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STUDY OF DOUBLE PHASE-CHOQUARD PROBLEM IN GENERALIZED ψ-HILFER FRACTIONAL DERIVATIVE SPACES WITH p-LAPLACIAN OPERATOR

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ABSTRACT. In this paper, our focus is on a specific class of non-linear ψ-Hilfer fractional generalized double phase-Choquard differential equations involving the p-Laplacian operator with Dirichlet boundary conditions. The equation is given by:

$$\begin{cases} \mathcal{L}^{\gamma,\beta;\psi} u = \left(\int_\Omega \frac{G\big(u(x)\big)}{|x-y|^\lambda} dx \right) g\big(u(y)\big), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with $\mathcal{L}^{\gamma,\beta;\psi}$ is defined as:

$$\mathcal{L}^{\gamma,\beta;\psi} u := \mathbb{D}_T^{\gamma,\beta;\psi} \Big(|\mathbb{D}_{0^+}^{\gamma,\beta;\psi} u|^{p-2}\, \mathbb{D}_{0^+}^{\gamma,\beta;\psi} u + \mathbf{a}(x) |\mathbb{D}_{0^+}^{\gamma,\beta;\psi} u|^{q-2}\, \mathbb{D}_{0^+}^{\gamma,\beta;\psi} u \Big),$$

where $\mathbb{D}_T^{\gamma,\beta;\psi}$ and $\mathbb{D}_{0^+}^{\gamma,\beta;\psi}$ are ψ-Hilfer fractional derivatives of order $\frac{1}{p} < \gamma < 1$ and type $0 \leq \beta \leq 1$ and $\mathbf{a}(\cdot)$ is non-negative weight function, and $G(\cdot)$ represents Choquard nonlinearities satisfying a certain growth conditions. By employing the mountain pass theorem without the Palais-Smale condition, along with the Hardy-Littlewood-Sobolev inequality, we establish the existence of a weak solution to the aforementioned problem. Our main results are novel and contribute to the literature on problems involving ψ-Hilfer derivatives with the p-Laplacian operator. This investigation enhances the scope of understanding in this specific class of problems.

Key words and phrases. Generalized ψ-Hilfer derivative, double phase-Choquard equation, mountain pass theorem, Hardy-Littlewood-Sobolev inequality.

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1. Introduction

The equation known as the Choquard equation, given by

$$-\Delta u+u=\left(\int_{\mathbb{R}^3}\frac{u^2(y)}{|x-y|}dy\right)u,\quad u\in H^1(\mathbb{R}^3), \tag{1.1}$$

was initially introduced by Choquard in 1976 and has since captured considerable attention in the realms of physics and mathematical analysis [35]. This equation serves as an approximation to the Hartree-Fock theory of a one-component plasma, providing insights into intricate interactions between particles. Lion in [33] studied the normalized solutions of the following problem

$$-\Delta u+\lambda u=\left(\int_{\mathbb{R}^3}u^2(y)V(|x-y|)dy\right)u(x),\quad \text{in } \mathbb{R}^3, \tag{1.2}$$

where V is some given positive function. In the special case where $V=1/|x|$, equation (1.2) returns to equation (1.1). Furthermore, Penrose proposed it as a model for elucidating the self-gravitational collapse of a quantum mechanical wave function, underscoring its significance in comprehending essential quantum phenomena [36].

In the context of Choquard equations driven by a p-Laplacian operator, P. Le in [32], established the existence of weak solutions to the following semilinear Choquard equation, which appears as a model in quantum mechanics,

$$-\Delta_p u=\left(\frac{1}{|x|^{n-\alpha}}*|u|^q\right)|u|^{q-2}u,\quad u\in\mathbb{R}^n,$$

where $2\leq p<q\leq n$ and $\max\{0,n-2p\}<\alpha<n$. In [1], the authors studied the existence of semiclassical ground state solutions to the following generalized Choquard equation

$$-\Delta_p u+|u|^{p-2}=\left(\int_{\mathbb{R}^3}\frac{F(u(y))}{|x-y|}dy\right)f(u(x)),\quad \text{in } \mathbb{R}^n.$$

Alternatively, the fractional diffusion integrodifferential equation problems are as follows

$$\begin{cases} a^2\dfrac{\partial^2}{\partial x^2}T(x,t)=\displaystyle\int_0^t\frac{T(x,\tau)(t-\tau)^{-(\lambda+1)}}{\Gamma(-\lambda)}d\tau, & x\in\mathbb{R}^+, a>0, 0<\lambda\leq 1,\\ T(x,0)=\theta(x),\quad T(0,t)=0,\quad t>0. \end{cases} \tag{1.3}$$

The authors in [46] discuss the exactly solution and the asymptotic behavior of the problem (1.3) for different values of λ, i.e., for $\lambda=1$ and $0<\lambda<1$. The authors in [31] solved exactly the following fractional diffusion equation based on Riemann-Liouville fractional derivatives

$$\mathbb{D}^{\alpha}_{0^+}f(r,t)=C_\alpha\Delta f(r,t),$$

where $f(r,t)$ denotes the unknown field and C_α denotes the fractional diffusion constant with dimensions $[cm/s^\alpha]$ and $D^\alpha_{0^+}$ is the Riemann-Liouville derivative of order α.

Numerous researchers have suggested employing fractional time derivatives to address problems involving linear or non-linear differential equations. The essential question is whether there exists a connection between fractional derivatives and gradient terms. The answer is provided in [45], where the authors extend gradient elasticity models to characterize materials exhibiting fractional non-locality and fractality. They derive a generalization of three-dimensional continuum gradient elasticity theory, starting from integral relations and assuming a weak non-locality of power-law (fractional) type. This results in constitutive relations featuring fractional Laplacian terms, achieved through the application of fractional Taylor series in wave-vector space. Subsequently, the authors explore non-linear field equations with fractional derivatives of non-integer order to describe nonlinear elastic effects in gradient materials with power-law long-range interactions within the framework of weak non-locality approximation. The specific constitutive relationship detailed in this study could serve as the foundation for developing a fractional extension of deformation theory in gradient plasticity. On the other hand, related to the double phase problem, the stationary general reaction diffusion double-phase is given by the form

$$u_t = \operatorname{div}[A(x)\nabla u] + b(x,u), \quad \text{with } A(x) = |\nabla u|^{p-2} + |\nabla u|^{q-2}, \tag{1.4}$$

where the function u represents a concentration, and $\operatorname{div}[A(x)\nabla u]$ relates to diffusion with diffusion coefficient $A(x)$. The term $b(x,u)$ corresponds to sources and loss processes, and this type of problem has applications in physics and allied fields such as biophysics, plasma physics, solid state physics, and chemical reaction design. For more information, refer to [4, 7]. One example of this type of problem is the following equation:

$$-\Delta_p u - \Delta_q u + |u|^{p-2}u + |u|^{q-2}u = f(x,u) \quad \text{in } \mathbb{R}^n, \quad 1 < p \le q < +\infty,$$

which is connected with the general reaction-diffusion system (1.4). The equation involves two distinct materials with power-hardening exponents p and q. Many authors have established existence, multiplicity, and regularity results for this type of problem in bounded or unbounded domains, as discussed in [23, 24, 30] and the references therein.

