Kragujevac Journal of Mathematics
Volume 51(4) (2027), Pages 547–556.

EQUIDISTANT DIMENSION OF JOHNSON AND KNESER GRAPHS

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Abstract. In this paper the recently introduced concept of equidistant dimension $\mathrm{eqdim}(G)$ of graph G is considered. Useful property of distance-equalizer set of arbitrary graph G has been established. For Johnson graphs $J_{n,2}$ and Kneser graphs $K_{n,2}$ exact values for $\mathrm{eqdim}(J_{n,2})$ and $\mathrm{eqdim}(K_{n,2})$ have been derived, while for Johnson graphs $J_{n,3}$ it is proved that $\mathrm{eqdim}(J_{n,3}) \leq n-2$. Finally, the exact value of $\mathrm{eqdim}(J_{2k,k})$ for odd k has been presented.

1. Introduction and Previous Work

The set of vertices S is a resolving (or locating) set of graph G if all other vertices are uniquely determined by their distances to the vertices in S. The metric dimension of G is the minimum cardinality of resolving sets of G. Resolving sets for graphs and the metric dimension were introduced by Slater [9] and, independently, by Harary and Melter [7]. The concept of doubly resolving set for G has been introduced by Caceres et al. [3].

However, recently, several authors have turned their attention in the opposite direction from resolvability, thus trying to study anonymization problems in networks instead of location aspects. A subset of vertices A is a 2-antiresolving set for G if, for every vertex $v \notin A$, there exists another different vertex $w \notin A$ such that v and w have the same vector of distances to the vertices of A [10]. The 2-metric antidimension of a graph is the minimum cardinality of 2-antiresolving sets for G. More about this topic can be found in [4, 8].

Key words and phrases. Distance-equalizer set, equidistant dimension, Johnson graphs, Kneser graphs.

2020 *Mathematics Subject Classification.* Primary: 05C12. Secondary: 05C69.

DOI

Received: April 26, 2025.

Accepted: October 28, 2025.

In the same spirit, paper [6] introduces new graph concepts that can also be applied to anonymization problems in networks: distance-equalizer set and equidistant dimension. The authors study the equidistant dimension of several classes of graphs, proving that in the case of paths and cycles this invariant is related to a classical problem of number theory. They also show that distance-equalizer sets can be used for constructing doubly resolving sets, and obtain a new bound for the minimum cardinality of doubly resolving sets of G in terms of the metric dimension and the equidistant dimension of G. In [5] it is proved that the equidistant dimension problem is NP-hard in the general case, and equidistant dimension of lexicographic product of graphs is considered.

1.1. **Definitions and basic properties.** All graphs considered in this paper are connected, undirected, simple, and finite. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The order of G is $|V(G)|$. For any vertex $v \in V(G)$, its open neighborhood is the set $N(v) = \{w \in V(G) \mid vw \in E(G)\}$ and its closed neighborhood is $N[v] = N(v) \bigcup \{v\}$.

The degree of a vertex v, denoted by $\deg(v)$, is defined as the cardinality of $N(v)$. If $\deg(v) = 1$, then we say that v is a leaf, in which case the only vertex adjacent to v is called its support vertex. When $\deg(v) = |V(G)| - 1$, we say that v is universal. The maximum degree of G is $\Delta(G) = \max\{\deg(v) \mid v \in V(G)\}$ and its minimum degree is $\delta(G) = \min\{\deg(v) \mid v \in V(G)\}$. If all vertices of G have the same degree r, i.e., $\Delta(G) = \delta(G) = r$, we say that graph G is r-regular. The distance between two vertices $v, w \in V(G)$, denoted by $d(v, w)$, is the leghth of a shortest $u - v$ path, and the diameter of G is $\mathrm{Diam}(G) = \max\{d(v, w) \mid v, w \in V(G)\}$. The set of vertices on equal distances from adjacent vertices u and v is denoted in the literature by ${}_uW_v$ ([2]). In general, the same notation can be used also for non-adjacent vertices, i.e. ${}_uW_v = \{x \in V(G) \mid d(u, x) = d(v, x)\}$.

