F(xy) = 0$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. Following the same arguments as in Theorem 3.11, instead of equation (3.57), we have

$$[d(x), \alpha(x)]\alpha(y)d(x) + d(x)\alpha(y)[d(x), \alpha(x)] = 0, \quad \text{for all } x, y \in I.$$

By Lemma 2.5, we find

$$(3.58) \quad d(x)\alpha(y)[d(x), \alpha(x)] = 0, \quad \text{for all } x, y \in I.$$

Taking xy in place of y in (3.58), we get

$$(3.59) \quad d(x)\alpha(x)\alpha(y)[d(x), \alpha(x)] = 0, \quad \text{for all } x, y \in I.$$

Left multiply (3.58) by $\alpha(x)$ to obtain

$$(3.60) \quad \alpha(x)d(x)\alpha(y)[d(x), \alpha(x)] = 0, \quad \text{for all } x, y \in I.$$

Comparing (3.59) and (3.60), we obtain $[d(x), \alpha(x)]\alpha(y)[d(x), \alpha(x)] = 0$ for all $x, y \in I$. By semiprimeness of $\alpha(I)$, we find $[\alpha(x), d(x)] = 0$ for all $x \in I$. It implies that $[x, \varphi(x)] = 0$ for all $x \in I$, where $\varphi = \alpha^{-1}d$, which is an ordinary derivation. Hence, in light of Lemma 2.3, we are done. \square

Now onwards, we consider that F is a two sided multiplicative generalized (α, α) -derivation associated with (α, α) -derivation d , i.e., F satisfies the following conditions:

$$F(xy) = F(x)\alpha(y) + \alpha(x)d(y) = d(x)\alpha(y) + \alpha(x)F(y), \quad \text{for all } x, y \in R.$$

Theorem 3.13. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R and α be an automorphism of R . If $[d(x), F(y)] \pm F([x, y]) = 0$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. Assume that

$$(3.61) \quad [d(x), F(y)] \pm F([x, y]) = 0, \quad \text{for all } x, y \in I.$$

Replacing y by yx in (3.61) and using it, we get

$$(3.62) \quad F(y)[d(x), \alpha(x)] + [d(x), \alpha(y)]d(x) \pm \alpha([x, y])d(x) = 0, \quad \text{for all } x, y \in I.$$

Replacing y by xy in (3.62), we have

$$\begin{aligned} & \alpha(x)F(y)[d(x), \alpha(x)] + d(x)\alpha(y)[d(x), \alpha(x)] + \alpha(x)[d(x), \alpha(y)]d(x) \\ & + [d(x), \alpha(x)]\alpha(y)d(x) \pm \alpha(x)\alpha([x, y])d(x) = 0, \quad \text{for all } x, y \in I. \end{aligned}$$

Using (3.62), we get

$$d(x)\alpha(y)[d(x), \alpha(x)] + [d(x), \alpha(x)]\alpha(y)d(x) = 0, \quad \text{for all } x, y \in I.$$

This expression also appeared in Theorem 3.12, so the result is followed in the same way. \square

Theorem 3.14. *Let R be a 2-torsion free semiprime ring and α be an automorphism of R . If $d(x) \circ F(y) \pm F(x \circ y) = 0$ for all $x, y \in R$, then d maps R into $Z(R)$ and R contains a nonzero central ideal.*

Proof. Assume that

$$(3.63) \quad (d(x) \circ F(y)) \pm F(x \circ y) = 0, \quad \text{for all } x, y \in I.$$

Replacing y by yx in (3.63) and using it, we get

$$(3.64) \quad -F(y)[d(x), \alpha(x)] + (d(x) \circ \alpha(y))d(x) \pm \alpha(x \circ y)d(x) = 0, \quad \text{for all } x, y \in I.$$

Replacing y by xy in (3.64), we find

$$\begin{aligned} & -\alpha(x)F(y)[d(x), \alpha(x)] - d(x)\alpha(y)[d(x), \alpha(x)] + \alpha(x)(d(x) \circ \alpha(y))d(x) \\ & + [d(x), \alpha(x)]\alpha(y)d(x) \pm \alpha(x)\alpha(x \circ y)d(x) = 0, \quad \text{for all } x, y \in I. \end{aligned}$$

Using (3.64), we obtain

$$d(x)\alpha(y)[d(x), \alpha(x)] = [d(x), \alpha(x)]\alpha(y)d(x), \quad \text{for all } x, y \in I.$$

This expression also appeared as equation (3.57) in Theorem 3.11, so the result is followed in the same way. \square

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¹DEPARTMENT OF MATHEMATICS,
PATEL MEMORIAL NATIONAL COLLEGE,
RAJPURA 140401, INDIA
Email address: gurninder_rs@pbi.ac.in

²DEPARTMENT OF MATHEMATICS,
ÇANAKKALE ONSEKİZ MART UNIVERSITY,
ÇANAKKALE 17020, TURKEY
Email address: ayseyran@stu.comu.edu.tr
Email address: neseta@comu.edu.tr