Kragujevac Journal of Mathematics
Volume 50(8) (2026), Pages 1353–1367.

STUDY OF KIRCHHOFF CURVATURE PROBLEM WITH ψ-HILFER DERIVATIVE AND $p(\cdot)$-LAPLACIAN OPERATOR

ELHOUSSAIN ARHRRABI[1,*] AND HAMZA EL-HOUARI[2]

Abstract. The paper focuses on the existence and multiplicity of weak solutions to nonlinear Kirchhoff-type equations involving ψ-Hilfer derivatives with $p(\cdot)$-Laplacian operators and Dirichlet boundary conditions. Through the application of a critical point approach, along with genus theory and variational techniques, we establish the existence and multiplicity results within appropriate fractional ψ-Hilfer derivative spaces. Our novel main results contribute to the advancement of the literature on differential equations involving fractional ψ-Hilfer generalized curvature phenomena.

1. Introduction

Fractional derivatives go beyond classical derivatives, extending the scope of traditional differentiation. Initially akin to ordinary differentiation, the concept of fractional differentiation has evolved in recent research, emphasizing its broader applicability. Within mathematical analysis, fractional analysis explores diverse methods of defining real number powers or complex powers of the differentiation operator and integration. The study extends to fractional order differential equations, which represent generalized and non-integer differential equations in both time and space domains. These equations are characterized by a power-law memory kernel, capturing nonlocal relationships, as highlighted by Kolma [29]. Several approaches exist for incorporating fractional integro-differentiation operations, including the Riemann-Liouville, Caputo and Grunwald-Letnikov methods, along with their modifications. It is crucial to note

Key words and phrases. Generalized ψ-Hilfer derivative, Kirchhoff problem, curvature phenomena, critical point theorem, genus theory, variational approach.

2020 *Mathematics Subject Classification.* Primary: 35J60. Secondary: 32C05, 35J50, 35J67, 46E35

DOI

Received: August 07, 2024.

Accepted: December 18, 2024.

that a significant portion of the literature on fractional differentiation primarily concentrates on Riemann-Liouville and Caputo fractional derivatives. However, there are alternative well-established definitions, such as the fractional derivative of Hadamard, the fractional derivative of Erdélyi-Kober, and others. Readers interested in delving further into fractional calculus are referred to additional resources like [47] for more comprehensive insights.

A fresh look at capillary phenomena in current literature highlights the need for progress in this area. Capillary action, the movement of liquids through narrow spaces without relying on external forces, is observed in various situations, like paint moving between bristles or liquids flowing through thin tubes. This action is powered by the forces between the liquid and solid surfaces. In small tubes, a combination of surface tension and adhesive forces moves the liquid. Studying phenomena like water rising in tubes or the formation of drops and bubbles involves using calculus variations, providing a unified way to address mathematical questions across different scenarios. Recently, interest in capillary phenomena has grown, driven by its relevance in fields like industry, healthcare, and micro-fluids. Given the breadth of the subject, we will touch on a few instances for those interested [8, 27, 32].

In the context of evolving trends in modern physics and mechanics, it becomes crucial to consider the dynamic landscape when formulating mathematical models. Drawing insights from pertinent research, it is essential to narrow our focus on specific domains in order to gain a deeper understanding of the underlying theory of the primary research problem. To address this, we employ the generalized ψ-Hilfer fractional derivative to analyze a nonlinear Kirchhoff equation featuring a positive parameter. This equation adheres to Dirichlet boundary conditions and is formulated as follows:

$$\begin{cases} -\left(\alpha+\beta\mathcal{L}(u)\right)\mathbf{L}^{\gamma,\kappa;\psi}_{\tau(x)}(u)=\xi|u|^{r(x)-2}u-h(x)|u|^{\tau(x)-2}u, \quad \text{in } \Omega, \\ u=0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with

$$\mathcal{L}(u):=\int_\Omega \frac{1}{\tau(x)}\left(\left|\mathrm{D}^{\gamma,\kappa;\psi}_{0^+}u\right|^{\tau(x)}+\sqrt{1+\left|\mathrm{D}^{\gamma,\kappa;\psi}_{0^+}u\right|^{2\tau(x)}}\right)dx$$

and

$$\mathbf{L}^{\gamma,\kappa;\psi}_{\tau(x)}(u):=\mathrm{D}^{\gamma,\kappa;\psi}_{T}\left(\left(1+\frac{\left|\mathrm{D}^{\gamma,\kappa;\psi}_{0^+}u\right|^{\tau(x)}}{\sqrt{1+\left|\mathrm{D}^{\gamma,\kappa;\psi}_{0^+}u\right|^{2\tau(x)}}}\right)\left|\mathrm{D}^{\gamma,\kappa;\psi}_{0^+}u\right|^{\tau(x)-2}\mathrm{D}^{\gamma,\kappa;\psi}_{0^+}u\right),$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N>3$, with smooth boundary $\partial\Omega$, α,β are positive constants, $h\in C(\overline{\Omega},\mathbb{R})$, ξ is a positive parameter, $\tau, r\in C^+(\overline{\Omega})$ such that:

$$1<r^-\leq r(x)\leq r^+<\tau^-\leq\tau(x)\leq\tau^+<2\tau^-<\tau^\star(x)=\frac{N\tau(x)}{N-\gamma\tau(x)}, \tag{1.2}$$

for all $x\in\Omega$, h satisfies the following hypothesis:

($\mathbf{H}_0$) $h:\Omega\to[0,\infty)$ such that $h\in L^\infty(\Omega)$,

$\mathrm{D}_T^{\gamma,\kappa;\psi}$ and $\mathrm{D}_{0+}^{\gamma,\kappa;\psi}$ are ψ-Hilfer fractional derivatives of order $\gamma \in (\frac{1}{\tau(x)}, 1)$ and type $\kappa \in [0,1]$ defined later, this type of derivative is introduced by Vanterler et al. [39], by means of the Gronwall inequality. They explored specific cases of the generalized fractional partial derivative, such as the ψ-Riemann-Liouville and ψ-Caputo fractional partial derivatives, discussed in detail in [36]. It's worth noting that when κ tends toward 1 and $\psi(x) = x$, our problem (1.1) simplifies to the integer case, elucidated on page 6 of [38]. This implies that our problem extends the scope of numerous Kirchhoff-type papers in the literature, especially in the context of fractional integration. Recent studies have investigated Kirchhoff equations using different operators. For instance, in [6], the authors employed a variational approach with the mountain pass theorem to establish the existence and multiplicity of solutions for a specific $p(\cdot)$-Kirchhoff type equation

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)\,|\nabla u|^{p(x)}}dx\right)\operatorname{div}(|\nabla u|^{p(x)-2}\,\nabla u) = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $p \in C(\overline{\Omega})$ with $1 < p(\cdot) < N$.

