ON A GENERALIZED DRYGAS FUNCTIONAL EQUATION AND ITS APPROXIMATE SOLUTIONS IN 2-BANACH SPACES

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Abstract. In this paper, we introduce and solve the following generalized Drygas functional equation
\[ f(x + ky) + f(x - ky) = 2f(x) + k^2 f(y) + k^2 f(-y), \]
where \( k \in \mathbb{N} \). Also, we discuss some stability and hyperstability results for the considered equation in 2-Banach spaces by using the fixed point approach.

1. Introduction and preliminaries

We begin this paper by some notations and symbols. We will denote the set of natural numbers by \( \mathbb{N} \), the set of real numbers by \( \mathbb{R} \), \( \mathbb{R}_+ = [0, \infty) \) and the set of all natural numbers greater than or equal to \( m \) will be denoted by \( \mathbb{N}_m, m \in \mathbb{N} \). We write \( B^A \) to mean the family of all functions mapping from a nonempty set \( A \) into a nonempty set \( B \).

S. Gähler [23,24] introduced the basic concept of linear 2-normed spaces. He gave some important facts concerning 2-normed spaces and some preliminary results as follows.

Definition 1.1. Let \( X \) be a real linear space with \( \dim X > 1 \) and \( \| \cdot, \cdot \| : X \times X \to [0, \infty) \) be a function satisfying the following properties:
\begin{itemize}
  \item[(a)] \( \| x, y \| = 0 \) if and only if \( x \) and \( y \) are linearly dependent;
  \item[(b)] \( \| x, y \| = \| y, x \| \);
  \item[(c)] \( \| \lambda x, y \| = |\lambda| \| x, y \| \);
\end{itemize}

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(d) \( \|x + y + z\| \leq \|x, y\| + \|x, z\| \),

for all \( x, y, z \in X \) and \( \lambda \in \mathbb{R} \). Then the function \( \|\cdot, \cdot\| \) is called the \textit{the 2-norm} on \( X \) and the pair \((X, \|\cdot, \cdot\|)\) is called \textit{the linear 2-normed space}. Sometimes the condition (d) is called the \textit{triangle inequality}.

\textbf{Example 1.1.} For \( x = (x_1, x_2), y = (y_1, y_2) \in X = \mathbb{R}^2 \), the Euclidean 2-norm \( \|x, y\|_{\mathbb{R}^2} \) is defined by

\[ \|x, y\|_{\mathbb{R}^2} = |x_1 y_2 - x_2 y_1|. \]

\textbf{Lemma 1.1.} Let \((X, \|\cdot, \cdot\|)\) be a 2-normed space. If \( x \in X \) and \( \|x, y\| = 0 \) for all \( y \in X \), then \( x = 0 \).

\textbf{Definition 1.2.} A sequence \( \{x_k\} \) in a 2-normed space \( X \) is called a \textit{convergent sequence} if there is an \( x \in X \) such that

\[ \lim_{k \to \infty} \|x_k - x, y\| = 0, \]

for all \( y \in X \). If \( \{x_k\} \) converges to \( x \), write \( x_k \to x \) with \( k \to \infty \) and call \( x \) the limit of \( \{x_k\} \). In this case, we also write \( \lim_{k \to \infty} x_k = x \).

\textbf{Definition 1.3.} A sequence \( \{x_k\} \) in a 2-normed space \( X \) is said to be a \textit{Cauchy sequence} with respect to the 2-norm if

\[ \lim_{k,l \to \infty} \|x_k - x_l, y\| = 0, \]

for all \( y \in X \). If every Cauchy sequence in \( X \) converges to some \( x \in X \), then \( X \) is said to be \textit{complete} with respect to the 2-norm. Any complete 2-normed space is said to be a \textit{2-Banach space}.

The following lemma is one of the tools whose we need in our main results.

\textbf{Lemma 1.2} ([31]). Let \( X \) be a 2-normed space. Then

(a) \( \|x, z\| - \|y, z\| \leq \|x - y, z\| \) for all \( x, y, z \in X \);

(b) if \( \|x, z\| = 0 \) for all \( z \in X \), then \( x = 0 \);

(c) for a convergent sequence \( x_n \) in \( X \)

\[ \lim_{n \to \infty} \|x_n, z\| = \|\lim_{n \to \infty} x_n, z\|, \]

for all \( z \in X \).

The problem of the stability of functional equations is caused by the question of S. M. Ulam [38] about the stability in group homomorphisms. The first affirmative partial answer to the Ulam’s problem for Banach spaces was provided by D. H. Hyers [28]. The result of Hyers was generalizable. Namely, it was generalized by T. Aoki [3] for additive mappings and by Th. M. Rassias [34] for linear mappings by considering an unbounded Cauchy difference. In 1994, P. Găvruţa [25] introduced the generalization of the Th. M. Rassias theorem was obtained by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias’ approach.
Within that, a special kind of stability was introduced. This kind is the hyperstability which was given by the following definition.