In the context of fractional differential equation boundary-value problems with p-Laplacian operator, for the existence and non-existence of weak solutions to the nonlinear examination of solution existence and stability can be found in [39], the equation as the form

$$\begin{cases} \mathbb{D}_T^{\alpha,\beta;\psi}\left(|\mathbb{D}_{0^+}^{\alpha,\beta;\psi}u(x)|^{p-2}\, \mathbb{D}_{0^+}^{\alpha,\beta;\psi}u(x)\right) = \lambda|u(x)|^{p-2}u(x) + b(x)|u(x)|^{q-1}u(x), \\ \mathbf{I}_{0^+}^{\beta(\beta-1);\psi}u(0) = \mathbf{I}_T^{\beta(\beta-1);\psi}u(T) = 0, \end{cases}$$

where $\mathbb{D}_{0^+}^{\alpha,\beta;\psi}$, $\mathbb{D}_T^{\alpha,\beta;\psi}$ are ψ-Hilfer fractional derivatives left-sided and right-sided of order $\frac{1}{p} < \alpha < 1$, type $0 \le \beta \le 1$, $1 < q < p-1 < +\infty$, $b \in L^\infty(\Omega)$ and $\mathbf{I}_{0^+}^{\beta(\beta-1);\psi}$,

$\mathbf{I}_{T}^{\beta(\beta-1);\psi}$ are ψ-Riemann-Liouville fractional integrals left-sided and right-sided, for all $x \in \Omega = [0,T]$.

In 2023, Sousa et al. [44], discussed the existence and regularity of weak solutions for ψ-Hilfer fractional boundary value problem by using an extension of the Lax-Milgram theorem to the following nonlinear boundary value problem

$$\begin{cases} \mathbb{D}_{T}^{\alpha,\beta;\psi}\Big(|\mathbb{D}_{0^+}^{\alpha,\beta;\psi}u(x)|^{p-2}\ \mathbb{D}_{0^+}^{\alpha,\beta;\psi}u(x)\Big) + u(x) = \lambda\Phi(t,u(x)), & t \in (0,T),\\ \mathbf{I}_{0^+}^{\beta(\beta-1);\psi}u(0) = \mathbf{I}_{T}^{\beta(\beta-1);\psi}u(T) = 0, \end{cases}$$

where $\mathbb{D}_{T}^{\alpha,\beta;\psi}$, $\mathbb{D}_{0^+}^{\alpha,\beta;\psi}$ are ψ-Hilfer fractional derivatives left-sided and right-sided of order $\frac{1}{2} < \alpha < 1$, type $0 \leq \beta \leq 1$, respectively, $\mathbf{I}_{0^+}^{\beta(\beta-1);\psi}$, $\mathbf{I}_{T}^{\beta(\beta-1);\psi}$ are ψ-Riemann-Liouville fractional integrals left-sided and right-sided of order $\beta(\beta-1)$, respectively, λ is a parameter and $\Phi : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In [38], the authors established the existence of solutions to the following new class of singular double phase p-Laplacian equation problems with a ψ-Hilfer fractional operator combined from a parametric term, namely:

$$\begin{cases} \mathbb{D}_{T}^{\alpha,\beta;\psi}\Big(|\mathbb{D}_{0^+}^{\alpha,\beta;\psi}u|^{p-2}\ \mathbb{D}_{0^+}^{\alpha;\beta;\psi}u + \mu(x)|\mathbb{D}_{0^+}^{\alpha,\beta;\psi}u|^{q-2}\ \mathbb{D}_{0^+}^{\alpha,\beta;\psi}u\Big) & \\ = \xi(x)u^{-\sigma} + \lambda u^{r-1}, & \text{in} \quad \Omega = [0,T] \times [0,T],\\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

We can not quote all reference in the existence of solution for fractional equation, for that we refer for interesting reader to [2,3,5,8–22,25–29,40–45].

Motivated by these results, we turn our attention to the exploration of the existence solution in a suitable fractional ψ-Hilfer derivative space for the double phase-Choquard problem with p-Laplacian operator presented in this paper, namely

$$\begin{cases} \mathcal{L}^{\gamma,\beta;\psi}u = \left(\int_{\Omega} \dfrac{G(u(x))}{|x-y|^{\lambda}}dx\right) g(u(y)), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

with

$$\mathcal{L}^{\gamma,\beta;\psi}u := \ \mathbb{D}_{T}^{\gamma,\beta;\psi}\Big(|\mathbb{D}_{0^+}^{\gamma,\beta;\psi}u|^{p-2}\ \mathbb{D}_{0^+}^{\gamma,\beta;\psi}u + \mathbf{a}(x)|\mathbb{D}_{0^+}^{\gamma,\beta;\psi}u|^{q-2}\ \mathbb{D}_{0^+}^{\gamma,\beta;\psi}u\Big),$$

where $\mathbb{D}_{T}^{\gamma,\beta;\psi}$ and $\mathbb{D}_{0^+}^{\gamma,\beta;\psi}$ are ψ-Hilfer fractional derivatives of order $\frac{1}{p} < \gamma < 1$ and type $0 \leq \beta \leq 1$ and $\mathbf{a}(x)$ is non-negative weight function with compact support in Ω and $g : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying:

(g_1) $N \geq 2$, $1 < p < N$, $p < q < p + \frac{\alpha p}{N}$ and $\mathbf{a}(\cdot) \in C_0^{\infty}(\Omega)$ with $0 < \alpha \leq 1$;

(g_2) the growth condition, i.e.,

$$|g(\xi)| \leq c_1\Big(|\xi|^{r_1-1} + |\xi|^{r_2-1}\Big), \quad \text{for all } \xi \in \mathbb{R}^N \text{ and } c_1 > 0,$$

where

$$p < \tau r_1 \leq \tau r_2 < p^*, \tag{1.6}$$

and $\tau = 2N/(2N-\lambda) > 1$ and p^* being the critical Sobolev exponent to p;

(g_3) there is $\alpha > q$ such that

$$0 < \alpha G(\xi) \leq 2g(\xi)\xi, \quad \text{where } G(\xi) := \int_0^\xi g(\tau)d\tau.$$

To our surprise, these results represent the first contributions available in the literature for the ψ-Hilfer fractional generalized double phase-Choquard differential equations involving the p-Laplacian operator with Dirichlet boundary conditions within the framework of ψ-fractional derivative space $\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$. Our approach to establishing existence results for problem (1.5) hinges on utilizing the mountain pass theorem without the Palais-Smale condition [6]. Initially, we demonstrate that the energy functional $\mathfrak{E}$ connected to the problem (1.5) adheres to the mountain pass geometry. We establish the boundedness of a sequence $\{u_k\}_{k\in\mathbb{N}}$ in $\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$ that does not satisfy the Palais-Smale condition. Ultimately, we leverage the properties of the transformed sequence along with technical skills to achieve the existence result for problem (1.5). One of the key challenges in this approach lies in utilizing the Hardy-Littlewood-Sobolev inequality for nonlinearities involving ψ-Hilfer fractional derivative.

This work is organized as follows. In Section 2, we provide a brief overview of the key features of Musielak spaces and ψ-fractional derivative spaces. Moving on to Section 3, we present the existing solutions to problems (1.5), along with their corresponding proofs.

2. Preliminary

In this section we refer to [38]. Consider the nonlinear function $\mathcal{H} : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\mathcal{H}(x,u) = u^p + \mathbf{a}(x)u^q.$$

Let $\mathbf{M}(\Omega)$ be the space of all measurable functions $u : \Omega \to \mathbb{R}$. Then, Musielak space $L^{\mathcal{H}}\big(\Omega\big)$ is given by

$$L^{\mathcal{H}}(\Omega) = \left\{ u \in \mathbf{M}(\Omega) : \varrho^{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x,|u|)dx < +\infty \right\},$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf\left\{ \delta > 0 : \varrho^{\mathcal{H}}\left(\frac{u}{\delta}\right) \leq 1 \right\}.$$

Moreover, we define the weighted space

$$L^q_{\mathbf{a}}(\Omega) = \left\{ \mathbf{f} \in \mathbf{M}\big(\Omega\big) : \int_\Omega \mathbf{a}(x)|\mathbf{f}|^q dx < +\infty \right\},$$

with the seminorm

$$\|\mathbf{f}\|_{\mathbf{a},q} = \left(\int_\Omega \mathbf{a}(x)|\mathbf{f}|^q \right)^{\frac{1}{q}}.$$

ψ-fractional derivative space. Let $A := [c, d]$, $-\infty \leq c < d \leq +\infty$, $n-1 < \gamma < n$, $n \in \mathbb{N}$, $\mathbf{f}$, $\psi \in C^n(I, \mathbb{R})$ such that ψ is increasing and $\psi'(x) \neq 0$, for all $u \in A$.