Chung in [7] used the mountain pass theorem combined with the minimum principle, to obtained at least two non-negative, non-trivial weak solutions for the following $p(x)$-Kirchhoff-type equations

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)|\nabla u|^{p(x)}}dx\right)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ u \geq 0, & \text{in } \Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $p \in C(\overline{\Omega})$ with $1 < p(\cdot) < N$ and λ is a positive real parameter and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ is a Caratheodory function that may change sign. In [25] Shapour et al., used variational methods and critical point theory to established the existence of multiple solutions for problem (1.3). For more examples of Kirchhoff-type problem, we refer the reader to [15–18, 46].

In the study of curvature problems related to the $p(\cdot)$-Laplacian, an example can be found in the work of authors in [26]. They employed Ricceri's variational principle, as developed by Bonanno and Molica Bisci, to establish the existence of at least one weak solution and infinitely many weak solutions for the following Neumann problem. This problem, originating from capillary phenomena, is a notable illustration in the context of $p(\cdot)$-Laplacian analysis

$$\begin{cases} -\operatorname{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right) + \alpha(x)|u|^{p(x)-2}u = \lambda f(x,u), & \text{in } \Omega, \\ \frac{\partial u}{\partial v} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary of class C^1, ν is the outer unit normal to $\partial\Omega$, $\alpha \in L^\infty(\Omega)$, f is an L^1-Caratheodory function. Furthermore, in [34], the authors studied the existence and multiplicity of solutions for the following nonlinear eigenvalue problems for $p(x)$-Laplacian-like operators

$$\begin{cases} -\operatorname{div}\left(\left(1+\dfrac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right) = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $p \in C(\Omega)$ and $p(x) > 2$, for all $x \in \Omega$, $\lambda > 0$ and f satisfies some growth condition and Ambrosetti-Rabinowitz type condition (AR).

In the context of Kirchhoff-type problem involving tempered fractional derivatives, we refer to [33]. The authors employed variational methods to establish the existence of infinitely many solutions to the following:

$$\begin{cases} M\left(\int_{\mathbb{R}} |D_+^{\alpha,\lambda}u(t)|^2dt\right) D_-^{\alpha,\lambda}(D_+^{\alpha,\lambda}u(t)) = f(t,u(t)), & t \in \mathbb{R}, \\ u \in W_\lambda^{\alpha,2}(\mathbb{R}), \end{cases}$$

where $D_\pm^{\alpha,\lambda}(\cdot)$ denote the left and right tempered fractional derivatives of order $\alpha \in (1/2, 1]$, $\lambda > 0$, $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $M \in C(\mathbb{R}^+,\mathbb{R}^+)$. In [31], the authors studied the existence and multiplicity of solutions to the following Kirchhoff equation with singular nonlinearity and Riemann-Liouville fractional derivative:

$$\begin{cases} \left(a + b\int_0^T |{}_0D_t^\alpha(u(t))|^p dt\right) {}_tD_T^\alpha(\Phi({}_0D_t^\alpha(u(t)))) = \dfrac{\lambda g(t)}{u^\gamma(t)} + f(t,u(t)), & t \in (0,T), \\ u(0) = u(T) = 0, \end{cases}$$

where $a \geq 1$, b, $\lambda > 0$, $p > 1$ are constants, $\frac{1}{p} < \lambda \geq 1$, $0 < \gamma < 1$, $g \in C([0,1])$ and $f \in C^1([0,T]\times\mathbb{R},\mathbb{R})$. Under appropriate assumptions on the function f, they employed variational methods to show the existence and multiplicity of positive solutions of the above problem with respect to the parameter λ. All the problems discussed above are associated to the stationary version of the Kirchhoff problem

$$\varrho\frac{\partial^2 u}{\partial\gamma^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.7}$$

presented by Kirchhoff in 1883 [28], which extends d'Alembert's wave equation. One notable feature of model (1.7) is that it contains a nonlocal term $\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2$. The parameters L, h, E, m, P_0 in model (1.7) represent different physical meanings, which we will not cover here. For an application related to Kirchhoff problems in capillary phenomena, we refer to [45], where the authors proposed a computational model to simulate the interaction between a complex fluid and a solid material. The complex fluid is represented using a diffuse-interface model, governed by the incompressible

Navier-Stokes-Cahn-Hilliard equations. Preferential-wetting boundary conditions are applied at the fluid-solid interface. The fluid-solid interaction involves fluid traction on the interface, incorporating a contribution from capillary stress. The dynamic interface condition considers the traction exerted by the non-uniform fluid-solid surface tension. The authors formulated the complex-fluid-solid interaction problem in a weak form, employing an Arbitrary-Lagrangian-Eulerian approach for the Navier-Stokes-Cahn-Hilliard equations. They also appropriately reformulated the complex-fluid traction and the fluid-solid surface tension. To validate their model, further details can be found in the cited paper [45].

Our approach to proving the existence and multiplicity results for problems (1.1) relies on the utilization of the critical points theorem together with genus theory and variational approach.

This work is organized as follows. In Section 2, we provide a brief overview of the key features of variable exponent Lebesgue spaces and ψ-Hilfer fractional derivative spaces. Moving on to Section 3, we present the existing solutions to problems (1.1), along with their corresponding proofs.

2. Preliminary

In this section we collect preliminary concepts of the theory of variable exponent Lebesgue space, classical and fractional ψ-Hilfer derivative space (see [2, 4, 5, 11–14, 19–24, 40–43]).

2.1. Variable exponent Lebesgue space.

In the following, we define

$$C^+(\mathbb{R}^N) = \Big\{\tau \in C(\mathbb{R}^N) : \ 1 < \tau^- \le \tau^+ < +\infty\Big\},$$

where

$$\tau^- := \inf_{x\in\mathbb{R}^N} \tau(x) \quad \text{and} \quad \tau^+ := \sup_{x\in\mathbb{R}^N} \tau(x).$$

Denote by $\mathbf{U}(\mathbb{R}^N)$ the set of all measurable real-valued functions defined in $\mathbb{R}^N$. For any $s \in C^+(\mathbb{R}^N)$, we denote the variable exponent Lebesgue space by

$$L^{\tau(x)}(\mathbb{R}^N) = \left\{u \in \mathbf{U}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u(x)|^{\tau(x)}\mathrm{d}x < +\infty\right\},$$

equipped with the Luxemburg norm

$$\|u\|_{L^{\tau(x)}} = \inf\left\{\nu > 0 : \ \int_{\mathbb{R}^N} \left|\frac{u(x)}{\nu}\right|^{\tau(x)}\mathrm{d}x \le 1\right\},$$

then, the variable exponent Lebesgue space $\left(L^{\tau(x)}(\mathbb{R}^N), \|\cdot\|_{L^{\tau(x)}}\right)$ becomes a Banach space.