Let n and k be positive integers ($n > k$) and $[n] = \{1, 2, \ldots, n\}$. Then k-subsets are subsets of $[n]$ which have cardinality equal to k. The Johnson graph $J_{n,k}$ is an undirected graph defined on all k-subsets of set $[n]$ as vertices, where two k-subsets are adjacent if their intersection has cardinality equal to $k - 1$. Mathematically, $V(J_{n,k}) = \{A \mid A \subset [n], |A| = k\}$ and $E(J_{n,k}) = \{AB \mid A, B \subset [n], |A| = |B| = k, |A \bigcap B| = k - 1\}$.

It is easy to see that $J_{n,k}$ and $J_{n,n-k}$ are isomorphic, so we shall only consider Johnson graphs with $n \geq 2k$. The distance between two vertices A and B in $J_{n,k}$ can be computed by Remark 1.1.

Remark 1.1. For $A, B \in V(J_{n,k})$ it holds $d(A, B) = |A \setminus B| = |B \setminus A| = k - |A \bigcap B|$.

In the special case when $n = 2k$ the distance between $\overline{A} = [n] \setminus A$ and B can be computed by Remark 1.2.

Remark 1.2. For $A, B \in V(J_{2k,k})$ it holds $d(\overline{A}, B) = k - d(A, B) = |A \bigcap B|$.

Considering Remark 1.1, it is easy to see that Johnson graph $J_{n,k}$ is a $k(n-k)$-regular graph of diameter k.

The Kneser graph $K_{n,k}$ is an undirected graph also defined on all k-subsets of set $[n]$ as vertices, where two k-subsets are adjacent if their intersection is empty set. Mathematically, $V(K_{n,k}) = \{A \mid A \subset [n], |A| = k\}$ and $E(K_{n,k}) = \{AB \mid A, B \subset [n], |A| = |B| = k, A \bigcap B = \emptyset\}$.

Kneser graph is connected only if $n > 2k$, it is also $\binom{n-k}{k}$-regular graph. Specially, for $k = 2$, Kneser graph $K_{n,2}$ is the complement of the corresponding Johnson graph $J_{n,2}$, and both graphs have diameter 2. Hence, if $d_{K_{n,2}}(A, B) = 1$, then $d_{J_{n,2}}(A, B) = 2$, and vice versa. Therefore, for $A \neq B$ it holds $d_{K_{n,2}}(A, B) = 3 - d_{J_{n,2}}(A, B)$.

Definition 1.1 ([6])**.** Let $u, v, x \in V(G)$. We say that x is equidistant from u and v if $d(u, x) = d(v, x)$.

Definition 1.2 ([6])**.** A subset S of vertices is called a distance-equalizer set for G if for every two distinct vertices $u, v \in V(G) \setminus S$ there exists a vertex $x \in S$ equidistant from u and v.

Definition 1.3 ([6])**.** The equidistant dimension of G, denoted by $\mathrm{eqdim}(G)$, is the minimum cardinality of a distance-equalizer set of G

Remark 1.3 ([6]). If v is a universal vertex of a graph G, then $S = \{v\}$ is a minimum distance-equalizer set of G, and so $\mathrm{eqdim}(G) = 1$.

Lemma 1.1 ([6])**.** *Let G be a graph. If S is a distance-equalizer set of G and v is a support vertex of G, then S contains v or all leaves adjacent to v.*

Consequently, the following corollary holds.

Corollary 1.1 ([6])**.** $\mathrm{eqdim}(G) \geq |\{v \in V(G) \mid v \text{ is a support vertex}\}|$.

Theorem 1.1 ([6])**.** *For every graph G of order $n \geq 2$, the following statements hold:*
- $\mathrm{eqdim}(G) = 1$ *if and only if* $\Delta(G) = n - 1$*;*
- $\mathrm{eqdim}(G) = 2$ *if and only if* $\Delta(G) = n - 2$*.*

Corollary 1.2 ([6])**.** *If G is a graph of order n with $\Delta(G) < n-2$, then* $\mathrm{eqdim}(G) \geq 3$.