**Definition 1.4 ([13]).** Let $S$ be a nonempty set, $(Y,d)$ be a metric space, $E \subset C \subset \mathbb{R}_+^S$ be nonempty, $T$ be an operator mapping $C$ into $\mathbb{R}_+^S$ and $F_1, F_2$ be operators mapping a nonempty set $D \subset Y^S$ into $Y^S$. We say that the operator equation

$$F_1\varphi(x_1, \ldots, x_n) = F_2\varphi(x_1, \ldots, x_n), \quad x_1, \ldots, x_n \in S,$$

is $(E,T)$-hyperstable provided for any $\varepsilon \in E$ and $\varphi_0 \in D$ with

$$d(F_1\varphi_0(x_1, \ldots, x_n), F_2\varphi_0(x_1, \ldots, x_n)) \leq \varepsilon(x_1, \ldots, x_n), \quad x_1, \ldots, x_n \in S,$$

there is a solution $\varphi \in D$ of equation (1.1) such that

$$d(\varphi(x), \varphi_0(x)) \leq T\varepsilon(x), \quad x \in S.$$

In [5] the first result of hyperstability has been published, however, the term hyperstability was first used in [30].

There are many papers concerning the hyperstability of functional equations, see for example [4,7–9,13,16–20,26,27,30,33]. In 2013, Brzdęk [6] gave an important result that will be a basic tool to study the stability and hyperstability of functional equations.

**Theorem 1.1 ([6]).** Let $X$ be a nonempty set, $(Y,d)$ a complete metric space $f_1, \ldots, f_s: X \rightarrow X$ and $L_1, \ldots, L_s: X \rightarrow \mathbb{R}_+$ be given mappings. Let $\Lambda: \mathbb{R}_+^X \rightarrow \mathbb{R}_+^X$ be a linear operator defined by

$$\Lambda\delta(x) := \sum_{i=1}^s L_i(x)\delta(f_i(x)),$$

for $\delta \in \mathbb{R}_+^X$ and $x \in X$. If $T: Y^X \rightarrow Y^X$ is an operator satisfying the inequality

$$d(T\xi(x), T\mu(x)) \leq \sum_{i=1}^s L_i(x)d(\xi(f_i(x)), \mu(f_i(x))), \quad \xi, \mu \in Y^X, x \in X,$$

and a function $\varepsilon: X \rightarrow \mathbb{R}_+$ and a mapping $\varphi: X \rightarrow Y$ satisfies

$$d(T\varphi(x), \varphi(x)) \leq \varepsilon(x), \quad x \in X,$$

$$\varepsilon^*(x) := \sum_{k=0}^{\infty} \Lambda^k\varepsilon(x) < \infty, \quad x \in X,$$

then for every $x \in X$ the limit

$$\psi(x) := \lim_{n \rightarrow \infty} T^n\varphi(x)$$

exists and the function $\psi \in Y^X$ is a unique fixed point of $T$ with

$$d(\varphi(x), \psi(x)) \leq \varepsilon^*(x), \quad x \in X.$$

In 2019, M. Almahalebi et al. [2] introduced and proved an analogue of Theorem 1.1 in 2-Banach spaces.
Theorem 1.2 ([2]). Let $X$ be a nonempty set, $(Y, \|\cdot\|)$ be a 2-Banach space, $g : X \to Y$ be a surjective mapping and let $f_1, \ldots, f_r : X \to X$ and $L_1, \ldots, L_r : X \to \mathbb{R}_+$ be given mappings. Suppose that $T : Y^X \to Y^X$ and $\Lambda : \mathbb{R}_+^{X \times X} \to \mathbb{R}_+^{X \times X}$ are two operators satisfying the conditions
\[
\|T \xi(x) - T \mu(x), g(z)\| \leq \sum_{i=1}^{r} L_i(x) \|\xi(f_i(x)) - \mu(f_i(x)), g(z)\|,
\]
for all $\xi, \mu \in Y^X$, $x, z \in X$ and
\[
\Lambda \delta(x, z) := \sum_{i=1}^{r} L_i(x) \delta(f_i(x), z), \quad \delta \in \mathbb{R}_+^{X \times X}, \quad x, z \in X.
\]
If there exist functions $\varepsilon : X \times X \to \mathbb{R}_+$ and $\varphi : X \to Y$ such that
\[
\|T \varphi(x) - \varphi(x), g(z)\| \leq \varepsilon(x, z)
\]
and
\[
\varepsilon^*(x, z) := \sum_{n=0}^{\infty} \left(\Lambda^n \varepsilon\right)(x, z) < \infty,
\]
for all $x, z \in X$, then the limit
\[
\lim_{n \to \infty} \left((T^n \varphi)\right)(x)
\]
exists for each $x \in X$. Moreover, the function $\psi : X \to Y$ defined by
\[
\psi(x) := \lim_{n \to \infty} \left((T^n \varphi)\right)(x)
\]
is a fixed point of $T$ with
\[
\|\varphi(x) - \psi(x), g(z)\| \leq \varepsilon^*(x, z),
\]
for all $x, z \in X$.