We have the following generalized Holder inequality

$$\left|\int_{\mathbb{R}^N} u(x)v(x)\mathrm{d}x\right| \le \ 2\|u\|_{L^{\tau(x)}}\|v\|_{L^{\tau'(x)}}, \tag{2.1}$$

for $u \in L^{\tau(x)}(\mathbb{R}^N)$, $v \in L^{\tau'(x)}(\mathbb{R}^N)$ such that $\frac{1}{\tau(x)} + \frac{1}{\tau'(x)} = 1$.

At this point, let define the following map $\sigma_{\tau(x)} : L^{\tau(x)}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\sigma_{\tau(x)}(u) = \int_{\mathbb{R}^N} |u(x)|^{\tau(x)} dx.$$

Then, we can see the important relationship between the norm $\| \cdot \|_{L^{\tau(x)}}$ and the corresponding modular function $\sigma_{\tau(x)}(\cdot)$ given in the next proposition.

Proposition 2.1. *If u and $(u_k)_{k\in\mathbb{N}} \in L^{\tau(x)}(\mathbb{R}^N)$, we have*

$$\|u\|_{L^{\tau(x)}} < 1\ (= 1, > 1) \text{ if and only if } \sigma_{\tau(x)}(u) < 1\ (= 1, > 1), \tag{2.2}$$

$$\|u\|_{L^{\tau(x)}} > 1, \text{ then } \|u\|^{\tau^-}_{L^{\tau(x)}} \leq \sigma_{\tau(x)}(u) \leq \|u\|^{\tau^+}_{L^{\tau(x)}}, \tag{2.3}$$

$$\|u\|_{L^{\tau(x)}} < 1, \text{ then } \|u\|^{\tau^+}_{L^{\tau(x)}} \leq \sigma_{\tau(x)}(u) \leq \|u\|^{\tau^-}_{L^{\tau(x)}}, \tag{2.4}$$

$$\lim_{k\longrightarrow+\infty} \|u_k - u\|_{L^{\tau(x)}} = 0 \text{ if and only if } \lim_{k\longrightarrow+\infty} \sigma_{\tau(x)}(u_k - u) = 0. \tag{2.5}$$

Remark 2.1. Note that, by (2.3) and (2.4), we can derive the two subsequent inequalities:

$$\|u\|_{L^{\tau(x)}} \leq \sigma_{\tau(x)}(u) + 1, \tag{2.6}$$

$$\sigma_{\tau(x)}(u) \leq \|u\|^{\tau^+}_{L^{\tau(x)}} + \|u\|^{\tau^-}_{L^{\tau(x)}}. \tag{2.7}$$

2.2. **ψ-Hilfer fractional derivative space.** Let $A := [c, d]$, $-\infty \leq c < d \leq +\infty$, $n-1 < \gamma < n$, $n \in \mathbb{N}$, $\mathbf{f}$, $\psi \in C^n(A, \mathbb{R})$ such that ψ is increasing and $\psi'(x) \neq 0$, for all $x \in A$. We recall the following definitions (see [39]).

- The left-sided fractional ψ-Hilfer integrals of a function $\mathbf{f}$ is given by

$$\mathbf{I}^{\gamma;\psi}_{c^+}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)} \int_c^x \psi'(y)(\psi(x) - \psi(y))^{\gamma-1}\mathbf{f}(y)\mathrm{d}y. \tag{2.8}$$

- The right-sided fractional ψ-Hilfer integrals of a function $\mathbf{f}$ is given by

$$\mathbf{I}^{\gamma;\psi}_{d^-}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)} \int_x^d \psi'(y)(\psi(y) - \psi(x))^{\gamma-1}\mathbf{f}(y)\mathrm{d}y. \tag{2.9}$$

- The left-sided ψ-Hilfer fractional derivatives for a function $\mathbf{f}$ of order γ and type $0 \leq \kappa \leq 1$ is defined by

$$\mathrm{D}^{\gamma,\kappa;\psi}_{c^+}\mathbf{f}(x) = \mathbf{I}^{\kappa(n-\gamma);\psi}_{c^+}\Big(\frac{1}{\psi'(x)} \cdot \frac{d}{\mathrm{d}x}\Big)^n \mathbf{I}^{(1-\kappa)(n-\gamma);\psi}_{c^+}\mathbf{f}(x).$$

- The right-sided ψ-Hilfer fractional derivatives for a function $\mathbf{f}$ of order γ and type $0 \leq \kappa \leq 1$ is defined by

$$\mathrm{D}^{\gamma,\kappa;\psi}_{c^+}\mathbf{f}(x) = \mathbf{I}^{\kappa(n-\gamma);\psi}_{d^-}\Big(-\frac{1}{\psi'(x)} \cdot \frac{d}{\mathrm{d}x}\Big)^n \mathbf{I}^{(1-\kappa)(n-\gamma);\psi}_{d^-}\mathbf{f}(x).$$

Choosing $\kappa \to 1$, we obtain ψ-Caputo fractional derivatives left-sided and right-sided, given by

$$\mathrm{D}^{\gamma;\psi}_{c^+}\mathbf{f}(x) = \mathbf{I}^{(n-\gamma);\psi}_{c^+}\Big(\frac{1}{\psi'(x)} \cdot \frac{d}{\mathrm{d}x}\Big)^n \mathbf{f}(x), \tag{2.10}$$

$$\mathrm{D}_{d^-}^{\gamma;\psi}\mathbf{f}(x) = \mathbf{I}_{d^-}^{(n-\gamma);\psi}\Big(-\frac{1}{\psi'(x)}\cdot\frac{d}{\mathrm{d}x}\Big)^n\mathbf{f}(x). \tag{2.11}$$

Remark 2.2. The ψ-Hilfer fractional derivatives defined as above can be written in the following form

$$\mathrm{D}_{c^+}^{\gamma,\kappa;\psi}\mathbf{f}(x) = \mathbf{I}_{c^+}^{\mu-\gamma;\psi}\mathrm{D}_{c^+}^{\gamma;\psi}\mathbf{f}(x)$$

and

$$\mathrm{D}_{d^-}^{\gamma,\kappa;\psi}\mathbf{f}(x) = \mathbf{I}_{d^-}^{\mu-\gamma;\psi}\mathrm{D}_{d^-}^{\gamma;\psi}\mathbf{f}(x),$$

with $\mu = \gamma + \kappa(n-\gamma)$ and $\mathbf{I}_{c^+}^{\mu-\gamma;\psi}$, $\mathbf{I}_{d^-}^{\mu-\gamma;\psi}$, $\mathrm{D}_{c^+}^{\gamma;\psi}$ and $\mathrm{D}_{d^-}^{\gamma;\psi}$ as defined in (2.8), (2.9), (2.10) and (2.11).