Theorem 1.2 ([6])**.** *For every graph G of order n, the following statements hold.*
- *If $n \geq 2$, then* $\mathrm{eqdim}(G) = n - 1$ *if and only if G is a path of order* 2.
- *If $n \geq 3$, then* $\mathrm{eqdim}(G) = n - 2$ *if and only if* $G \in \{P_3, P_4, P_5, P_6, C_3, C_4, C_5\}$.

Corollary 1.3 ([6])**.** *If G is a graph of order $n \geq 7$, then* $1 \leq \mathrm{eqdim}(G) \leq n - 3$.

Proposition 1.1 ([6])**.** *For any positive integer k, it holds that* $\mathrm{eqdim}(J_{n,k}) \leq n$ *whenever* $n \in \{2k-1, 2k+1\}$ *or* $n > 2k^2$.

In [1] the exact value of metric dimension for $J_{n,2}$ for $n \geq 6$ and an upper bound of metric dimension for $J_{n,k}$ for $k \geq 3$ are given.

2. New Results

This section gives new results:

- an useful property of distance-equalizer set of arbitrary graph G;
- the equidistant dimension of $J_{n,2}$, $K_{n,2}$ and $J_{2k,k}$ for odd k;
- a tight upper bound for $\text{eqdim}(J_{n,3})$.

In order to illustrate the structure of Johnson and Kneser graphs, Figure 1 and Figure 2 display graphs $J_{5,2}$ and $K_{5,2}$. It should be noted that $K_{5,2}$ is isomorphic to the well-known Petersen graph. By Remark 2.1 in Subsection 2.2 and Theorem 2.4 in Subsection 2.5, the equidistant dimension of both $J_{5,2}$ and $K_{5,2}$ is equal to 3, with the corresponding minimal distance-equalizer set $S = \{\{1,2\},\{1,3\},\{2,3\}\}$.

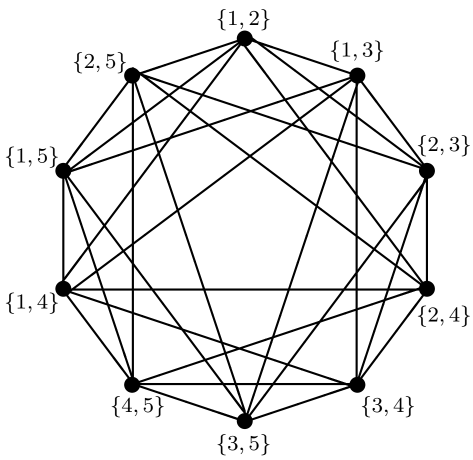

Figure 1. Johnson graph $J_{5,2}$

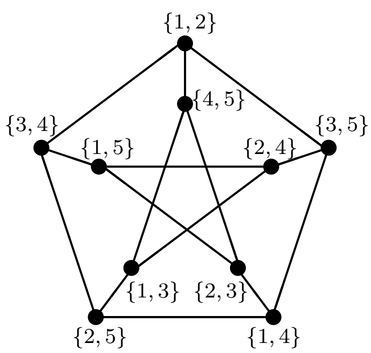

Figure 2. Kneser graph $K_{5,2}$

2.1. Some properties of distance-equalizer set of graph G.

Lemma 2.1. *Let G be a graph. Set S is a distance-equalizer set of G if and only if $(\forall u, v \in V(G))\ S \bigcap (\{u, v\} \bigcup {}_uW_v) \neq \emptyset$.*

Proof. $(\Rightarrow)$ Case 1: $u \in S$.

Since $u \in S$ and $u \in \{u, v\} \bigcup {}_uW_v$, then u is also member of their intersection, i.e., $u \in S \bigcap (\{u, v\} \bigcup {}_uW_v) \neq \emptyset$.

Case 2: $v \in S$.

Similarly as in Case 1, since $v \in S$ and $v \in \{u, v\} \bigcup {}_uW_v$, then v is also member of their intersection, i.e., $v \in S \bigcap (\{u, v\} \bigcup {}_uW_v) \neq \emptyset$.