Another version of Theorem 1.2 in 2-Banach space can be found in [14]. Also, J. Brzdęk and K. Ciepliński extended their fixed point result to the $n$-normed spaces in [15].

In this paper, we consider and solve the following equation
\[
f(x + ky) + f(x - ky) = 2f(x) + k^2f(y) + k^2f(-y),
\]
where $k \in \mathbb{N}$. This equation can be reduced to the Drygas equation
\[
f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y).
\]
In addition, we use Theorem 1.2 to investigate some stability and hyperstability results of equation (1.3) in 2-Banach spaces.
2. Solution of (1.3)

Throughout this section, $X$ and $Y$ will be real vector spaces. The functional equation (1.3) is connected with the functional equation (1.4) as it is shown below.

**Theorem 2.1.** A function $f : X \rightarrow Y$ satisfies the functional equation (1.3) if and only if $f$ satisfies the Drygas functional equation (1.4) for all $x, y \in X$.

**Proof.** Suppose that $f : X \rightarrow Y$ satisfies (1.3) for all $x, y \in X$. Letting $x = y = 0$ in (1.3), we get $f(0) = 0$. Also, by setting $x = 0$ in (1.3), we obtain that
\[
 f(ky) + f(-ky) = k^2f(y) + k^2f(-y), \quad y \in X.
\]
To prove that $f$ satisfies the Drygas functional equation (1.4) for all $x, y \in X$, we assume that $x' = x$ and $y' = ky$ be two elements in $X$. Then we get
\[
 f(x' + y') + f(x' - y') = f(x + ky) + f(x - ky)
\]
\[
 = 2f(x) + k^2f(y) + k^2f(-y)
\]
\[
 = 2f(x) + f(ky) + f(-ky)
\]
\[
 = 2f(x') + f(y') + f(-y'), \quad x, y \in X,
\]
which means that $f$ satisfies the Drygas functional equation (1.4) for all $x, y \in X$. On the other hand, let $f$ be a function satisfying the Drygas functional equation (1.4) for all $x, y \in X$ with $f(0) = 0$ and $f(x) = B(x, x) + A(x)$. Then
\[
 f(x + ky) + f(x - ky) = 2f(x) + f(ky) + f(-ky)
\]
\[
 = 2f(x) + B(ky, ky) + A(ky) + B(-ky, -ky) + A(-ky)
\]
\[
 = 2f(x) + k^2B(y, y) + k^2B(-y, -y) + A(ky) + A(-ky)
\]
\[
 = 2f(x) + k^2B(y, y) + k^2B(-y, -y) + k^2\left(2A(y) + A(-y)\right)
\]
\[
 = 2f(x) + k^2B(y, y) + k^2B(-y, -y) + k^2\left(A(y) + A(-y)\right)
\]
which means that $f$ satisfies (1.3) for all $x, y \in X$. \hfill \Box

3. Stability Results

In this section, we give some investigations on the stability and hyperstability results of the equation (1.3) by using Theorem 1.2 in 2-Banach spaces.

**Theorem 3.1.** Let $X$ be a normed space, $\left(Y, \| \cdot \| \right)$ be a 2-Banach space and $h_1, h_2 : X^2 \rightarrow \mathbb{R}_+$ be two functions such that
\[
 U := \{ n \in \mathbb{N} : \alpha_n < 1 \} \neq \emptyset,
\]
where
\[
\alpha_n = \frac{1}{2} \lambda_1(1+kn)\lambda_2(1+kn) + \frac{1}{2} \lambda_1(1-\lambda_2(1-\lambda_2(n) + \frac{k^2}{2} \lambda_2(n) + \frac{k^2}{2} \lambda_1(-n)\lambda_2(-n)
\]
and
\[
\lambda_i(n) := \inf \{ t \in \mathbb{R}_+ : h_i(nx, z) \leq th_i(x, z), x, z \in X_0 \},
\]
for all \( n \in \mathbb{N} \) with \( i \in \{1,2\} \). Assume that \( f : X \to Y \) satisfies the inequality
\[
\|f(x + ky) + f(x - ky) - 2f(x) - k^2f(y) - k^2f(-y), g(z)\| \leq h_1(x, z)h_2(y, z),
\]
for all \( x, y, z \in X_0 \) such that \( x + ky \neq 0 \) and \( x - ky \neq 0 \), where \( g : X \to Y \) is a surjective mapping. Then there exists a unique function \( D : X \to Y \) that satisfies the equation (1.3) such that
\[
\|f(x) - D(x), g(z)\| \leq \lambda_0h_1(x, z)h_2(x, z),
\]
for all \( x, z \in X_0 \), where
\[
\lambda_0 = \frac{\lambda_2(n)}{2(1-\alpha_m)}.
\]