In this paper we take $\Omega = A_1\times\cdots\times A_N = [c_1,d_1]\times\cdots\times[c_N,d_N]$ where $0<c_i<d_i$ for all $i\in\mathbb{N}$, $0<\gamma_1,\dots,\gamma_N<1$.

- The ψ-Riemann-Liouville fractional partial integral of order γ of N-variables $\mathbf{f} = (\mathbf{f}_1,\dots,\mathbf{f}_N)$ is defined by

$$\mathbf{I}_{c,x}^{\gamma;\psi}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)}\int_{A_1}\int_{A_2}\cdots\int_{A_N}\psi'(y)(\psi(x)-\psi(y))^{\gamma-1}\mathbf{f}(y)\mathrm{d}y,$$

with $\psi'(y)(\psi(x)-\psi(y))^{\gamma-1} = \psi'(y_1)(\psi(x_1)-\psi(y_1))^{\gamma_1-1}\cdots\psi'(y_N)(\psi(x_N)-\psi(y_N))^{\gamma_N-1}$ and $\Gamma(\gamma) = \Gamma(\gamma_1)\Gamma(\gamma_2)\cdots\Gamma(\gamma_N)$, $x_i = x_1x_2\cdots x_N$ and $\mathrm{d}y_i = \mathrm{d}y_1\mathrm{d}y_2\cdots\mathrm{d}y_N$, for all $i\in\{1,2,\dots,N\}$.

- $\mathrm{D}_{c,x_i}^{\gamma,\kappa;\psi}$ is defined by

$$\mathrm{D}_{c,x_i}^{\gamma,\kappa;\psi}\mathbf{f}(x_i) = \mathbf{I}_{c,x_i}^{\kappa(n-\gamma);\psi}\Big(\frac{1}{\psi'(x_i)}\cdot\frac{\partial^N}{\partial x_i}\Big)\mathbf{I}_{c,x_i}^{(1-\kappa)(n-\gamma);\psi}\mathbf{f}(x_i),$$

with $\partial x_i = \partial x_1,\partial x_2,\dots,\partial x_N$ and $\psi'(x_i) = \psi'(x_1)\psi'(x_2)\cdots\psi'(x_N)$ for all $i\in\{1,2,\dots,N\}$. Analogously, it is defined $\mathrm{D}_{d,x_i}^{\gamma,\kappa;\psi}(\cdot)$.

Now that we have all the necessary tools, we are ready to commence our study. To facilitate this, we define the ψ-Hilfer fractional derivative space $\mathcal{H}_{\tau(x)}^{\gamma,\kappa,\psi}(\Omega)$ as follow

$$\mathcal{H}_{\tau(x)}^{\gamma,\kappa,\psi}(\Omega) = \mathcal{H}_{\tau(x)}(\Omega) := \Big\{u\in L^{\tau(x)}(\Omega) : |\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u|\in L^{\tau(x)}(\Omega)\Big\},$$

equipped with the norm

$$\|u\|_{\mathcal{H}_{\tau(x)}} = \|u\|_{L^{\tau(x)}} + \|\mathrm{D}_{0^+}^{\gamma,\kappa,\psi}u\|_{L^{\tau(x)}}.$$

Proposition 2.2 ([44])**.** *Let $0<\gamma\le 1$, $0\le\kappa\le 1$ and $1<\tau(x)$. $\mathcal{H}_{\tau(x)}(\Omega)$ is a reflexive and separable Banach space.*

Remark 2.3. We can define $\mathcal{H}_{\tau(x),0}^{\gamma,\kappa,\psi}(\Omega) := \mathcal{H}_{\tau(x),0}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $\mathcal{H}_{\tau(x)}^{\gamma,\kappa,\psi}(\Omega)$ which can be renormed by the equivalent norm $\|u\| := \Big\|\mathrm{D}_{0^+}^{\gamma,\kappa,\psi}u\Big\|_{L^{\tau(x)}}$. This space is a separable and reflexive Banach space (see [44]).

Proposition 2.3 ([30]). *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$. Let $p \in C^0(\overline{\Omega})$. If $r : \overline{\Omega} \to (1, +\infty)$ such that*

$$1 \leq r(x) < \tau^\star(x) := \begin{cases} \frac{N\tau(x)}{N-\gamma\tau(x)}, & \text{if } \gamma\tau(x) < N, \\ +\infty, & \text{if } \gamma\tau(x) \geq N, \end{cases} \quad \text{for all } x \in \overline{\Omega},$$

then, the embedding

$$\mathcal{H}_{\tau(x),0}(\Omega) \hookrightarrow L^{r(x)}(\Omega), \tag{2.12}$$

is compact and there is a constant $c_0 > 0$, such that $\|u\|_{L^{r(x)}} \leq c_0 \|u\|$.

2.3. **Genus Theory.** We introduce fundamental concepts related to Krasnoselskii's genus (refer to [9]) which will be employed in the proof of our main results. Let X be a real Banach space and

$$\mathfrak{R} := \{A \subset X \backslash \{0\} : A \text{ is compact and symmetric}\}.$$

Definition 2.1. Let $A \in \mathfrak{R}$ and $X = \mathbb{R}^k$. We define the genus of A as follows:

$$\mathfrak{G}(A) := \inf \left\{ k \geq 1 : \text{exists } g \in C\left(A, \mathbb{R}^k \backslash \{0\}\right), g \text{ is odd} \right\}$$

and $\mathfrak{C}(A) = +\infty$, if does not exist such a map for any $k > 0$.

Theorem 2.1 ([9]). *Let $\Omega \subset \mathbb{R}^N$ be bounded symmetric with boundary $\partial\Omega$. Assume that $0 \in \Omega$, then $\mathfrak{G}(\partial\Omega) = N$.*

Corollary 2.1 ([9]). *The genus of unit sphere $\mathbb{S}^{N-1}$ of the space $\mathbb{R}^N$ is N, i.e., $\mathfrak{G}(\mathbb{S}^{N-1}) = N$.*

Definition 2.2. Let X be a real Banach space, and $\Upsilon \in C^1(X, \mathbb{R})$. We say that Υ satisfies the Palais-Smale condition ((PS) for short) if any sequence $\{u_n\}_{n \in \mathbb{N}} \subset X$ such that $\{\Upsilon(u_n)\}_{n \in \mathbb{N}}$ is bounded and $\Upsilon'(u_n) \to 0$ as $n \to +\infty$, admits a convergent subsequence.

Theorem 2.2 ([10]). *Let $\Upsilon \in C^1(X, \mathbb{R})$ and let it satisfies the (PS) condition. Additionally, we assume the following conditions.*

(i) Υ is bounded from below and even.

(ii) There is a compact set $N \in \mathfrak{R}$ such that $\mathfrak{G}(N) = k$ and $\sup_{x \in N} \Upsilon(x) < \Upsilon(0)$.