Case 3: $u, v \notin S$.

Since S is a distance-equalizer set of G, and $u, v \in V(G) \setminus S$, then $(\exists x \in S)\, d(u, x) = d(v, x)$. Therefore, $x \in S$ and $x \in {}_uW_v$ so $S \bigcap {}_uW_v$ is not empty (since it contains x) implying $S \bigcap (\{u, v\} \bigcup {}_uW_v) \neq \emptyset$.

$(\Leftarrow)$ Let $S \subset V(G)$ and $(\forall u, v \in V(G))\ S \bigcap (\{u, v\} \bigcup {}_uW_v) \neq \emptyset$. Suppose that $u, v \in V(G) \setminus S$. From $\emptyset \neq S \bigcap (\{u, v\} \bigcup {}_uW_v) = (S \bigcap (\{u, v\}) \bigcup (S \bigcap {}_uW_v) = S \bigcap {}_uW_v$. It follows that there exists $x \in S$ such that $d(u, x) = d(v, x)$, i.e., S is a distance-equalizer set of G. □

Corollary 2.1. *Let G be a graph, and u and v any vertices from $V(G)$. If S is a distance-equalizer set of G and ${}_uW_v = \emptyset$, then $u \in S$ or $v \in S$.*

It should be noted that Lemma 1.1 from [6] is a consequence of Corollary 2.1. Indeed, if v is a support vertex of G and u is one of leaves adjacent to v, it is obvious that

$$(\forall x \in V(G) \setminus \{u\})\ d(u, x) = d(v, x) + 1$$

and $1 = d(u, v) = d(u, u) + 1$, and, therefore, ${}_uW_v = \emptyset$. If S is a distance-equalizer set of G, by Corollary 2.1, S contains v or all leaves adjacent to v.

2.2. Equidistant dimension of $J_{n,2}$.

The exact value of $\mathrm{eqdim}(J_{n,2})$ for $n \geq 4$ is given by Remark 2.1 and Theorem 2.1.

Remark 2.1. By a total enumeration, it is found that

- $\mathrm{eqdim}(J_{4,2}) = 2$ with distance-equalizer set $S = \{\{1, 2\}, \{3, 4\}\}$;
- $\mathrm{eqdim}(J_{5,2}) = 3$ with distance-equalizer set $S = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$.

Theorem 2.1. *For $n \geq 6$ it holds* $\mathrm{eqdim}(J_{n,2}) = 3$.

Proof. Step 1: $\mathrm{eqdim}(J_{n,2}) \geq 3$.

Since $J_{n,k}$ is $k(n-k)$-regular graph, so $\Delta(J_{n,k}) = \delta(J_{n,k}) = k(n-k)$. For $k = 2$ it follows that $\Delta(J_{n,2}) = 2(n-2)$. Since $|V(J_{n,2})| = \binom{n}{2} = \frac{n(n-1)}{2}$ it is obvious that for $n \geq 5$ it holds $\Delta(J_{n,2}) = 2(n-2) < |J_{n,2}| - 2 = \frac{n(n-1)}{2} - 2$, so by Corollary 1.2, it follows that $\mathrm{eqdim}(J_n) \geq 3$.

Step 2: $\mathrm{eqdim}(J_{n,2}) \leq 3$.

Let $S = \{\{1,2\},\{1,3\},\{2,3\}\}$. We will prove that set S is a distance-equalizer set by checking all pairs of vertices X and Y from $V(J_{n,2}) \setminus S$.

Case 1: $\{1,2,3\} \bigcap X = \emptyset$ and $\{1,2,3\} \bigcap Y = \emptyset$.

Let $Z = \{1,2\}$. Then, $d(X,Z) = 2 - |X \bigcap Z| = 2 = 2 - |Y \bigcap Z| = d(Y,Z)$.

Case 2: $\{1,2,3\} \bigcap X = \emptyset$ and $\{1,2,3\} \bigcap Y \neq \emptyset$.