Proof. Let us fix \( m \in \mathbb{N} \). Replacing \( x \) by \( mx \), where \( x \in X_0 \), in the inequality (3.1), we obtain
\[
\left\| \frac{1}{2} f\left((1 + km)x\right) + \frac{1}{2} f\left((1 - km)x\right) - \frac{k^2}{2} f(mx) - \frac{k^2}{2} f(-mx) - f(x), g(z) \right\|
\]
\[
\leq \frac{1}{2} h_1(x, z)h_2(mx, z),
\]
for all \( x, z \in X_0 \). Define the operator \( T_m : Y^{X_0} \to Y^{X_0} \) by
\[
T_m\xi(x) := \frac{1}{2} \xi\left((1 + km)x\right) + \frac{1}{2} \xi\left((1 - km)x\right) - \frac{k^2}{2} \xi(mx) - \frac{k^2}{2} \xi(-mx),
\]
for all \( x \in X_0 \) and \( \xi \in Y^{X_0} \). Further put
\[
\varepsilon_m(x, z) := \frac{1}{2} h_1(x, z)h_2(mx, z), \quad x, z \in X_0,
\]
and observe that
\[
\varepsilon_m(x, z) = \frac{1}{2} h_1(x, z)h_2(mx, z) \leq \frac{1}{2} \lambda_2(m)h_1(x, z)h_2(x, z), \quad x, z \in X_0, \quad m \in \mathbb{N}.
\]
Thus, the inequality (3.2) becomes
\[
\|T_mf(x) - f(x), g(z)\| \leq \varepsilon_m(x, z), \quad x, z \in X_0.
\]
Furthermore, for every \( x, z \in X_0 \) and \( \xi, \mu \in Y^{X_0} \), we have
\[
\left\| T_m\xi(x) - T_m\mu(x), g(z) \right\|
\]
\[
= \left\| \frac{1}{2} \xi\left((1 + km)x\right) + \frac{1}{2} \xi\left((1 - km)x\right) - \frac{k^2}{2} \xi(mx) \right\|
\]
\begin{align*}
  &-\frac{k^2}{2}\xi(-mx) - \frac{1}{2}\mu((1+km)x) - \frac{1}{2}\mu((1-km)x) + \frac{k^2}{2}\mu(mx) + \frac{k^2}{2}\mu(-mx), g(z)\right)
  \leq \frac{1}{2}\|\xi - \mu((1+km)x), g(z)\| + \frac{1}{2}\|\xi - \mu((1-km)x), g(z)\|
  + \frac{k^2}{2}\|\xi - \mu(mx), g(z)\| + \frac{k^2}{2}\|\xi - \mu(-mx), g(z)\|
  \end{align*}

for all \(x, z \in X_0\) and \(\xi, \mu \in Y^{X_0}\). It means that the condition (1.2) is satisfied and this brings us to define the operator \(\Lambda_m : \mathbb{R}^{X_0 \times X_0} \to \mathbb{R}^{X_0 \times X_0}\) by

\[\Lambda_m \delta(x, z) := \frac{1}{2}\delta((1+km)x, z) + \frac{1}{2}\delta((1-km)x, z) + \frac{k^2}{2}\delta(mx, z) + \frac{k^2}{2}\delta(-mx, z),\]

for all \(x, z \in X_0\) and \(\delta \in \mathbb{R}^{X_0 \times X_0}\). This operator has the form given by (1.2) with \(f_1(x) = (1+km)x, f_2(x) = (1-km)x, f_3(x) = mx, f_4(x) = -mx, L_1(x) = L_2(x) = \frac{1}{2}\) and \(L_3(x) = L_4(x) = \frac{k^2}{2}\) for all \(x \in X_0\).

By induction on \(n \in \mathbb{N}\), it is easy to show that

\begin{equation}
(\Lambda^0_{m,\varepsilon_m})(x, z) \leq \frac{1}{2}\lambda_2(m)\alpha_m h_1(x, z) h_2(x, z),
\end{equation}

for all \(x, z \in X_0\) and all \(m \in \mathcal{U}\), where

\[\alpha_m = \frac{1}{2}\lambda_1(1+km)\lambda_2(1+km) + \frac{1}{2}\lambda_1(1-km)\lambda_2(1-km) + \frac{k^2}{2}\lambda_1(m)\lambda_2(m) + \frac{k^2}{2}\lambda_1(-m)\lambda_2(-m).\]