• The left-sided fractional ψ-Hilfer integrals of a function $\mathbf{f}$ is given by

$$\mathbf{I}_c^{\gamma;\psi}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)} \int_0^x \psi'(y)(\psi(x) - \psi(y))^{\gamma-1}\mathbf{f}(y)\mathrm{dv}. \tag{2.1}$$

• The right-sided fractional ψ-Hilfer integrals of a function $\mathbf{f}$ is given by

$$\mathbf{I}_d^{\gamma;\psi}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)} \int_x^d \psi'(y)(\psi(y) - \psi(x))^{\gamma-1}\mathbf{f}(y)\mathrm{dv}. \tag{2.2}$$

• The left-sided ψ-Hilfer fractional derivatives for a function $\mathbf{f}$ of order γ and type $0 \leq \beta \leq 1$ is defined by

$$\mathbb{D}_c^{\gamma,\beta;\psi}\mathbf{f}(x) = \mathbf{I}_c^{\beta(n-\gamma);\psi}\Big(\frac{1}{\psi'(x)} \cdot \frac{d}{dx}\Big)^n \mathbf{I}_c^{(1-\beta)(n-\gamma);\psi}\mathbf{f}(x).$$

• The right-sided ψ-Hilfer fractional derivatives for a function $\mathbf{f}$ of order γ and type $0 \leq \beta \leq 1$ is defined by

$$\mathbb{D}_c^{\gamma,\beta;\psi}\mathbf{f}(x) = \mathbf{I}_d^{\beta(n-\gamma);\psi}\Big(-\frac{1}{\psi'(x)} \cdot \frac{d}{dx}\Big)^n \mathbf{I}_d^{(1-\beta)(n-\gamma);\psi}\mathbf{f}(x).$$

Choosing $\beta \to 1$, we obtain ψ-Caputo fractional derivatives left-sided and right-sided, given by

$$\mathbb{D}_c^{\gamma;\psi}\mathbf{f}(x) = \mathbf{I}_c^{(n-\gamma);\psi}\Big(\frac{1}{\psi'(x)} \cdot \frac{d}{dx}\Big)^n \mathbf{f}(x), \tag{2.3}$$

$$\mathbb{D}_d^{\gamma;\psi}\mathbf{f}(x) = \mathbf{I}_d^{(n-\gamma);\psi}\Big(-\frac{1}{\psi'(x)} \cdot \frac{d}{dx}\Big)^n \mathbf{f}(x). \tag{2.4}$$

Remark 2.1. The ψ-Hilfer fractional derivatives defined as above can be written in the following form

$$\mathbb{D}_c^{\gamma,\beta;\psi}\mathbf{f}(x) = \mathbf{I}_c^{\mu-\gamma;\psi}\mathbb{D}_c^{\gamma;\psi}\mathbf{f}(x)$$

and

$$\mathbb{D}_d^{\gamma,\beta;\psi}\mathbf{f}(x) = \mathbf{I}_d^{\mu-\gamma;\psi}\mathbb{D}_d^{\gamma;\psi}\mathbf{f}(x),$$

with $\mu = \gamma + \beta(n-\gamma)$ and $\mathbf{I}_c^{\mu-\gamma;\psi}$, $\mathbf{I}_d^{\mu-\gamma;\psi}$, $\mathbb{D}_c^{\gamma;\psi}$ and $\mathbb{D}_d^{\gamma;\psi}$ as defined in (2.1), (2.2), (2.3) and (2.4).

In this paper, we take $\Omega = A_1 \times \cdots \times A_N = [c_1, d_1] \times \cdots \times [c_N, d_N]$ where $0 < c_i < d_i$ for all $i \in \mathbb{N}$, $0 < \gamma_1, \ldots, \gamma_N < 1$. Consider also $\psi(\cdot)$ to be an increasing and positive monotone function on $(c_1, d_1), \ldots, (c_N, d_N)$, having a continuous derivative $\psi'(\cdot)$ on $(c_1, d_1], \ldots, (c_N, d_N]$.

• The ψ-Riemann-Liouville fractional partial integral of order γ of N-variables $\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_N)$ is defined by

$$\mathbf{I}_{c,x}^{\gamma;\psi}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)} \int_{A_1}\int_{A_2} \cdots \int_{A_N} \psi'(y)(\psi(x) - \psi(y))^{\gamma-1}\mathbf{f}(y)\mathrm{d}v,$$

with $\psi'(y)(\psi(x)-\psi(y))^{\gamma-1} = \psi'(v_1)(\psi(x_1)-\psi(v_1))^{\gamma_1-1}\cdots\psi'(v_N)(\psi(x_N)-\psi(v_N))^{\gamma_N-1}$ and $\Gamma(\gamma) = \Gamma(\gamma_1)\Gamma(\gamma_2)\cdots\Gamma(\gamma_N)$, $u_i = u_1u_2\cdots u_N$ and $dv_i = dv_1dv_2\cdots dv_N$, for all $i \in \{1, 2, \dots, N\}$.

• The ψ-Hilfer fractional partial derivative of N-variables of order γ and type β $(0 \le \beta \le 1)$ is defined by

$$\mathbb{D}^{\gamma,\beta;\psi}_{c,x_i}\mathbf{f}(x_i) = \mathbf{I}^{\beta(n-\gamma);\psi}_{c,x_i}\Big(\frac{1}{\psi'(x_i)}\cdot\frac{\partial^N}{\partial x_i}\Big)\mathbf{I}^{(1-\beta)(n-\gamma);\psi}_{c,x_i}\mathbf{f}(x_i),$$

with $\partial x_i = \partial x_1, \partial x_2, \dots, \partial x_N$ and $\psi'(x_i) = \psi'(x_1)\psi'(x_2)\cdots\psi'(x_N)$ for all $i \in \{1, 2, \dots, N\}$. Analogously, it is defined $\mathbb{D}^{\gamma,\beta;\psi}_{d,x_i}(\cdot)$.

• The left-sided ψ-fractional derivative space $\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$ is defined by

$$\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega) = \Big\{u \in \mathbf{L}^{\mathcal{H}}(\Omega) :\ |\mathbb{D}^{\gamma,\beta;\psi}_{0+}u| \in \mathbf{L}^{\mathcal{H}}(\Omega); u = 0 \text{ a.e } \Omega \setminus 0\Big\},$$

equipped with the norm

$$\|u\|_{0,\mathcal{H}} = \|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u\|_{\mathcal{H}} = \||\mathbb{D}^{\gamma,\beta;\psi}_{0+}u|\|_{\mathcal{H}}$.

Remark 2.2. Note that $\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega) := \overline{C_0^\infty(\Omega)}^{\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)}$, and the equivalent norm on $\mathbb{H}^{\gamma,\beta;\psi}_{0,\mathcal{H}}(\Omega)$ is given by $\|u\|_{0,\mathcal{H}} = \|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u\|_{\mathcal{H}}$.

The results below will be needed for our purposes.

Proposition 2.1 ([39])**.** *Let (g_1) be satisfied. Then, the following embeddings hold:*
(i) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ and $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ are continuous for all $r \in [1, p]$;
(ii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for all $r \in [1, p^*]$;
(iii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ is compact for all $r \in [1, p^*)$;
(iy) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^q_{\mathbf{a}}(\Omega)$ is continuous;
(vi) $L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ is continuous.