Since $Y \notin S$, then $|Y \bigcap \{1,2,3\}| = 1$. Let $Z = \{1,2,3\} \setminus Y$. It is obvious that $Z \subset \{1,2,3\}$ and $|Z| = 2$ implying $Z \in S$. Since $\{1,2,3\} \bigcap X = \emptyset$ and $Y \bigcap Z = \emptyset$ then $d(X,Z) = 2 - |X \bigcap Z| = 2 = 2 - |Y \bigcap Z| = d(Y,Z)$.

Case 3: $\{1,2,3\} \bigcap X \neq \emptyset$ and $\{1,2,3\} \bigcap Y = \emptyset$.

This case is analogous as Case 2, only swap sets X and Y.

Case 4: $\{1,2,3\} \bigcap X \neq \emptyset$ and $\{1,2,3\} \bigcap Y \neq \emptyset$ and $X \bigcap Y \bigcap \{1,2,3\} = \emptyset$.

Let $Z = \{1,2,3\} \bigcap (X \bigcup Y)$. It is obvious that $Z \subseteq \{1,2,3\}$. Since $X, Y \notin S$ then $|X \bigcap \{1,2,3\}| = 1$ and $|Y \bigcap \{1,2,3\}| = 1$ it holds $|Z| = 2$ so, $Z \in S$. Therefore, $d(X,Z) = 2 - |X \bigcap Z| = 1 = 2 - |Y \bigcap Z| = d(Y,Z)$.

Case 5: $\{1,2,3\} \bigcap X \neq \emptyset$ and $\{1,2,3\} \bigcap Y \neq \emptyset$ and $X \bigcap Y \bigcap \{1,2,3\} \neq \emptyset$.

Since $X, Y \notin S$ it holds $|X \bigcap Y \bigcap \{1,2,3\}| = 1$. Let $Z = \{1,2,3\} \setminus X$. It is obvious that $Z = \{1,2,3\} \setminus Y$ and $X \bigcap Z = Y \bigcap Z = \emptyset$. Therefore, $d(X,Z) = 2 - |X \bigcap Z| = 2 = 2 - |Y \bigcap Z| = d(Y,Z)$.

Since

$$(\forall X, Y \in V(J_{n,2}) \setminus S)(\exists Z \in S) d(X,Z) = d(Y,Z),$$

then S is a distance-equalizer set for $J_{n,2}$ and thus $\text{eqdim}(J_{n,2}) \leq |S| = 3$. From Step 1 and Step 2 it holds $\text{eqdim}(J_{n,2}) = 3$ for all $n \geq 6$. □

2.3. **An upper bound of equidistant dimension of $J_{n,3}$.** The next theorem gives a tight upper bound of $\text{eqdim}(J_{n,3})$ for $n \geq 9$. The remaining cases when $n \in \{6,7,8\}$ are resolved by Theorem 2.3 for $n = 6$ and Table 1 for $n = 7$ and $n = 8$.

Theorem 2.2. *For $n \geq 9$ it holds* $\text{eqdim}(J_{n,3}) \leq n - 2$.

Proof. Let $S = \{\{1,2,j\} \mid 3 \leq j \leq n\}$. It can be proved that set S is a distance-equalizer set for $J_{n,3}$, i.e., for each two vertices X and Y from $V(J_{n,3}) \setminus S$, there exists a vertex $Z = \{1,2,l\}$ from S, such that $d(X,Z) = d(Y,Z)$. We will consider four cases:

Case 1: $\{1,2\} \bigcap X = \emptyset$ and $\{1,2\} \bigcap Y = \emptyset$.

It is easy to see that $|\{1,2\} \bigcup X \bigcup Y| \leq 8$. As $n \geq 9$, then there exists $l \in \{3,4,\ldots,n\}$ such that $l \notin X \bigcup Y$. Now, for vertex $Z = \{1,2,l\}$ from S, $d(X,Z) = 3 - |X \bigcap Z| = 3 = 3 - |Y \bigcap Z| = d(Y,Z)$.

Case 2: $\{1,2\} \bigcap X \neq \emptyset$ and $\{1,2\} \bigcap Y \neq \emptyset$.