Indeed, (3.3) and (3.4) imply that the inequality (3.5) holds for \(n = 0\). Next, we assume that (3.5) holds for \(n = r\), where \(r \in \mathbb{N}_1\). Then we obtain

\begin{align*}
(\Lambda^{r+1}_{m,\varepsilon_m})(x, z) &= \Lambda_m \left( (\Lambda^r_{m,\varepsilon_m})(x, z) \right)
  \leq \frac{1}{2}\left( \Lambda^r_{m,\varepsilon_m}((1+km)x, z) + \frac{k^2}{2}\left( \Lambda^r_{m,\varepsilon_m}(mx, z) + \frac{k^2}{2}\left( \Lambda^r_{m,\varepsilon_m}(-mx, z) \right) \right) \right)
  \leq \frac{1}{4}\lambda_2(m)\alpha^r_m h_1((1+km)x, z) h_2((1+km)x, z)
  + \frac{1}{4}\lambda_2(m)\alpha^r_m h_1((1-km)x, z) h_2((1-km)x, z)
  + \frac{k^2}{4}\lambda_2(m)\alpha^r_m h_1(mx, z) h_2(mx, z)
  + \frac{k^2}{4}\lambda_2(m)\alpha^r_m h_1(-mx, z) h_2(-mx, z)
  \leq \frac{1}{2}\lambda_2(m)h_1(\frac{1}{2}\lambda_1(1+km)\lambda_2(1+km) + \frac{1}{2}\lambda_1(1-km)\lambda_2(1-km)}
\end{align*}
To prove that (3.5) holds for all \( n \in \mathbb{N} \), we have

\[
\frac{1}{2} \lambda_2(m) \alpha_m h_1(x, z) h_2(x, z),
\]
for all \( x, z \in X_0 \) and all \( m \in \mathbb{U} \). It means that (3.5) holds for \( n = r + 1 \) which implies that (3.5) holds for all \( n \in \mathbb{N} \). Hence, in view of (3.5), we obtain

\[
\varepsilon^*(x, z) := \sum_{n=0}^{\infty} \lambda_n \varepsilon_m (x, z)
\]

\[
\leq \sum_{n=0}^{\infty} \frac{1}{2} \lambda_2(m) \alpha_m h_1(x, z) h_2(x, z)
\]

\[
= \frac{\lambda_2(m) h_1(x, z) h_2(x, z)}{2(1 - \alpha_m)} < \infty,
\]
for all \( x, z \in X_0 \) and all \( m \in \mathbb{U} \). Therefore, according to Theorem 1.2, with \( \varphi = f \) and using the surjectivity of \( g \), we get that the limit

\[
\lim_{n \to \infty} (f^n_m f)(x)
\]

exists and defined a function \( D_m : X \to Y \) such that

\[
(3.6) \quad \|f(x) - D_m(x), g(z)\| \leq \frac{\lambda_2(m) h_1(x, z) h_2(x, z)}{2(1 - \alpha_m)}, \quad x, z \in X_0, m \in \mathbb{U}.
\]

To prove that \( F_m \) satisfies the functional equation (1.3), just prove the following inequality by the induction on \( n \in \mathbb{N}_0 \)

\[
(3.7) \quad \|m f^n_m f(x + ky) + m f^n_m f(x - ky) - 2m f^n_m f(x) - k^2f^n_m f(y) - k^2f^n_m f(-y), g(z)\| \\
\leq \alpha_m h_1(x, z) h_2(y, z),
\]
for every \( x, y, z \in X_0 \) such that \( x + ky \neq 0, x - ky \neq 0 \) and every \( m \in \mathbb{U} \).

First, for \( n = 0 \), we just find (3.1). Next, take \( r \in \mathbb{N} \) and assume that (3.7) holds for \( n = r \) and every \( x, y, z \in X_0 \) such that \( x + ky \neq 0 \) and \( x - ky \neq 0 \), \( m \in \mathbb{U} \). Then, for each \( x, y, z \in X_0 \) and \( m \in \mathbb{U} \), we have

\[
\|m f^{r+1} f(x + ky) + m f^{r+1} f(x - ky) - 2m f^{r+1} f(x) \\
- k^2f^{r+1} f(y) - k^2f^{r+1} f(-y), g(z)\| \\
= \|\frac{1}{2} f^r m f((1 + km)(x + ky)) + \frac{1}{2} f^r m f((1 - km)(x + ky)) \\
- k^2 f^r m f(m(x + ky)) - k^2 f^r m f(-m(x + ky)) \\
+ \frac{1}{2} f^r m f((1 + km)(x - ky)) + \frac{1}{2} f^r m f((1 - km)(x - ky))\|
\]
Thus, by induction, we have shown that (3.7) holds for every $\alpha, \beta \leq 1$.