Then, Υ has at least k pairs of distinct critical points, and their corresponding critical values are less than $\Upsilon(0)$.

3. Main Result

In this section, we will proving the existence and multiplicity results for problems (1.1) relies on the utilization of the critical points theorem together with genus theory and variational approach.

Definition 3.1. We say that $u \in \mathcal{H}_{\tau(x),0}(\Omega)$ is a weak solution of problem (1.1) if

$$\Big(\alpha + \beta\mathcal{L}(u)\Big)\int_\Omega \left(\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u\right|^{\tau(x)-2} \mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u + \frac{\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u\right|^{2\tau(x)-2} \mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u\right|^{2\tau(x)}}} \right) \mathrm{D}_{0^+}^{\gamma,\kappa;\psi} \varphi \, dx$$

$$= \xi \int_\Omega |u|^{r(x)-2} u\varphi dx - \int_\Omega h(x)|u|^{\tau(x)-2}\varphi \, dx,$$

for all $\varphi \in \mathcal{H}_{\tau(x),0}(\Omega)$.

Let us introduce the energy functional $\mathfrak{E} : \mathcal{H}_{\tau(x),0}(\Omega) \to \mathbb{R}$ associated to problem (1.1)

$$\mathfrak{E}(u) = \alpha\mathcal{L}(u) + \frac{\beta}{2}\left(\mathcal{L}(u)\right)^2 - \xi\int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx + \int_\Omega \frac{h(x)}{\tau(x)}|u|^{\tau(x)}dx,$$

for all $u \in \mathcal{H}_{\tau(x),0}(\Omega)$.

Observe that $\mathfrak{E} \in C^1\left(\mathcal{H}_{\tau(x),0}(\Omega), \mathbb{R}\right)$ and it is noteworthy that the critical points of $\mathfrak{E}$ correspond to weak solutions of (1.1) and its Gateaux derivative is

$$\begin{aligned}&\langle \mathfrak{E}'(u), v\rangle \\ =&\Big(\alpha + \beta\mathcal{L}(u)\Big)\int_\Omega \left(\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u\right|^{\tau(x)-2} \mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u + \frac{\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u\right|^{2\tau(x)-2} \mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u\right|^{2\tau(x)}}} \right) \mathrm{D}_{0^+}^{\gamma,\kappa;\psi} v \, dx \\ &- \xi\int_\Omega |u|^{r(x)-2}uv dx + \int_\Omega h(x)|u|^{\tau(x)-2}uv \, dx,\end{aligned}$$

for all $u, v \in \mathcal{H}_{\tau(x),0}(\Omega)$.

Note that the functional $\mathcal{L} \in C^1\left(\mathcal{H}_{\tau(x),0}(\Omega), \mathbb{R}\right)$ and its derivative operator in weak sense $\mathcal{L}' : \mathcal{H}_{\tau(x),0}(\Omega) \to \left(\mathcal{H}_{\tau(x),0}(\Omega)\right)^*$ is such that

$$\langle \mathcal{L}'(u), v\rangle = \int_\Omega \left(\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u\right|^{\tau(x)-2} \mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u + \frac{\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u\right|^{2\tau(x)-2} \mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u\right|^{2\tau(x)}}} \right) \mathrm{D}_{0^+}^{\gamma,\kappa;\psi} v \, dx,$$

for all $u, v \in \mathcal{H}_{\tau(x),0}(\Omega)$.

Theorem 3.1. *Problem* (1.1) *admits at least k pairs of different critical points if* (1.2) *holds.*

Proposition 3.1 ([1])**.** *The functional $\mathcal{L}$ is a convex. The mapping $\mathcal{L}' : \mathcal{H}_{\tau(x),0}(\Omega) \to (\mathcal{H}_{\tau(x),0}(\Omega))^*$ is bounded homeomorphism and strictly monotone operator, and is a mapping of type (S_+), i.e., if $u_n \rightharpoonup u$ in $\mathcal{H}_{\tau(x),0}(\Omega)$ and $\overline{\lim}_{n\to+\infty}\langle \mathcal{L}'(u_n) - \mathcal{L}'(u), u_n - u\rangle \leq 0$, then $u_n \to u$ in $\mathcal{H}_{\tau(x),0}(\Omega)$.*

Lemma 3.1. *The functional $\mathfrak{E}$ satisfies the (PS) condition.*

Proof. Let show that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_{\tau(x),0}(\Omega)$. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_{\tau(x),0}(\Omega)$ be a (PS) sequence. Employing a proof by contradiction, we assume that, possibly after considering a sub-sequence, still denote by $\{u_n\}_{n\in\mathbb{N}}$, one has $\|u_n\| \to +\infty$ as $n \to +\infty$. Let us choose $0 < \omega < \left\{\frac{1}{r^+}, \frac{1}{\tau^+}, \frac{\tau^-}{2(\tau^+)^2}\right\}$. According to Proposition 2.3, for sufficiently large n, one has

$$\begin{aligned}
c + \|u_n\| \geq & \mathfrak{E}(u_n) - \omega \left\langle \mathfrak{E}'\left(u_n\right), u_n \right\rangle \\
\geq & \alpha\left(\frac{1}{\tau^+} - \omega\right) \int_\Omega |\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u_n|^{\tau(x)} dx + \beta\left(\frac{1}{2(\tau^+)^2} - \frac{\omega}{\tau^-}\right)\left(\int_\Omega |\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u_a|^{\tau(x)} dx\right)^2 \\
& - \xi\left(\frac{1}{r^-} - \omega\right)\int_\Omega |u_n|^{r(x)} dx + \left(\frac{1}{\tau^+} - \omega\right)\int_\Omega h(x)|u_n|^{\tau(x)} dx \\
\geq & \alpha\left(\frac{1}{\tau^+} - \omega\right)\|u_n\|^{\tau^-} + \beta\left(\frac{1}{2(\tau^+)^2} - \frac{\omega}{\tau^-}\right)\|u_n\|^{2\tau^-} - \xi c_0\left(\frac{1}{r^-} - \omega\right)\|u_n\|.
\end{aligned}$$

Dividing the aforementioned inequality by $\|\mathrm{u}_n\|$ and taking the limit as $n \to +\infty$, we arrive at a contradiction. It is implied by (1.2) that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_{\tau(x),0}(\Omega)$.