As $X \notin S$ and $Y \notin S$, then $|\{1,2\} \bigcap X| = 1$ and $|\{1,2\} \bigcap Y| = 1$ and, consequently, $|\{1,2\} \bigcup X \bigcup Y| \leq 6$. As $n \geq 9$, then there exists $l \in \{3,4,\ldots,n\}$ such that $l \notin X \bigcup Y$. Now, for vertex $Z = \{1,2,l\}$ from S, $d(X,Z) = 3 - |X \bigcap Z| = 2 = 3 - |Y \bigcap Z| = d(Y,Z)$.

Case 3: $\{1,2\} \bigcap X \neq \emptyset$ and $\{1,2\} \bigcap Y = \emptyset$.

As $X \notin S$, then $|\{1,2\} \bigcap X| = 1$ and, consequently, $(Y \setminus X) \bigcap \{1,2\} = \emptyset$ and $|Y \setminus X| \geq 1$. It means that there exists $l \in \{3, 4, \ldots, n\}$ such that $l \notin Y \setminus X$. Now, for vertex $Z = \{1, 2, l\}$ from S, $d(X, Z) = 3 - |X \bigcap Z| = 2 = 3 - |Y \bigcap Z| = d(Y, Z)$.

Case 4: $\{1,2\} \bigcap X = \emptyset$ and $\{1,2\} \bigcap Y \neq \emptyset$.

This case can be reduced to Case 3.

Based on all previous cases, for each pair of vertices from $V(J_{n,3}) \setminus S$ there exists a vertex $Z \in S$ such that $d(X,Z) = d(Y,Z)$. Therefore, set S is a distance-equalizer set for $J_{n,3}$. As $|S| = n-2$, then $\text{eqdim}(J_{n,3}) \leq |S| = n - 2$. □

2.4. **Equidistant dimension of $J_{2k,k}$, for odd k.** Since $\binom{2k}{k}$ is even, then it is possible to make a partitition (P_1, P_2) of $V(J_{2k,k})$, such that $P_1 \bigcap P_2 = \emptyset$, $P_1 \bigcup P_2 = V(J_{2k,k})$ and $|P_1| = |P_2| = \frac{1}{2}\binom{2k}{k}$. In the sequel we will use the following partition: $P_1 = \{X \in V(J_{2k,k}) \mid |X \bigcap \{1, 2, \ldots, k\}| > |X \bigcap \{k+1, k+2, \ldots, 2k\}|\}$, and $P_2 = V(J_{2k,k}) \setminus P_1$. It should be noted that for odd k it holds

$$|X \bigcap \{1, 2, \ldots, k\}| \neq |X \bigcap \{k+1, k+2, \ldots, 2k\}|,$$

so $P_2 = \{X \in V(J_{2k,k}) \mid |X \bigcap \{1, 2, \ldots, k\}| < |X \bigcap \{k+1, k+2, \ldots, 2k\}|\}$ and, consequently, $|P_1| = |P_2| = \frac{1}{2}\binom{2k}{k}$.

Theorem 2.3. *For any odd $k \geq 3$ it holds* $\text{eqdim}(J_{2k,k}) = \frac{1}{2}\binom{2k}{k}$.

Proof. Step 1: $\text{eqdim}(J_{2k,k}) \geq \frac{1}{2}\binom{2k}{k}$.

Let us consider $\frac{1}{2}\binom{2k}{k}$ pairs of vertices (X, Y) from $V(J_{2k,k})$, such that $X \in P_1$ and $Y = [2k] \setminus X \in P_2$. For any vertex $Z \in V(J_{2k,k})$ it holds $|Z \bigcap X| + |Z \bigcap Y| = k$. Since k is odd, $|Z \bigcap X|$ is odd and $|Z \bigcap Y|$ is even, or vice versa. Therefore, $|Z \bigcap X| \neq |Z \bigcap Y|$ implying $d(X, Z) = k - |Z \bigcap X| \neq k - |Z \bigcap Y| = d(Y, Z)$, so ${}_XW_Y = \emptyset$. According to Corollary 2.1, if S is a distance-equalizer set for graph $J_{2k,k}$ then either $X \in S$ or $Y \in S$, for each pair (X, Y). Since the number of pairs is $\frac{1}{2}\binom{2k}{k}$, then $|S| \geq \frac{1}{2}\binom{2k}{k}$.