\[
\begin{align*}
-k^2 \left( \mathcal{T}_m f \right)(m(x - ky)) &- k^2 \left( \mathcal{T}_m f \right)(m(x - ky)) \\
-k^2 \left( \mathcal{T}_m f \right)((1 + km)(x)) &+ \left( \mathcal{T}_m f \right)((1 - km)(x)) \\
+k^2 \left( \mathcal{T}_m f \right)(my) &- k^2 \left( \mathcal{T}_m f \right)((1 + km)(y)) \\
- \frac{k^2}{2} \left( \mathcal{T}_m f \right)((1 + km)(x)) &+ \frac{k^4}{2} \left( \mathcal{T}_m f \right)((1 - km)(y)) \\
+k^2 \left( \mathcal{T}_m f \right)(my) &- k^2 \left( \mathcal{T}_m f \right)((1 + km)(y)) \\
- k^2 \left( \mathcal{T}_m f \right)((1 + km)(x)) &+ \left( \mathcal{T}_m f \right)((1 - km)(y)) \\
- k^2 \left( \mathcal{T}_m f \right)((1 + km)(x)) &+ \left( \mathcal{T}_m f \right)((1 - km)(y)) \\
\leq & \frac{1}{2} \left\| \left( \mathcal{T}_m f \right)((1 + km)(x + ky)) + \left( \mathcal{T}_m f \right)((1 + km)(x - ky)) \right\| \\
- 2 \left( \mathcal{T}_m f \right)((1 + km)(x)) &- k^2 \left( \mathcal{T}_m f \right)((1 + km)(y)) \\
- k^2 \left( \mathcal{T}_m f \right)((1 + km)(y)), g(z) \right\| \\
+ & \frac{1}{2} \left\| \left( \mathcal{T}_m f \right)((1 - km)(x + ky)) + \left( \mathcal{T}_m f \right)((1 - km)(x - ky)) \right\| \\
- 2 \left( \mathcal{T}_m f \right)((1 - km)(x)) &- k^2 \left( \mathcal{T}_m f \right)((1 - km)(y)) \\
- k^2 \left( \mathcal{T}_m f \right)((1 - km)(x)) &+ \left( \mathcal{T}_m f \right)((1 - km)(y)) \\
+ & \frac{k^2}{2} \left\| \left( \mathcal{T}_m f \right)(m(x + ky)) + \left( \mathcal{T}_m f \right)(m(x - ky)) - 2 \left( \mathcal{T}_m f \right)(mx) \right\| \\
- k^2 \left( \mathcal{T}_m f \right)(my) &- k^2 \left( \mathcal{T}_m f \right)(-my), g(z) \right\| \\
+ & \frac{k^2}{2} \left\| \left( \mathcal{T}_m f \right)(-m(x + ky)) + \left( \mathcal{T}_m f \right)(-m(x - ky)) - 2 \left( \mathcal{T}_m f \right)(-mx) \right\| \\
- k^2 \left( \mathcal{T}_m f \right)(-my) &- k^2 \left( \mathcal{T}_m f \right)(my), g(z) \right\| \\
\leq & \frac{1}{2} \alpha^r_m h_1((1 + km)x, z) h_2((1 + km)y, z) + \frac{1}{2} \alpha^r_m h_1((1 - km)x, z) h_2((1 - km)y, z) + \frac{k^2}{2} \alpha^r_m h_1(mx, z) h_2(my, z) + \frac{k^2}{2} \alpha^r_m h_1(-mx, z) h_2(-my, z) \\
= & \alpha^r_{m+1} h_1(x, z) h_2(y, z).
\end{align*}
\]

Thus, by induction, we have shown that (3.7) holds for every $x, y, z \in X_0$, $n \in \mathbb{N}_0$, and $m \in \mathcal{U}$ such that $x + ky \neq 0$ and $x - ky \neq 0$. Letting $n \to \infty$ in (3.7), we obtain the equality

\[
D_m(x + ky) + D_m(x - ky) = 2D_m(x) + k^2 D_m(y) + k^2 D_m(-y),
\]
for all \(x, y \in X_0\) and \(m \in U\) such that \(x + ky \neq 0\) and \(x - ky \neq 0\). This implies that \(D_m : X \to Y\), defined in this way, is a solution of the equation

\[
D(x) = \frac{1}{2}D((1 + km)x) + \frac{1}{2}D((1 - km)x) - \frac{k^2}{2}D(mx) - \frac{k^2}{2}D(-mx),
\]

for all \(x \in X_0\) and all \(m \in U\). Next, we will prove that each cubic functional equation \(D : X \to Y\) satisfying (3.9). From (3.6), for each \(x \in X_0\), we get

\[
\|D(x) - D_{m_0}(x), g(z)\| \leq L h_1(x, z)h_2(x, z), \quad x, z \in X_0,
\]

with some \(L > 0\), is equal to \(D_m\) for each \(m \in U\). To this end, we fix \(m_0 \in U\) and \(D : X \to Y\) satisfying (3.9). From (3.6), for each \(x \in X_0\), we get

\[
\|D(x) - D_{m_0}(x), g(z)\| \leq \|D(x) - f(x), g(z)\| + \|f(x) - D_{m_0}(x), g(z)\| \\
\leq L h_1(x, z)h_2(x, z) + \varepsilon_{m_0}(x, z)
\]