Proposition 2.2 ([39])**.** *Let (g_1) be satisfied, $v \in L^{\mathcal{H}}(\Omega)$, $c > 0$ and $\varrho^{\mathcal{H}}$ previously defined. Then, the following hold:*
(i) if $v \neq 0$, then $\|v\|_{\mathcal{H}} = c$ if and only if $\varrho^{\mathcal{H}}\left(\frac{v}{c}\right) = 1$;
(ii) $\|v\|_{\mathcal{H}} < 1$ (resp. $> 1, = 1$) if and only if $\varrho^{\mathcal{H}}(v) < 1$ (resp. $> 1, = 1$);
(iii) if $\|v\|_{\mathcal{H}} < 1$, then $\|v\|^q_{\mathcal{H}} \le \varrho^{\mathcal{H}}(v) \le \|v\|^p_{\mathcal{H}}$;
(iy) if $\|v\|_{\mathcal{H}} > 1$, then $\|v\|^p_{\mathcal{H}} \le \varrho^{\mathcal{H}}(v) \le \|\boldsymbol{y}\|^q_{\mathcal{H}}$;
(y) $\|v\|_{\mathcal{H}} \to 0$ if and only if $\varrho^{\mathcal{H}}(v) \to 0$;
(vi) $\|v\|_{\mathcal{H}} \to +\infty$ if and only if $\varrho^{\mathcal{H}}(v) \to +\infty$.

Proposition 2.3 ([39])**.** *Let (g_1) be satisfied. Then, the embedding $\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for all $r \in [p, p^*]$.*

Remark 2.3. From (1.6) and Proposition 2.3, we have

$$\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega) \hookrightarrow L^{\tau r_i}(\Omega), \quad \text{with } r_i \in [p, p^*], i = 1, 2. \tag{2.5}$$

Proposition 2.4 (Hardy-Littlewood-Sobolev inequality)**.** *Let* $p, q > 1$ *and* $0 < \lambda < N$ *with* $1/p + \lambda/N + 1/q = 2$, $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$*. Then, there exists a sharp constant* $C(p, N, \lambda, q)$*, independent of* f, g*, such that*

$$\left|\int_{\Omega\times\Omega} \frac{f(x)g(y)}{|x-y|^\lambda} dxdy\right| \leq C(p, N, \lambda, q)||f||_{L^p(\Omega)}||g||_{L^q(\Omega)}. \tag{2.6}$$

3. Main Result

Our first main result is the following.

Theorem 3.1. *The problem* (1.5) *has a nontrivial solution under the conditions* (g_1)*-*(g_3)*.*

In the proof of Theorem 3.1 we will use variational methods. The energy functional $\mathfrak{E} : \mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega) \to \mathbb{R}$ associated with (1.5) is given by

$$\mathfrak{E}(u) = \frac{1}{p}\|\mathbb{D}_{0+}^{\gamma,\beta;\psi} u\|_p^p + \frac{1}{q}\|\mathbb{D}_{0+}^{\gamma,\beta;\psi} u\|_{\mathbf{a},q}^q - \Psi(u),$$

where

$$\Psi(u) = \frac{1}{2}\int_\Omega\int_\Omega \frac{G(u(x))G(u(y))}{|x-y|^\lambda} dxdy.$$

Theorem 3.1 is proved in several steps.

Step 1. The energy functional $\mathfrak{E}$ satisfies the mountain pass geometry, i.e., satisfies the following lemma.

Lemma 3.1. *The functional* $\mathfrak{E}$ *exhibits the following characteristics.*

(*i*) *For sufficiently small* $\rho > 0$, $\mathfrak{E}(u) \geq \eta$ *holds for* $u \in \mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$ *with* $\|u\|_{0,\mathcal{H}} = \rho$*, where* $\eta > 0$*.*

(*ii*) *There exists an element* $e \in \mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$ *such that* $\|e\|_{0,\mathcal{H}} > \rho$ *and* $\mathfrak{E}(e) < 0$*.*

First, we need to demonstrate the following useful property.

Proposition 3.1. *For each* $v \in \mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$*, we have the following property:* G *and* gv *are belong to* $L^\tau(\Omega)$*.*

Proof. Due to (g_2), if $u \in \mathbb{R}$ and $u(x) \neq 0$, then

$$|g(u(x))| \leq c_1\Big(|u(x)|^{r_1-1} + |u(x)|^{r_2-1}\Big). \tag{3.1}$$

Also, from the last inequality, we deduce

$$|G(v)| = \left|\int_0^v g(x)dx\right| \leq c_1\int_0^v \Big(|u|^{r_1-1} + |u|^{r_2-1}\Big)dx \leq c_1\Big(|v|^{r_1} + |v|^{r_2}\Big).$$

Hence,

$$|G(v)|^\tau \leq c_2\Big(|v|^{\tau r_1} + |v|^{\tau r_2}\Big), \tag{3.2}$$

where $c_2 = c_1^\tau$. Now, utilizing (2.5), we deduce that $G \in L^\tau(\Omega)$. By a similar argument as above, which applies to $g(u)v$, we deduce that $gv \in L^\tau(\Omega)$. □

Lemma 3.2. *For each $u \in \mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$. We have the following properties:*

(*i*)

$$\int_{\Omega\times\Omega} \frac{|G(u(x))g(u(y))v(y)|}{|x-y|^{\lambda}} dxdy < +\infty, \quad \textit{for all } v \in C_0^{\infty}(\Omega);$$

(*ii*)

$$\int_{\Omega\times\Omega} \frac{|G(u(x))g(u(y))v(y)|}{|x-y|^{\lambda}} dxdy \leq C\|G(u)\|_{L^{\tau}}\|g(u)v\|_{L^{\tau}}, \quad \textit{for all } v \in C_0^{\infty}(\Omega).$$

Proof. Taking into account Proposition 3.1, equations (2.5), (3.1), and the fact that $(a+b)^p \leq 2^{p-1}(a^p+b^p)$, for every $a, b \geq 0$ and $1 \leq p < +\infty$, as well as Proposition 2.4, we arrive at the proof of Lemma 3.2. ☐

Corollary 3.1. *For each $v \in \mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$, with $||v||_{0,\mathcal{H}} \leq 1$, the sequence $\{||g(u_k)v||_{L^{\tau}}\}\ k \in \mathbb{N}$ is bounded.*

Proof. Due to (3.1) and (2.5), we obtain

$$\begin{aligned}\int_{\Omega} |g(u_k(y))v(y)|^{\tau} dy \leq & c_2 \int_{\Omega} \Big(|u_k(y)|^{\tau(r_1-1)} + |u_k(y)|^{\tau(r_2-1)}\Big)|v(y)|^{\tau} dy \\ \leq & c_2 \Bigg[\Big(\int_{\Omega} |u_k|^{\tau r_1} dy\Big)^{\frac{r_1-1}{r_1}} \Big(\int_{\Omega} |v(y)|^{\tau r_1} dy\Big)^{\frac{1}{r_1}} + \Big(\int_{\Omega} |u_k|^{\tau r_2} dy\Big)^{\frac{r_2-1}{r_2}} \\ & \times \Big(\int_{\Omega} |v(y)|^{\tau r_2} dy\Big)^{\frac{1}{r_2}}\Bigg] \\ = & c_2\Big(||u_k(y)||^{\tau(r_1-1)}_{L^{\tau r_1}} ||v(y)||^{\tau}_{L^{\tau r_1}} + ||u_k(y)||^{\tau(r_2-1)}_{L^{\tau r_2}} ||v(y)||^{\tau}_{L^{\tau r_2}}\Big) \\ \leq & c_2\Big(||u_k(y)||^{\tau(r_1-1)}_{L^{\tau r_1}} + ||u_k(y)||^{\tau(r_2-1)}_{L^{\tau r_2}}\Big) < +\infty. \qquad (3.3)\end{aligned}$$

☐

Corollary 3.2. *The function $\mathfrak{E}$ belongs to $C^1\left(\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega), \mathbb{R}\right)$, and we can express it as follows:*

$$\begin{aligned}\mathfrak{E}'(u)v = & \int_{\Omega} \Big(|\mathbb{D}^{\gamma,\beta;\psi}_{0^+} u|^{p-2}\ \mathbb{D}^{\gamma,\beta;\psi}_{0^+} u + \mathbf{a}(x)|\mathbb{D}^{\gamma,\beta;\psi}_{0^+} u|^{q-2}\ \mathbb{D}^{\gamma,\beta;\psi}_{0^+} u\Big)\ \mathbb{D}^{\gamma,\beta;\psi}_{0^+} v dx \\ & - \int_{\Omega}\int_{\Omega} \frac{G(u(x))g(u(y))v(y)}{|x-y|^{\lambda}} dxdy, \qquad (3.4)\end{aligned}$$

for all $u, v \in \mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$.