Moreover, based on Proposition 2.3, we can assume that

$$\left\{\begin{array}{l}
u_n \to u \text{ strongly in } L^{r(x)}(\Omega), \\
u_n(x) \to u(x) \text{ a.e in } \Omega, \\
u_n \rightharpoonup u \text{ weakly in } \mathcal{H}_{\tau(x),0}(\Omega).
\end{array}\right. \tag{3.1}$$

Using Holder's inequality and (3.1), one has

$$\begin{aligned}
\left|\int_\Omega |u_n|^{\tau(x)-2} u_n(u_n-u)dx\right| &\leq \int_\Omega |u_n|^{\tau(x)-1}|u_n-u|dx \\
&\leq \|u_n\|^{\tau^+-1}_{\frac{\tau(x)}{\tau(x)-1}} \|u_n-u\|_{\tau(x)} \to 0, \quad \text{as } n \to +\infty.
\end{aligned}$$

Therefore,

$$\int_\Omega |u_n|^{\tau(x)-2} u_n(u_n-u)dx \to 0, \quad \text{as } n \to +\infty, \tag{3.2}$$

and

$$\int_\Omega |u_n|^{r(x)-2} u_n(u_n-u)dx \to 0, \quad \text{as } n \to \infty. \tag{3.3}$$

Then,

$$\langle \mathfrak{E}'(u_n), u_n - u\rangle \to 0, \quad \text{as } n \to +\infty.$$

This implies that

$$\begin{aligned}
&\langle \mathfrak{E}'(u_n), u_n - u\rangle \\
=&(\alpha + \beta\mathcal{L}(u_n))\int_\Omega \left(\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u_n\right|^{\tau(x)-2} \mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u_n + \frac{\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u_n\right|^{2\tau(x)-2} \mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u_n}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u_n\right|^{2\tau(x)}}}\right) \mathrm{D}_{0^+}^{\gamma,\kappa;\psi}(u_n-u)dx \\
&- \xi\int_\Omega |u_n|^{r(x)-2} u_n(u_n-u)dx + \int_\Omega h(x)|u_n|^{\tau(x)-2} u_n(u_n-u)dx \to 0, \quad \text{as} \quad n \to +\infty.
\end{aligned}$$

Thus, we can infer from equations (3.2) and (3.3) that

$$(\alpha+\beta\mathcal{L}(u_n))\int_\Omega\left(\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u_n\right|^{\tau(x)-2}\ \mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u_n+\frac{\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u_n\right|^{2\tau(x)-2}\ \mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u_n}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u_n\right|^{2\tau(x)}}}\right)$$
$$\times\left(\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u_n-\ \mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u\right)dx\to 0,\quad \text{as } n\to+\infty.$$

Therefore, according to Proposition 3.1, $u_n\to u$ in $\mathcal{H}_{\tau(x),0}(\Omega)$. Hence, we conclude the proof. $\square$

Lemma 3.2. *The functional $\mathfrak{E}$ is coercive and bounded from below.*

Proof. For any $u\in\mathcal{H}_{\tau(x),0}(\Omega)$, we have

$$\begin{aligned}\mathfrak{E}(u)\geq&\frac{\alpha}{\tau^+}\int_\Omega\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u\right|^{\tau(x)}+\frac{\beta}{2(\tau^+)^2}\left(\int_\Omega\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u\right|^{\tau(x)}dx\right)^2-\frac{\xi}{r^-}\int_\Omega|u|^{r(x)}dx\\&+\frac{1}{\tau^+}\int h(x)|u|^{\tau(x)}dx.\end{aligned}$$

Using Propositions 2.1 and 2.3, we have two cases.

Case 1. If $\|u\|_{L^{\tau(x)}}>1$, then

$$\mathfrak{E}(u)\geq\frac{\alpha}{\tau^+}\|u\|^{\tau^-}+\frac{\beta}{2(\tau^+)^2}\|u\|^{2\tau^-}-\frac{\xi c_0}{r^-}\|u\|^{r^+}.$$

According to (1.2), $\mathfrak{E}$ is coercive and bounded from below.

Case 2. If $\|u\|_{L^{\tau(x)}}<1$, then

$$\mathfrak{E}(u)\geq\frac{\alpha}{\tau^+}\|u\|^{\tau^+}+\frac{\beta}{2(\tau^+)^2}\|u\|^{2\tau^+}-\frac{\xi c_0}{r^-}\|u\|^{r^-}.$$

Since $2\tau^+>\tau^+$ and $2\tau^+>\tau^-$, this implies that $\mathfrak{E}$ is coercive and bounded from below. $\square$

Proof of Theorem 3.1. Let $(s_n)_{n=1}^\infty$ be a Schauder basis for the space $\mathcal{H}_{\tau^+,0}(\Omega)$ and $Y_k=\operatorname{span}\{s_1,s_2,\dots,s_k\}$, the subspace of $\mathcal{H}_{\tau^+,0}(\Omega)$ generated by $s_1,s_2,\dots,s_k$. Clearly, Y_k is subspace of $\mathcal{H}_{\tau^+,0}(\Omega)$. Then, since $\mathcal{H}_{\tau^+,0}(\Omega)\subset\mathcal{H}_{\tau(x),0}(\Omega)\subset L^{r(x)}(\Omega)$ we have $Y_k\subset L^{r(x)}(\Omega)$. Also, since Y_k is a finite-dimensional space, the norms $\|\cdot\|$ and $\|\cdot\|_{L^{r(x)}}$ are equivalent on Y_k. Therefore, there exists a positive constant c_k such that

$$\|u\|_{L^{r(x)}}\geq c_k\|u\|,\qquad \text{for all } u\in Y_k.$$

Note that since $\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u\right|^{2\tau(x)}}<2\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u\right|^{\tau(x)}$, we have

$$\mathcal{L}(u)\leq\frac{3}{\tau^-}\int_\Omega\left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u\right|^{\tau(x)}dx.\tag{3.4}$$

Let $u\in Y_k$ such that $\|u\|<1$, then using (3.4) and $(\mathbf{H}_0)$ one has

$$\mathfrak{E}(u)=\alpha\mathcal{L}(u)+\frac{\beta}{2}\left(\mathcal{L}(u)\right)^2-\xi\int_\Omega\frac{1}{r(x)}|u|^{r(x)}dx+\int_\Omega\frac{h(x)}{\tau(x)}|u|^{\tau(x)}dx$$

$$
\begin{aligned}
\leq &\frac{3\alpha}{\tau^-}\int_\Omega \left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u\right|^{\tau(x)}dx+\frac{9\beta}{2(\tau^-)^2}\int_\Omega \left|\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u\right|^{2\tau(x)}dx\\
&-\frac{\xi}{r^+}\int_\Omega |u|^{r(x)}dx+\frac{1}{\tau^-}\int_\Omega h(x)|u|^{\tau(x)}dx\\
\leq &c_1\Big(\|u\|^{\tau^-}+\|u\|^{2\tau^-}\Big)-c_2\|u\|^{r^+}_{L^{r(x)}}+\frac{1}{\tau^-}\|h\|_{L^\infty}\|u\|^{\tau^-}\\
\leq &c_3\Big(\|u\|^{\tau^-}+\|u\|^{2\tau^-}\Big)-c_4\|u\|^{r^+}\\
= &\|u\|^{r^+}\Big[c_3\Big(\|u\|^{\tau^--r^+}+\|u\|^{2\tau^--r^+}\Big)-c_4\Big].
\end{aligned}
$$