Step 2: $\text{eqdim}(J_{2k,k}) \leq \frac{1}{2}\binom{2k}{k}$.

We shall prove that P_1 is a distance-equalizer set for $J_{2k,k}$. For any two vertices Y and Z from $P_2 = V(J_{2k,k}) \setminus P_1$, let us construct $X \in P_1$ such that $d(Y, X) = d(Z, X)$. Since $|Y| = |Z| = k$ it follows that $|Y \setminus Z| = |Y| - |Y \bigcap Z| = |Z| - |Y \bigcap Z| = |Z \setminus Y|$. Additionally, as $Y, Z \in V(J_{2k,k})$ then $|Y \bigcap Z| = |\overline{Y} \bigcap \overline{Z}|$. Let $U_1 = (Y \bigcap Z) \bigcup (\overline{Y} \bigcap \overline{Z})$. It is easy to see that $U_1 \bigcap Y = U_1 \bigcap Z$ and $|U_1|$ is even so $k + 1 - |U_1|$ is also even.

Case 1. If $|U_1| < k$, let $a \in U_1$ be an arbitrary index and $U_2 = U_1 \setminus \{a\}$. Let W_1 and W_2 be any subsets of $Y \setminus Z$ and $Z \setminus Y$ of cardinality $\frac{k+1-|U_1|}{2}$ elements, respectively. Now let $U_3 = U_2 \bigcup W_1 \bigcup W_2$. It is obvious that $W_1 \subset Y$, $W_1 \bigcap Z = \emptyset$, $W_2 \subset Z$, $W_2 \bigcap Y = \emptyset$. Moreover, $|W_1| = |W_2|$, and therefore $|U_3 \bigcap Y| = |U_2 \bigcap Y| + |W_1 \bigcap Y| = |U_2 \bigcap Y| + |W_1| = |U_2 \bigcap Z| + |W_2| = |U_2 \bigcap Z| + |W_2 \bigcap Z| = |U_3 \bigcap Z|$.

Case 2. If $|U_1| > k$, let U_3 be any subset of U_1 of cardinality k. It is obvious that $U_3 \subset (Y \bigcap Z) \bigcup (\overline{Y} \bigcap \overline{Z})$ so $|U_3 \bigcap Y| = |U_3 \bigcap Z|$.

In both cases $|U_3| = k$ so $U_3 \in V(J_{2k,k})$. Therefore, in both cases $|U_3 \bigcap Y| = |U_3 \bigcap Z|$ and hence $d(U_3, Y) = k - |U_3 \bigcap Y| = k - |U_3 \bigcap Z| = d(U_3, Z)$.

Finally, we construct X as follows. If $U_3 \in P_1$, then $X = U_3$. Otherwise, if $U_3 \in P_2$, then $X = \overline{U_3} \in P_1$, and by Remark 1.2 it holds $d(\overline{U_3}, Y) = k - d(U_3, Y) = |U_3 \bigcap Y| = |U_3 \bigcap Z| = k - d(U_3, Z) = d(\overline{U_3}, Z)$. As, $d(Y, X) = d(Z, X)$ and $X \in P_1$, it follows that P_1 is a distance-equalizer set for graph $J_{2k,k}$. Therefore, $\text{eqdim}(J_{2k,k}) \leq |P_1| = \frac{1}{2}\binom{2k}{k}$. □

2.5. **Equidistant dimension of $K_{n,2}$.** The exact value for $\text{eqdim}(K_{n,2})$ is given in Theorem 2.4, and it is equal to $\text{eqdim}(J_{n,2}) = 3$.

Theorem 2.4. $\text{eqdim}(K_{n,2}) = 3$.

Proof. Step 1: $\text{eqdim}(K_{n,2}) \geq 3$.