Proof. Using the analysis presented earlier, along with a similar approach as in Lemma 3.2 of [1], we confirm the validity of this corollary. ☐

Proof of the main Lemma 3.1. Let prove (i). Applying Propositions 3.1, Lemma 3.4 and Proposition 2.4, we have

$$\Big|\int_{\Omega\times\Omega}\frac{G(u(x))G(u(y))}{|x-y|^{\lambda}}dxdy\Big|\leq c_3\|G(u)\|^2_{L^{\tau}(\Omega)},$$

for all $u\in\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$. Due to (3.1) and (3.2), we have

$$\|G(u)\|_{L^{\tau}(\Omega)}\leq c_4\Big(\|u\|^{r_1}_{L^{\tau r_1}}+\|u\|^{r_2}_{L^{\tau r_2}}\Big)\leq c_5\Big(\|u\|^{r_1}_{0,\mathcal{H}}+\|u\|^{r_2}_{0,\mathcal{H}}\Big), \tag{3.5}$$

where c_5 is a constant that does not depend on $u\in\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$ with $\|u\|_{0,\mathcal{H}}=\|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u\|_{\mathcal{H}}+\|u\|_{\mathcal{H}}<1$ we get that

$$\begin{aligned}\mathfrak{E}(u)&\geq\frac{1}{p}\|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u\|^p_p+\frac{1}{q}\|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u\|^q_{\mathbf{a},q}-c_6\Big(\|u\|^{r_1}_{0,\mathcal{H}}+\|u\|^{r_2}_{0,\mathcal{H}}\Big)\\&\geq c_6\|u\|^p_{0,\mathcal{H}}-c_7\Big(\|u\|^{r_1}_{0,\mathcal{H}}+\|u\|^{r_2}_{0,\mathcal{H}}\Big),\end{aligned}$$

where c_6 and c_7 are constants that do not depend on u. The fact that $r_2>p/2$, then the result follows by fixing $\|u\|_{0,\mathcal{H}}=\rho$ with $\rho>0$ small enough.

Let prove (ii). Let us fix $u_0\in\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)\backslash\{0\}$ with $u_0>0$ and define

$$J(t)=G\Big(x,\frac{tu_0}{\|u_0\|_{0,\mathcal{H}}}\Big),\quad\text{for } t>0, u\in\Omega,$$

The condition (g_2) implies that

$$\frac{J'(t)}{J(t)}\geq\frac{\alpha}{2t},\quad\text{for } t>0,$$

Integrating this over $[1,s\|u_0\|_{0,\mathcal{H}}]$ with $s>\frac{1}{\|u_0\|_{0,\mathcal{H}}}$, we get

$$G(x,su_0)\geq G\Big(x,\frac{u_0}{\|u_0\|_{0,\mathcal{H}}}\Big)\Big(s\|u_0\|_{0,\mathcal{H}}\Big)^{\alpha/2}.$$

With this, we are able to compose

$$\begin{aligned}\mathfrak{E}(su_0)\leq&\frac{s^p}{p}\|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u_0\|^p_p+\frac{s^q}{q}\|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u_0\|^q_{\mathbf{a},q}\\&-\frac{s^{\alpha}\|u_0\|^{\alpha}_{0,\mathcal{H}}}{2}\int_{\Omega}\Bigg(\int_{\Omega}\frac{G\Big(v,\frac{u_0}{\|u_0\|}\Big)}{|x-y|^{\lambda}}dy\Bigg)G\Big(x,\frac{u_0}{\|u_0\|}\Big)dx.\end{aligned}$$

Since $\alpha>q>p$ we can choose $s>\frac{1}{\|u_0\|_{0,\mathcal{H}}}$ large enough such that $e=su_0$ with $\|e\|_{0,\mathcal{H}}>\rho$ and $\mathfrak{E}(e)<0$. This finishes the proof of the main Lemma 3.1. □

Step 2. The sequence $\{u_k\}_{k\in\mathbb{N}}$, which does not satisfy the (PS)-condition, is a bounded sequence in $\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$. Recalling that the mountain pass theorem without the

Palais-Smale condition (refer to [6], Theorem 5.4.1) states the existence of a sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$ such that:

$$\mathfrak{E}(u_k) \to \theta \tag{3.6}$$

and

$$\mathfrak{E}'(u_k) \to 0, \tag{3.7}$$

where $\theta > 0$ is the mountain pass level defined by

$$\theta := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \mathfrak{E}(\gamma(t)),$$

with

$$\Gamma := \Big\{\gamma \in C\Big(\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega), \mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)\Big) : \gamma(0) = 0, \gamma(1) = e\Big\}.$$

Concerning the sequence mentioned earlier, we observe the following auxiliary characteristic.

Lemma 3.3. *The sequence* $\{u_k\}_{k\in\mathbb{N}}$ *is bounded in* $\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$.

Proof. Note that

$$\mathfrak{E}(u_k) - \frac{\mathfrak{E}'(u_k)u_k}{\alpha} \le \theta + 1 + \|u_k\|_{0,\mathcal{H}}, \tag{3.8}$$

for k large enough. Moreover, from Proposition 2.2 and (g_3) we have for $\|u_k\|_{0,\mathcal{H}} \ge 1$ that

$$\begin{aligned}\mathfrak{E}(u_k) - \frac{\mathfrak{E}'(u_k)u_k}{\alpha} =& \Big(\frac{1}{p} - \frac{1}{\alpha}\Big)\|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u_k\|^p_p + \Big(\frac{1}{q} - \frac{1}{\alpha}\Big)\|\mathbb{D}^{\gamma,\beta;\psi}_{0+}u_k\|^q_{\mathbf{a},q}\\ &+ \int_\Omega\int_\Omega \frac{G(x,u_k(x))}{|x-y|^\lambda}\Big(\frac{g(v,u_k(y))u_k(y)}{\alpha} - \frac{G(v,u_k(y))}{2}\Big)dxdy\\ \ge& \Big(\frac{1}{q} - \frac{1}{\alpha}\Big)\|u_k\|_{0,\mathcal{H}}.\end{aligned} \tag{3.9}$$

Hence, (3.8) and (3.9) owing to the boundedness of $\{u_k\}_{k\in\mathbb{N}}$ in $\mathbb{H}^{\gamma,\beta,\psi}_{0,\mathcal{H}}(\Omega)$. □

Step 3. Existence of solution, i.e., critical point of $\mathfrak{E}$.
First, we need the following two lemmas.