There exists $\lambda\in(0,1)$ sufficiently small such that $\lambda^{r^+}<1$ and $c_3\lambda^{\tau^--r^+}+c_3\lambda^{2\tau^--r^+}\leq \frac{c_4}{2}$. Let consider $\mathbf{S}^k_\lambda:=\{u\in Y_k:\|u\|=\lambda\}$. We have

$$\mathfrak{E}(u)\leq \lambda^{\tau^+}\left(c_3\lambda^{\tau^--r^+}+c_3\lambda^{2\tau^--r^+}-c_4\right),$$

for all $u\in\mathbf{S}^k_\lambda$. Thus,

$$\sup_{u\in\mathbf{S}^k_\lambda}\mathfrak{E}(u)\leq\left(\frac{c_4}{2}-c_4\right)=-\frac{c_4}{2}<0=\mathfrak{E}(0).$$

Since Y_k and $\mathbb{R}^k$ are isomorphic, $\mathbf{S}^k_\lambda$ and $\mathbb{S}^{k-1}$ are homeomorphic, thus $\mathfrak{G}\left(\mathbf{S}^k_\lambda\right)=k$. According to Theorem 2.2, $\mathfrak{E}$ has at least k pairs of different critical points. □

References

[1] E. Arhrrabi and H. El-Houari, *On a class of generalized capillarity phenomena involving fractional ψ-Hilfer derivative with $p(\cdot)$-Laplacian operator*, Kragujevac J. Math. (to appear).

[2] E. Arhrrabi and H. El-Houari, *Study of double phase-choquard problem in generalized ψ-Hilfer fractional derivative spaces with p-Laplacian operator*, Kragujevac J. Math. (to appear).

[3] E. Arhrrabi, H. El-Houari and J. V. da C. Sousa, *On a class of capillarity phenomenon with logarithmic nonlinearity involving $\theta(\cdot)$-Laplacian operator*, Comp. Appl. Math. **43**(6) (2024), Article ID 344. https://doi.org/10.1007/s40314-024-02863-8

[4] E. Arhrrabi and H. El-Houari, *Fractional Sobolev space: Study of Kirchhoff-Schrödinger systems with singular nonlinearity*, CUBO (2024), 407–430. https://cubo.ufro.cl/index.php/cubo/article/view/3818

[5] E. Arhrrabi and H. El-Houari, *On a class of generalized capillarity system involving fractional ψ-Hilfer derivative with $p(\cdot)$-Laplacian operator*, Math. Methods Appl. Sci. (2024). https://doi.org/10.1002/mma.10495

[6] M. Avci, B. Cekic and R. A. Mashiyev, *Existence and multiplicity of the solutions of the $p(x)$-Kirchhoff type equation via genus theory*, Math. Methods Appl. Sci. **34**(14) (2011), Article ID 1751. https://doi.org/10.1002/mma.1485

[7] N. T. Chung, *Multiple solutions for a $p(x)$-Kirchhoff-type equation with sign-changing nonlinearities*, Complex Var. Elliptic Equ. **58**(12) (2013), 1637–1646. https://www.tandfonline.com/doi/abs/10.1080/17476933.2012.701289

[8] L. Cueto-Felgueroso and R. Juanes, *Macroscopic phase-field model of partial wetting: bubbles in a capillary tube*, Phys. Rev. Lett. **108**(14) (2012), Article ID 144502. https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.108.144502

[9] M. A. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, 1964. https://cir.nii.ac.jp/crid/1370567920009303426

[10] D. C. Clark and D. Gilbarg, *A variant of the Ljusternik-Schnirelman theory*, Indiana Univ. Math. J. **22**(1) (1972), 65–74. https://www.jstor.org/stable/24890396

[11] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136**(5) (2012), 521–573. https://www.sciencedirect.com/science/article/pii/S0007449711001254

[12] D. Edmunds and J. Rakonik, *Sobolev embeddings with variable exponent*, Studia Math. **3**(143) (2000), 267–293. https://www.infona.pl/resource/bwmeta1.element.bwnjournal-article-smv143i3p267bwm

[13] A. Elhoussain and E. H. Hamza, *Fractional Sobolev space with variable exponents: Study of Kirchhoff problem by Berkovits degree theory*, Nonlinear Stud. **31**(4) (2024), 1135–1147. https://www.nonlinearstudies.com/index.php/nonlinear/article/view/3494

[14] A. Elhoussain and E. H. Hamza, *A singular ψ-Hilfer generalized fractional differential system problems with $p(\cdot)$-Laplacian operator*, Math. Sci. (N.Y.) (2024). 1–16. https://link.springer.com/article/10.1007/s10958-024-07408-2

[15] H. El-Houari, L. S. Chadli and H. Moussa, *Existence of a solution to a nonlocal Schrodinger system problem in fractional modular spaces*, Adv. Oper. Theory **7**(1) (2022), 1–30. https://doi.org/10.1007/s43036-021-00166-x

[16] H. El-Houari, L. S. Chadli and H. Moussa, *Existence of solution to M-Kirchhoff system type*, 7th International Conference on Optimization and Applications (ICOA), IEEE, (2022), 1–6. https://doi.org/10.1109/ICOA51614.2021.9442669

[17] H. El-Houari, L. S. Chadli and H. Moussa, *Multiple solutions in fractional Orlicz-Sobolev spaces for a class of nonlocal Kirchhoff systems*, Filomat **38**(8) (2024), 2857–2875. https://doi.org/10.2298/FIL2408857E

[18] E. H. Hamza, A. Elhoussain and J. V. D. D. C. Sousa, *On a class of Kirchhoff problems with nonlocal terms and logarithmic nonlinearity*, J. Pseudo-Differ. Oper. Appl. **15**(3) (2024), 52 pages. https://link.springer.com/article/10.1007/s11868-024-00624-z

[19] X. Fan, Q. Zhang and D. Zhao, *Eigenvalues of $p(x)$-Laplacian Dirichlet problem*, J. Math. Anal. Appl. **302**(2) (2005), 306–317. https://www.sciencedirect.com/science/article/pii/S0022247X03008606

[20] X. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263**(2) (2001), 424–446.