(3.10)

\[
\leq L_0 h_1(x, z)h_2(x, z) \sum_{n=0}^{\infty} \alpha^n_{m_0},
\]

where \(L_0 := 2(1 - \alpha_{m_0})L + \lambda_2(m_0) > 0\) and we exclude the case that \(h_1(x, z) \equiv 0\) or \(h_2(x, z) \equiv 0\) which is trivial. Observe that \(D\) and \(D_{m_0}\) are solutions to equation (3.8) for all \(m \in U\). Next, we show that, for each \(j \in \mathbb{N}_0\), we have

\[
\|D(x) - D_{m_0}(x), g(z)\| \leq L_0 h_1(x, z)h_2(x, z) \sum_{n=j}^{\infty} \alpha^n_{m_0}, \quad x, z \in X_0.
\]

The case \(j = 0\) is exactly (3.10). We fix \(r \in \mathbb{N}\) and assume that (3.11) holds for \(j = r\). Then, in view of (3.10), for each \(x, z \in X_0\), we get

\[
\|D(x) - D_{m_0}(x), g(z)\| = \left\| \frac{1}{2}D((1 + km_0)x) + \frac{1}{2}D((1 - km_0)x) - \frac{k^2}{2}D(m_0x) \\
- \frac{k^2}{2}D(-m_0x) - \frac{1}{2}D_{m_0}((1 + km_0)x) \\
- \frac{1}{2}D_{m_0}((1 - km_0)x) + \frac{k^2}{2}D_{m_0}(m_0x) \\
+ \frac{k^2}{2}D_{m_0}(-m_0x), g(z) \right\| \\
\leq \frac{1}{2}\|D((1 + km_0)x) - D_{m_0}((1 + km_0)x), g(z)\| \\
+ \frac{1}{2}\|D((1 - km_0)x) - D_{m_0}((1 - km_0)x), g(z)\| \\
+ \frac{k^2}{2}\|D(m_0x) - D_{m_0}(m_0x), g(z)\| \\
+ \frac{k^2}{2}\|D(-m_0x) - D_{m_0}(-m_0x), g(z)\|.
\]
Then every (3.11) holds for all \( U \) and Theorem 4.1. This implies (1.3) with (3.12)

\[
\frac{1}{2} L_0 h_1((1 + km_0)x, z) h_2((1 + km_0)x, z) \sum_{n=r}^\infty \alpha_m^n + \frac{1}{2} L_0 h_1((1 - km_0)x, z) h_2((1 - km_0)x, z) \sum_{n=r}^\infty \alpha_m^n + \frac{k^2}{2} L_0 h_1(m_0x, z) h_2(m_0x, z) \sum_{n=r}^\infty \alpha_m^n + \frac{k^2}{2} L_0 h_1(-m_0x, z) h_2(-m_0x, z) \sum_{n=r}^\infty \alpha_m^n \\
\leq L_0 \alpha_{m_0} h_1(x, z) h_2(x, z) \sum_{n=r}^\infty \alpha_m^n \\
= L_0 h_1(x, z) h_2(x, z) \sum_{n=r+1}^\infty \alpha_m^n.
\]

This shows that (3.11) holds for \( j = k + 1 \). Now we can conclude that the inequality (3.11) holds for all \( j \in \mathbb{N}_0 \). Now, letting \( j \to \infty \) in (3.11), we get

(3.12)

\[ D = D_{m_0}. \]

Thus, we have also proved that \( D_m = D_{m_0} \) for each \( m \in \mathbb{U} \), which (in view of (3.6)) yields

\[
\left\| f(x) - D_{m_0}(x), g(z) \right\| \leq \frac{\lambda_2(m) h_1(x, z) h_2(x, z)}{2(1 - \alpha_m)}, \quad x, z \in X_0, \ m \in \mathbb{U}.
\]

This implies (1.3) with \( D = D_{m_0} \) and (3.12) confirms the uniqueness of \( D \).

\[ \square \]

4. Hyperstability Results

The following theorems and corollaries concern the \( \eta \)-hyperstability of (1.3) in 2-Banach spaces. Namely, we consider functions \( f : X \to Y \) fulfilling (1.3) approximately, i.e., satisfying the inequality

(4.1)

\[
\left\| f(x + ky) + f(x - ky) - 2f(x) - k^2 f(y) - k^2 f(-y), g(z) \right\| \leq \eta(x, y, z),
\]

for all \( x, y, z \in X_0 \) such that \( x + ky \neq 0 \) and \( x - ky \neq 0 \) with \( \eta : X_0 \times X_0 \times X_0 \to \mathbb{R}_+ \) is a given mapping. Then we find a unique cubic function \( F : X \to Y \) which is close to \( f \). Then, under some additional assumptions on \( \eta \), we prove that the conditional functional equation (1.3) is \( \eta \)-hyperstable in the class of functions \( f : X \to Y \), i.e., each \( f : X \to Y \) satisfying inequality (4.1), with such \( \eta \), must fulfill equation (1.3).