Lemma 3.4. *The following limits hold for a subsequence:*
(i)

$$\int_{\Omega\times\Omega} \frac{G(x,u_k(x))g(v,u(y))v(y)}{|x-y|^\lambda}dxdv \to \int_{\Omega\times\Omega} \frac{G(x,u(x))g(v,u(y))v(y)}{|x-y|^\lambda}dxdv,$$

for all $v \in C^\infty_c(\Omega)$*;*
(ii)

$$\int_\Omega\int_\Omega \frac{G(x,u_k(x))\Big(g(v,u_k(y))v(y) - g(v,u(y))v(y)\Big)}{|x-y|^\lambda}dxdy \to 0,$$

for all $v \in C_c^\infty(\Omega)$*;*

(iii)

$$\int_\Omega \int_\Omega \frac{G(u_k(x))g(u_k(y))v(y)}{|x-y|^\lambda}dxdy \to \int_\Omega \int_\Omega \frac{G(u(x))g(u(y))v(y)}{|x-y|^\lambda}dxdv,$$

for all $v \in C_c^\infty(\Omega)$*.*

Proof. (i) Lemma 3.3, (2.5), and Proposition 3.1 collectively establish that $\{G(u_k)\}_{k\in\mathbb{N}}$ forms a bounded sequence in $L^\tau(\mathbb{R}^n)$. Leveraging the continuity of G, along with the previously mentioned information and the pointwise convergence $G(u_k(x)) \to G(u(x))$ almost everywhere in $\mathbb{R}^n$, we deduce that $G(u_k) \rightharpoonup G(u)$ in $L^\tau(\mathbb{R}^n)$. By virtue of Proposition 2.4, it follows that the function

$$H(w) := \int_\Omega \int_\Omega \frac{w(x)g(u(y))v(y)}{|x-y|^\lambda}, \quad w \in L^\tau(\Omega),$$

defines a continuous linear functional. Since $G(u_k) \rightharpoonup G(u)$ in $L^\tau(\Omega)$, it follows that

$$\int_\Omega \int_\Omega \frac{G(u_k(x))g(u_k(y))v(y)}{|x-y|^\lambda}dxdy \to \int_\Omega \int_\Omega \frac{G(u(x))g(u(y))v(y)}{|x-y|^\lambda}dxdy,$$

which proves (i).

For (ii), since $\{G(u_k)\}_{k\in\mathbb{N}}$ is bounded in $L^\tau(\Omega)$, we have

$$\begin{aligned}&\left|\int_{\Omega\times\Omega} \frac{G(u_k(x))(g(u_k(y))v(y) - g(u(y))v(y))}{|x-y|^\lambda}dxdy\right| \\ \leq & c_8\Big\|G(u_k)\Big\|_{L^\tau(\Omega)}\Big\|g(u_k)v - g(u)v\Big\|_{L^\tau(\Omega)} \\ \leq & c_9\Big\|g(u_k)v - g(u)v\Big\|_{L^\tau(\Omega)}.\end{aligned}$$

Let $v \in C_c^\infty(\Omega)$. Since Ω is bounded the compactness of the embeddings $\mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega) \hookrightarrow L^{\tau r_2}(\Omega)$, implies that there exist $h \in L^{\tau r_1}(\Omega)$ and $w \in L^{\tau r_2}(\Omega)$ such that

$$u_k(x) \to u(x) \text{ a.e in } \Omega, \quad |u_k(x)| \leq h(x) \quad \text{and} \quad |u_k(x)| \leq w(x) \text{ a.e in } \Omega. \tag{3.10}$$

Combining (3.10) with Lebesgue's dominated convergence theorem, we infer that

$$\|g(u_k)v - g(u)v\|_{L^\tau(\Omega)} = \|g(u_k)v - g(u)v\|_{L^\tau(\Omega)} \to 0.$$

The proof for (ii) is now complete, and (iii) directly follows from both (i) and (ii). ☐

Proposition 3.2. *Let hypotheses* (g_1)*-*(g_3) *be satisfied. For a subsequence of* $\{u_k\}_{k\in\mathbb{N}}$*, we have*

$$\mathbb{D}_{0^+}^{\gamma,\beta,\psi}u_k \to \mathbb{D}_{0^+}^{\gamma,\beta,\psi}u, \quad \textit{pointwise a.e. in } \Omega.$$

Consequently, it holds

$$\left|\mathbb{D}_{0^+}^{\gamma,\beta,\psi}u_k\right|^{p-2}\mathbb{D}_{0^+}^{\gamma,\beta,\psi}u_k \rightharpoonup \left|\mathbb{D}_{0^+}^{\gamma,\beta,\psi}u\right|^{p-2}\mathbb{D}_{0^+}^{\gamma,\beta,\psi}u, \quad \textit{in } \left[L^{\frac{p}{p-1}}(\Omega)\right]^N, \tag{3.11}$$

$$\left|\mathbb{D}_{0^+}^{\gamma,\beta,\psi}u_k\right|^{q-2}\mathbb{D}_{0^+}^{\gamma,\beta,\psi}u_k \rightharpoonup \left|\mathbb{D}_{0^+}^{\gamma,\beta,\psi}u\right|^{q-2}\mathbb{D}_{0^+}^{\gamma,\beta,\psi}u, \quad in \ \left[L^{\frac{q}{q-1}}(\Omega)\right]^N. \tag{3.12}$$

Proof. We refer to the proof of Lemma 13 in [37]. □

Lemma 3.5. *The function u is a critical point of $\mathfrak{E}$.*

Proof. First of all, we claim that

$$\mathfrak{E}'(u_k)v \to \mathfrak{E}'(u)v, \quad \text{for all } v \in C_c^\infty(\Omega).$$

To verify such limit, note that

$$\begin{aligned}\mathfrak{E}'(u)v =& \int_\Omega \left(|\mathbb{D}_{0^+}^{\gamma,\beta;\psi}u|^{p-2}\,\mathbb{D}_{0^+}^{\gamma,\beta;\psi}u + \mathbf{a}(x)|\mathbb{D}_{0^+}^{\gamma,\beta;\psi}u|^{q-2}\,\mathbb{D}_{0^+}^{\gamma,\beta;\psi}u\right)\cdot \mathbb{D}_{0^+}^{\gamma,\beta;\psi}v dx \\ &- \int_\Omega\int_\Omega \frac{G(u(x))g(u(y))v(y)}{|x-y|^\lambda}dxdy,\end{aligned}$$

Lemma 3.4, Proposition 3.2 owing to

$$\int_{\Omega\times\Omega}\frac{G(u_k(x))g(u_k(y))v(y)}{|x-y|^\lambda}dxdy \to \int_{\Omega\times\Omega}\frac{G(u(x))g(u(y))v(y)}{|x-y|^\lambda}dxdy \tag{3.13}$$

and

$$\begin{aligned}&|\mathbb{D}_{0^+}^{\gamma,\beta;\psi}u_k|^{p-2}\,\mathbb{D}_{0^+}^{\gamma,\beta;\psi}u_k + \mathbf{a}(x)\,|\mathbb{D}_{0^+}^{\gamma,\beta;\psi}u_k|^{q-2}\,\mathbb{D}_{0^+}^{\gamma,\beta;\psi}u_k \\ \to &|\mathbb{D}_{0^+}^{\gamma,\beta;\psi}u|^{p-2}\,\mathbb{D}_{0^+}^{\gamma,\beta;\psi}u + \mathbf{a}(x)|\,\mathbb{D}_{0^+}^{\gamma,\beta;\psi}u|^{q-2}\,\mathbb{D}_{0^+}^{\gamma,\beta;\psi}u.\end{aligned} \tag{3.14}$$

From relations (3.13) and (3.14), the claim follows. As $\mathfrak{E}'(u_k)v \to 0$, this claim implies that $\mathfrak{E}'(u)v = 0$ for all $v \in C_c^\infty(\mathbb{R}^n)$. With the knowledge that $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$, the lemma follows. □

Proof of Theorem 3.1. If $u \neq 0$, then u serves as a nontrivial solution, concluding the theorem. However, if $u = 0$, the task is to locate another solution $v \in \mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)\backslash\{0\}$ for equation (1.5). In pursuit of this objective, the assertion presented below plays a pivotal role in our reasoning.