As stated in Section 1, Kneser graph $K_{n,k}$ is connected only for $n > 2k$ implying that for $k = 2$ all Kneser graphs $K_{n,2}$ satisfy $n \geq 5$. Similarly as for Johnson graphs, Kneser graph $K_{n,k}$ is $\binom{n-k}{k}$-regular graph, so $\Delta(K_{n,k}) = \delta(K_{n,k}) = \binom{n-k}{k}$. For $k = 2$ it follows that $\Delta(K_{n,2}) = \frac{(n-2)(n-3)}{2}$. Since $|K_{n,2}| = \binom{n}{2} = \frac{n(n-1)}{2}$ it is obvious that for $n \geq 5$ it holds $4n > 10$ so $(n-2)(n-3) = n^2 - 5n + 6 < n^2 - n - 4 = n(n-1) - 4$ implying $\binom{n-2}{2} < \binom{n}{2} - 2$ which means $\Delta(K_{n,2}) = \binom{n-2}{2} < |K_{n,2}| - 2 = \binom{n}{2} - 2$, so by Corollary 1.2 it follows that $\text{eqdim}(K_{n,2}) \geq 3$.

Step 2: $\text{eqdim}(J_{n,2}) \leq 3$.

As already noticed $\overline{J_{n,2}} = K_{n,2}$ and $\text{Diam}(J_{n,2}) = \text{Diam}(K_{n,2}) = 2$, so $V(J_{n,2}) = V(K_{n,2})$ and for each two vertices $A, B \in V(K_{n,2})$ with $A \neq B$ it holds $d(A, B) = 3 - d_{J_{n,2}}(A, B)$, where $d(A, B)$ and $d_{J_{n,2}}(A, B)$ are distances between A and B in Kneser graph $K_{n,2}$ and Johnson graph $J_{n,2}$, respectively.

Let $S = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, and X and Y are any vertices from $V(K_{n,2}) \setminus S$. Since $V(J_{n,2}) = V(K_{n,2})$, and by Theorem 2.1, the same set $S = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ is proved to be a distance-equalizer set for graph $J_{n,2}$, then

$$(\forall X, Y \in V(J_{n,2}) \setminus S)(\exists Z \in S)\, d_{J_{n,2}}(X, Z) = d_{J_{n,2}}(Y, Z).$$

It follows that $d(X, Z) = 3 - d_{J_{n,2}}(X, Z) = 3 - d_{J_{n,2}}(Y, Z) = d(Y, Z)$. Therefore, the same set S is also a distance-equalizer set for graph $K_{n,2}$. From Step 1 and Step 2 it holds $\text{eqdim}(K_{n,2}) = 3$. □

2.6. **Some other individual exact values.** It is interesting to examine values of $\text{eqdim}(J_{n,k})$ and $\text{eqdim}(K_{n,k})$ in cases that are not covered by the obtained theoretical results presented above. Table 1 contains such values for Johnson and Kneser graphs up to 84 vertices obtained by a total enumeration. Since Kneser graphs are not connected for $n = 2k$, graph $K_{8,4}$ is not connected, which is denoted by ”-”.

Table 1. $\mathrm{eqdim}(J_{n,k})$ and $\mathrm{eqdim}(K_{n,k})$ for $k \geq 3$

n	k	$\mathrm{eqdim}(J_{n,k})$	$\mathrm{eqdim}(K_{n,k})$
7	3	5	5
8	3	8	3
8	4	7	-
9	3	7	3

3. Conclusions

In this paper, equidistant dimensions of Johnson and Kneser graphs are considered. Exact values $\mathrm{eqdim}(J_{n,2}) = 3$, $\mathrm{eqdim}(J_{2k,k}) = \frac{1}{2}\binom{2k}{k}$ for odd k and $\mathrm{eqdim}(K_{n,2}) = 3$ are found. Moreover, it is proved that $n-2$ is a tight upper bound for $\mathrm{eqdim}(J_{n,3})$.

Further work can be directed to finding the equidistant dimension of other interesting classes of graphs. Also, it would be interesting to develop exact and/or heuristic approaches for solving the equidistant dimension problem.

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