[21] A. Ghanmi and K. Saoudi, *The Nehari manifold for a singular elliptic equation involving the fractional Laplace operator*, Fract. Differ. Calc. **6**(2) (2016), 201–217. https://dx.doi.org/10.7153/fdc-06-13

[22] A. Ghanmi and Z. Zhang, *Nehari manifold and multiplicity results for a class of fractional boundary value problems with p-Laplacian*, Bull. Korean Math. Soc. **56**(5) (2019), 1297–1314. https://koreascience.kr/article/JAKO201928463078719.page

[23] A. Ghanmi, M. Kratou, K. Saoudi and D. D. Repovs, *Nonlocal p-Kirchhoff equations with singular and critical nonlinearity terms*, Asymptot. Anal. **131**(1) (2022), 125–143. https://journals.sagepub.com/doi/full/10.3233/ASY-221769

[24] A. Ghanmi, *Existence of nonnegative solutions for a class of fractional p-Laplacian problems*, Nonlinear Stud. **22**(3) (2015), 373–379.

[25] S. Heidarkhani, A. Ghobadi and M. Avci, *Multiple solutions for a class of $p(x)$-Kirchhoff-type equations*, Appl. Math. E-Notes **22** (2022), 160–168.

[26] S. Heidarkhani, G. A. Afrouzi and S. Moradi, *Variational approaches to $p(x)$-Laplacian-like problems with Neumann condition originated from a capillary phenomena*, Int. J. Math. Math. Sci. **19**(2) (2018), 189–203.

[27] A. E. Herr, J. I. Molho, J. G. Santiago, M. G. Mungal, T. W. Kenny and M. Garguilo, *Electroosmotic capillary flow with nonuniform zeta potential*, Analytical Chemistry **72**(5) (2000), 1053–1057.

[28] G. Kirchhoff, *Vorlesungen Huber. Mechanik*, Leipzig, Teubner, 1883.

[29] V. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*, Springer Science and Business Media, 1999.

[30] O. Kovacik and J. Rakosnik, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41**(116) (1991) 592–618. `http://dml.cz/dmlcz/102493`

[31] M. Kratou, *Ground state solutions of p-Laplacian singular Kirchhoff problem involving a Riemann-Liouville fractional derivative*, Filomat **33**(7) (2019), 2073–2088. `https://doi.org/10.2298/FIL1907073K`

[32] E. E. Miller and R. D. Miller, *Physical theory for capillary flow phenomena*, J. Appl. Phys. **27**(4) (1956), 324–332. `https://doi.org/10.1063/1.1722370`

[33] N. Nyamoradi, Y. Zhou, B. Ahmad and A. Alsaedi, *Variational methods for Kirchhoff type problems with tempered fractional derivative*, Electron. J. Differ. Equ. **34** (2018), 13 pages.

[34] M. M. Rodrigues, *Multiplicity of solutions on a nonlinear eigenvalue problem for $p(x)$-Laplacian-like operators*, Mediterr. J. Math. **9** (2012), 211–223. `https://link.springer.com/article/10.1007/s00009-011-0115-y`

[35] J. Simon, *Regularite de la Solution d'une Equation non Lineaire Dans $\mathbb{R}^N$*, Journées d'Analyse Non Linéaire, Springer, Berlin, Heidelberg, 205–227.

[36] J. V. D. C. Sousa, and E. C. de Oliveira, *On the stability of a hyperbolic fractional partial differential equation*, Differ. Equ. Dyn. Syst. **31**(1) (2023). 31–52. `https://link.springer.com/article/10.1007/s12591-019-00499-3`

[37] J. V. D. C. Sousa, K. D. Kucche and J. J. Nieto, *Existence and multiplicity of solutions for fractional $k(\xi)$-Kirchhoff-type equation*, Qual. Theory Dyn. Syst. **23**(1) (2024), Article ID 27. `https://link.springer.com/article/10.1007/s12346-023-00877-x`

[38] J. Sousa, K. B. Lima and L. S. Tavares, *Existence of solutions for a singular double phase problem involving a ψ-Hilfer fractional operator via Nehari manifold*, Qual. Theory Dyn. Syst. **22**(3) (2023). 1–26. `https://link.springer.com/article/10.1007/s12346-023-00794-z`

[39] J. V. D. C. Sousa and E. C. De Oliveira, *On the ψ-Hilfer fractional derivative*, Commun. Nonlinear Sci. Numer. Simul. **60** (2018). 72–91. `https://doi.org/10.1016/j.cnsns.2018.01.005`

[40] J. V. da C. Sousa, *Existence of nontrivial solutions to fractional Kirchhoff double phase problems*, Comput. Appl. Math. **43**(2) (2024), Article ID 93. `https://link.springer.com/article/10.1007/s40314-024-02599-5`

[41] J. V. D. C. Sousa, D. S. Oliveira and L. S. Tavares, *Solutions of the mean curvature equation with the Nehari manifold*, Comput. Appl. Math. **43**(1) (2024), ARticle ID 24. `https://link.springer.com/article/10.1007/s40314-023-02534-0`

[42] J. V. D. C. Sousa, D. S. Oliveira and P. R. Agarwal, *Existence and multiplicity for fractional Dirichlet problem with $\gamma(\xi)$-Laplacian equation and Nehari manifold*, Appl. Anal. Disc. Math. **17**(2) (2023). 480–495. `https://www.jstor.org/stable/27281422`

[43] J. V. D. C. Sousa, A. Elhoussain, E. H. Hamza and L. S. Tavares, *Basic results for fractional anisotropic spaces and applications*, J. Pseudo-Differ. Oper. Appl. **15**(4) (2024), Article ID 71. `https://link.springer.com/article/10.1007/s11868-024-00641-y`

[44] H. M. Srivastava and V. da Costa Sousa, *Multiplicity of solutions for fractional-order differential equations via the $\kappa(x)$-Laplacian operator and the Genus theory*, Fractal and Fractional **6**(9) (2022), Article ID 481. `https://doi.org/10.3390/fractalfract6090481`

[45] E. H. Van Brummelen, M. Shokrpour-Roudbari and G. J. Van Zwieten, *Elasto-capillarity simulations based on the Navier-Stokes-Cahn-Hilliard equations*, in: *Advances in Computational*

Fluid-Structure Interaction and Flow Simulation: New Methods and Challenging Computations, 2016, 451–462. https://link.springer.com/chapter/10.1007/978-3-319-40827-9_35

[46] Z. Yucedag, M. Avci and R. Mashiyev, *On an elliptic system of $p(x)$-Kirchhoff-type under Neumann boundary condition*, Math. Model. Anal. **17**(2) (2012). 161–170. https://doi.org/10.3846/13926292.2012.655788

[47] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, 2023.

[1]Laboratory of Systems, Control and Decision (LSCD), School of New Engineering Sciences (ENSI), Tangier, Morocco.
* Corresponding author.

[2]Laboratory of Applied Mathematics, Computer Science and Systems, Group AMNEA. Moulay Is-mail University, Faculty of Science and Technology. Errachidia, Morocco.
Email address: arhrrabi.elhoussain@gmail.com
ORCID iD: https://orcid.org/0000-0002-9239-4603
Email address: h.elhouari94@gmail.com
ORCID iD: https://orcid.org/0000-0003-4240-1286