Claim. There exist $s > 0$, $\vartheta > 0$ and a sequence $(v_n)_n \subset \Omega$ such that

$$\liminf_{n\to+\infty}\int_{B_s(v_n)}|u_k(x)|^p dx \geq \vartheta > 0. \tag{3.15}$$

Proof. In fact, if the above claim does not hold, by Lions's lemma ([34], Lemmma I.1), one has

$$u_k \to 0, \quad \text{in } L^\tau(\Omega). \tag{3.16}$$

Moreover, Proposition 2.4 owing to,

$$\left|\int_\Omega\int_\Omega \frac{G(u_k(x))g(u_k(y))u_k(y)}{|x-y|^\lambda}dxdy\right| \leq C\|G(u_k(x))\|_{L^r(\Omega)}\|g(u_k(y))u_k(y)\|_{L^r(\Omega)}.$$

By (3.3), (3.5), and 3.16, we obtain that

$$\int_{\Omega} |G(u_k(x))|^{\tau} dx \to 0 \quad \text{and} \quad \int_{\Omega} |g(u_k(y))u_k(y)|^{\tau} dy \to 0.$$

Therefore,

$$\int_{\Omega} \frac{G(u_k(x))g(u_k(y))u_k(y)}{|x-y|^{\lambda}} dxdy \to 0. \tag{3.17}$$

Using (3.7) together with (3.17) give

$$\int_{\Omega} \left(|\mathbb{D}_{0+}^{\gamma,\beta;\psi} u|^{p-2}\ \mathbb{D}_{0+}^{\gamma,\beta;\psi} u + \mathbf{a}(x)\ |\mathbb{D}_{0+}^{\gamma,\beta;\psi} u|^{q-2}\ \mathbb{D}_{0+}^{\gamma,\beta;\psi} u\right) \mathbb{D}_{0+}^{\gamma,\beta;\psi} v dx \to 0.$$

This limit leads to $\mathfrak{E}(u_k) \to 0$, which contradicts (3.6). □

Due to the next lemma, we finish the prove of Theorem 3.1.

Lemma 3.6. *Let $\{u_k\}_{k\in\mathbb{N}} \subset \mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$ be such that $\mathfrak{E} \to \theta$. Then, there exists $\{\tilde{y}_k\}_{k\in\mathbb{N}} \subset \Omega$ such that the translated sequence*

$$\tilde{v} := u_k(x + \tilde{y}_k)$$

has a subsequence which converges in $\mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$.

Proof. By utilizing the fact that $\mathfrak{E}'(u_k)u_k \to 0$ and $\mathfrak{E}(u_k) \to \theta$, we can employ the same reasoning as in the proof of Lemma 3.3 to demonstrate that the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$. Then, considering $\tilde{u}_k(x) = u_k(x + \tilde{y}_k)$ and a subsequence, we can find $\tilde{u} \in \mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$ such that $\tilde{u}_k \rightharpoonup \tilde{u}$ in $\mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$ and $\tilde{u} \neq 0$ according to (3.15). Furthermore, for $(t_k)_{k\in\mathbb{N}} > 0$, we can construct $\tilde{v}_k = t_k\tilde{u}_k \in \mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$. Then,

$$\mathfrak{E}(\tilde{v}_k) \leq \max_{t\geq 0} \mathfrak{E}(tu_k) = \mathfrak{E}(u_k),$$

and so

$$\mathfrak{E}(\tilde{v}_k) \to \theta. \tag{3.18}$$

Since (3.18) holds, we have that $\{\tilde{v}_k\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$, which implies that we can assume $\tilde{v}_k \rightharpoonup \tilde{v}$ in $\mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$. Moreover, $(t_k)_{k\in\mathbb{N}}$ is bounded and converges to $t_0 > 0$. Suppose for contradiction that $t_0 = 0$. Then, by the boundedness of $\{\tilde{u}_k\}_{k\in\mathbb{N}}$, we have $\|\tilde{v}_k\|_{0,\mathcal{H}} = t_k\|\tilde{u}_k\|_{0,\mathcal{H}} \to 0$, which contradicts $\mathfrak{E}(\tilde{v}_k) \to \theta > 0$. Hence, $t_0 > 0$. Since the weak limit is unique, we have $\tilde{v} = t_0\tilde{u}$ and $\tilde{u} \neq 0$. Thus, $\tilde{v}_k \to \tilde{v}$ in $\mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$, and consequently $\tilde{u}_k \to \tilde{u}$ in $\mathbb{H}_{0,\mathcal{H}}^{\gamma,\beta,\psi}(\Omega)$. □

References

[1] C. O. Alves and M. Yang, *Existence of semi classical ground state solutions for a generalized Choquard equation*, J. Differential Equations **257**(11) (2014), 4133–4164. https://doi.org/10.1016/j.jde.2014.08.004

[2] E. Arhrrabi and H. El-Houari, *On a class of generalized capillarity system involving fractional ψ-Hilfer derivative with $p(\cdot)$-Laplacian operator*, Math. Math. Methods Appl. Sci. (2024), 1–23, https://doi.org/10.1002/mma.10495.

[3] E. Arhrrabi and H. El-Houari, *On a class of generalized capillarity phenomena involving fractional ψ-Hilfer derivative with $p(\cdot)$-Laplacian operator*, Kragujevac J. Math. (2026) (to appear).
[4] G. M. Bisci, V. D. Radulescu, and R. Servadei, *Variational Methods for Nonlocal Fractional Problems*, Cambridge University Press, 2016.
[5] L. S. Chadli, H. El-Houari and H. Moussa, *Multiplicity of solutions for nonlocal parametric elliptic systems in fractional Orlicz-Sobolev spaces*, J. Elliptic Parabol. Equ. **9**(2) (2023), 1131–1164. https://doi.org/10.1007/s41808-023-00238-4
[6] J. H. Chabrowski, *Variational Methods for Potential Operator Equations: with Applications to Nonlinear Elliptic Equations*, Walter de Gruyter, 2011.
[7] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136**(5) (2012), 521–573. https://doi.org/10.1016/j.bulsci.2011.12.004
[8] H. El-Houari, H. Moussa and H. Sabiki, *Multiplicity and concentration properties of solutions for double-phase problem in fractional modular spaces*, J. Elliptic Parabol. Equ. (2024), 1–47. https://doi.org/10.1007/s41808-024-00278-4
[9] H. El-Houari, L. S. Chadli and H. Moussa, *Existence of a solution to a nonlocal Schrödinger system problem in fractional modular spaces*, Advances in Operator Theory **7**(1) (2022), 1–30. https://doi.org/10.1007/s43036-021-00166-x
[10] H. El-Houari, L. S. Chadli and H. Moussa, *Existence of solution to M-Kirchhoff system type*, 7th International Conference on Optimization and Applications (ICOA), IEEE, (2022), 1–6. https://doi.org/10.1109/ICOA51614.2021.9442669
[11] H. El-houari, H. Moussa and L. S. Chadli, *A class of elliptic inclusion in fractional Orlicz-Sobolev spaces*, Complex Var. Elliptic Equ. **69**(5) (2024), 755–772. https://doi.org/10.1080/17476933.2022.2159955
[12] H. El-Houari, L. S. Chadli and H. Moussa, *A class of non-local elliptic system in non-reflexive fractional Orlicz-Sobolev spaces*, Asian-Eur. J. Math. **16**(07) (2023), Article ID 2350114. https://doi.org/10.1142/S1793557123501140
[13] H. El-Houari, H. Sabiki and H. Moussa, *On topological degree for pseudomonotone operators in fractional Orlicz-Sobolev spaces: study of positive solutions of non-local elliptic problems*, Adv. Oper. Theory **9**(2) (2024), 16 pages. https://doi.org/10.1007/s43036-023-00313-6
[14] H. El-Houari, L. S. Chadli and H. Moussa, *A weak solution to a non-local problem in fractional Orlicz-Sobolev spaces*, Asia Pac. J. Math. **10**(2) (2023), 17 pages. https://doi.org/10.28924/APJM/10-2
[15] H. El-Houari, L. S. Chadli and H. Moussa, *Ground state solutions for a nonlocal system in fractional Orlicz-Sobolev spaces*, Int. J. Differ. Equ. **2022**(1) (2022), Article ID 3849217.
[16] H. El-Houari, H. Moussa, S. Kassimi and H. Sabiki, *Fractional Musielak spaces: a class of non-local problem involving concave-convex nonlinearity*, J. Elliptic Parabol. Equ. (2023), 1–39. https://doi.org/10.1007/s41808-023-00252-6
[17] H. El-Houari, L. S. Chadli and H. Moussa, *Nehari manifold and fibering map approach for fractional $p(\cdot)$-Laplacian Schrödinger system*, SeMA J (2023), 1–23. https://doi.org/10.1007/s40324-023-00343-3
[18] H. El-Houari and E. Arhrrabi, *Fractional ψ-Hilfer derivative spaces: study of Kirchhoff Problem with $p(\cdot)$-Laplacian Operator*, Bull. Transilv. Univ. Brabësov Ser. III. Math. Comput. Sci. (2024) (to appear).
[19] E. H. Hamza, A. Elhoussain and J. V. D. D. C. Sousa, *On a class of Kirchhoff problems with nonlocal terms and logarithmic nonlinearity*, J. Pseudo-Differ. Oper. Appl. **15**(3) (2024), 52 pages. https://link.springer.com/article/10.1007/s11868-024-00624-z
[20] A. Elhoussain, E. H. Hamza and J. V. da C. Sousa, *On a class of capillarity phenomenon with logarithmic nonlinearity involving $\theta(\cdot)$-Laplacian operator*, Comput. Appl. Math. **43**(6) (2024), Article ID 344. https://link.springer.com/article/10.1007/s40314-024-02863-8

[21] H. El-Houari, L. S. Chadli and H. Moussa, *Multiple solutions in fractional Orlicz-Sobolev spaces for a class of nonlocal Kirchhoff systems*, Filomat **38**(8) (2024), 2857–2875. https://doi.org/10.2298/FIL2408857E

[22] H. El-Houari, H. Sabiki and H. Moussa, *Multivalued elliptic inclusion in fractional Orlicz-Sobolev spaces*, Complex Anal. Oper. Theory **18**(4) (2024), 94. https://doi.org/10.1007/s11785-024-01541-1

[23] G. M. Figueiredo, *Existence of positive solutions for a class of p&q elliptic problems with critical growth on* $\mathbb{R}^N$, J. Math. Anal. Appl. **378**(2) (2011), 507–518. https://doi.org/10.1016/j.jmaa.2011.02.017

[24] G. M. Figueiredo, *Existence and multiplicity of solutions for a class of p&q elliptic problems with critical exponent*, Math. Nachr. **286**(11–12) (2013), 1129–1141. https://doi.org/10.1002/mana.201100237

[25] A. Ghanmi, H. Mguagli, S. Turki and N. Zeddini, *Existence of positive bounded solutions for some nonlinear elliptic systems*, J. Math. Anal. Appl. **352**(1) (2009), 440–448. https://doi.org/10.1016/j.jmaa.2008.04.029

[26] A. Ghanmi, *Nontrivial solution for Kirchhoff-type problem involving the* $p(x)$*-Laplace operator*, J. Math. **48**(4) (2018), 1145–1158. https://doi.org/10.1216/RMJ-2018-48-4-1145

[27] A. Ghanmi and S. Horrigue, *Existence of positive solutions for a coupled system of nonlinear fractional differential equations*, Ukr. Math. J. **71**(2) (2019), 39–49. https://doi.org/10.1007/s11253-019-01623-w

[28] H. El-Houari, L. S. Chadli and H. Moussa, *Existence of ground state solutions of elliptic system in fractional Orlicz-Sobolev spaces*, Res. Nonlinear Analysis **5**(2) (2022), 112–130. https://doi.org/10.53006/rna.1021871

[29] H. El-Houari, L. S. Chadli and H. Moussa, *On a class of fractional* $\Gamma(\cdot)$*-Kirchhoff-Schrödinger system type*, Cubo **26**(1) (2024), 53–73. http://dx.doi.org/10.56754/0719-0646.2601.053

[30] C. He and G. Li, *The regularity of weak solutions to nonlinear scalar field elliptic equations containing p&q-Laplacians*, Ann. Acad. Sci. Fenn. Math. **33**(2) (2008), 337–371.

[31] R. Hilfer, *Fractional diffusion based on Riemann-Liouville fractional derivatives*, J. Phy. Chem. B **104**(16) (2000), 3914–3917. https://doi.org/10.1021/jp9936289

[32] P. Le, *Liouville theorems for a p-Laplace equation with Hartree type nonlinearity*, Vietnam J. Math. (2023) 1–14. https://doi.org/10.1007/s10013-021-00508-5

[33] P. L. Lions, *The Choquard equation and related questions*, Nonlinear Anal. **4**(6) (1980), 1063–1072. https://doi.org/10.1016/0362-546X(80)90016-4

[34] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, Part* 2, Ann. Inst. H. Poincaré C Anal. Non Linéaire **4**(1) (1984), 223–283. https://doi.org/10.1016/S0294-1449(16)30428-0

[35] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Stud. Appl. Math. **57**(2) (1977), 93–105. https://doi.org/10.1002/sapm197757293

[36] R. Penrose, *On gravity's role in quantum state reduction*, Gen. Relativity Gravitation **28** (1996), 581–600. https://doi.org/10.1007/BF02105068

[37] H. M. Srivastava and J. V. da Costa Sousa, *Multiplicity of solutions for fractional-order differential equations via the* $\kappa(x)$*-Laplacian operator and the Genus theory*, Fract. Fract. **6**(9) (2022), Article ID 481. https://doi.org/10.3390/fractalfract6090481

[38] J. Sousa, K. B. Lima and L. S. Tavares, *Existence of solutions for a singular double phase problem involving a* ψ*-Hilfer fractional operator via Nehari manifold*, Qual. Theory Dyn. Syst. **22**(3) (2023), 1–26. https://doi.org/10.1007/s12346-023-00794-z

[39] J. Sousa, C. Vanterler da, J. Zuo and D. O'Regan, *The Nehari manifold for a* ψ*-Hilfer fractional p-Laplacian*, Applicable Anal. **101**(14) (2022), 5076–5106. https://doi.org/10.1080/00036811.2021.1880569

[40] C. E. T. Ledesma and J. V. D. C. Sousa, *Fractional integration by parts and Sobolev type inequalities for ψ-fractional operators*, Math. Meth. Appl. Sci. **45**(16) (2022), 9945–9966. https://doi.org/10.1002/mma.8348

[41] J. V. D. C. Sousa, D. S. Oliveira and S. L. Tavares, *Solutions of the mean curvature equation with the Nehari manifold*, Comput. Appl. Math. **43**(1) (2024), 24 pages. https://doi.org/10.1007/s40314-023-02534-0

[42] J. V. D. C. Sousa, K. D. Kucche and J. J. Nieto, *Existence and multiplicity of solutions for ractional $\kappa(\xi)$-Kirchhoff-type equation*, Qual. Theory Dyn. Sys. **23**(1) (2024), 27 pages. https://doi.org/10.1007/s12346-023-00877-x

[43] J. V. D. C. Sousa, D. S. Oliveira and R. P. Agarwal, *Existence and multiplicity for fractional Dirichlet problem with $\gamma(\xi)$-Laplacian equation and Nehari manifold*, Appl. Anal. Disc. Math. **17**(2) (2023), 480–495. https://www.jstor.org/stable/27281422

[44] J. Sousa, C. Vanterler da, M. A. P. Pulido and E. C. D. Oliveira, *Existence and regularity of weak solutions for ψ-Hilfer fractional boundary value problem*, Mediter. J. Math. **18**(4) (2021), Article ID 147. https://doi.org/10.1007/s00009-021-01789-3

[45] V. E. Tarasov and E. C. Aifantis, *On fractional and fractal formulations of gradient linear and nonlinear elasticity*, Acta Mech. **230** (2019), 2043–2070. https://doi.org/10.1007/s00707-019-2373-x

[46] W. Wyss, *The fractional diffusion equation*, J. Math. Phy. **27**(11) (1986), 2782–2785.

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