**Theorem 4.1.** Let \( X \) be a normed space, \( (Y, \| \cdot, \|) \) be a real 2-Banach space, \( h_1, h_2 \) and \( \mathbb{U} \) be as in Theorem 3.1. Assume that

(4.2)

\[
\left\{ \begin{array}{l}
\lim_{n \to \infty} \lambda_2(n) = \lim_{n \to \infty} \lambda_1(1 + kn) \lambda_2(1 + kn) = \lim_{n \to \infty} \lambda_1(-n) \lambda_2(-n) = 0, \\
\lim_{n \to \infty} \lambda_1(1 - kn) \lambda_2(1 - kn) = \lim_{n \to \infty} \lambda_1(n) \lambda_2(n) = 0.
\end{array} \right.
\]

Then every \( f : X \to Y \) satisfying (3.1) is a solution of (1.3) on \( X_0 \).
Proof. Suppose that \( f : X \to Y \) satisfies (3.1). Then, by Theorem 3.1, there exists a mapping \( D : X \to Y \) satisfying (1.3) and

\[
\|f(x) - D(x), g(z)\| \leq \lambda_0 h_1(x, z) h_2(x, z),
\]

for all \( x, z \in X_0 \), where \( g : X \to Y \) is a surjective mapping and

\[
\lambda_0 = \frac{\lambda_2(n)}{2(1 - \alpha_m)},
\]

with

\[
\alpha_n = \frac{1}{2} \lambda_1(1 + kn) \lambda_2(1 + kn) + \frac{1}{2} \lambda_1(1 - kn) \lambda_2(1 - kn) + \frac{k^2}{2} \lambda_1(n) \lambda_2(n) + \frac{k^2}{2} \lambda_1(-n) \lambda_2(-n).
\]

Since, in view of (4.2), \( \lambda_0 = 0 \), this means that \( f(x) = D(x) \) for all \( x \in X_0 \), whence

\[
f(x + ky) + f(x - ky) = 2f(x) + k^2 f(y) + k^2 f(-y),
\]

for all \( x, y \in X_0 \) such that \( x + ky \neq 0 \) and \( x - ky \neq 0 \), which implies that \( f \) satisfies the functional equation (1.3) on \( X_0 \).

Corollary 4.1. Let \( (X, \| \cdot \|) \) be a normed space, \( (Y, \| \cdot \|) \) be a real 2-Banach space and \( \theta \geq 0, s \geq 0, p, q \in \mathbb{R} \) such that \( p + q < 0 \). Suppose that \( f : X \to Y \) such that \( f(0) = 0 \) satisfies the inequality

\[
\|f(x + ky) + f(x - ky) - 2f(x) - k^2 f(y) - k^2 f(-y), g(z)\| \leq \|x\|^p \|y\|^q \|z\|^s,
\]

for all \( x, y, z \in X_0 \) such that \( x + ky \neq 0 \) and \( x - ky \neq 0 \), where \( g : X \to Y \) is a surjective mapping. Then \( f \) satisfies (1.3) on \( X_0 \).

Proof. The proof follows from Theorem 3.1 by defining \( h_1, h_2 : X_0 \times X_0 \to \mathbb{R}_+ \) by

\[
h_1(x, z) = \theta_1 \|x\|^p \|z\|^s, \quad h_2(y, z) = \theta_2 \|y\|^q \|z\|^s \quad \text{and} \quad h_1(0, z) = h_2(0, z) = 0
\]

with \( \theta_1, \theta_2 \in \mathbb{R}_+ \), \( s_1, s_2 \in \mathbb{R}_+ \) and \( p, q \in \mathbb{R} \) such that \( \theta_1 \theta_2 = \theta \), \( s_1 + s_2 = s \) and \( p + q < 0 \). For each \( n \in \mathbb{N} \), we have

\[
\lambda_1(n) = \inf \{ t \in \mathbb{R}_+: h_1(nx, z) \leq t h_1(x, z), x, z \in X_0 \}
\]

\[
= \inf \{ t \in \mathbb{R}_+: \theta_1 \|nx\|^p \|z\|^s \leq t \theta_1 \|x\|^p \|z\|^s, x, z \in X_0 \}
\]

\[
= n^p.
\]

Also, we have \( \lambda_2(n) = n^q \) for all \( n \in \mathbb{N} \). Clearly, we can find \( n_0 \in \mathbb{N} \) such that

\[
\frac{1}{2} \lambda_1(1 + kn) \lambda_2(1 + kn) + \frac{1}{2} \lambda_1(1 - kn) \lambda_2(1 - kn) + \frac{k^2}{2} \lambda_1(n) \lambda_2(n) + \frac{k^2}{2} \lambda_1(-n) \lambda_2(-n)
\]

\[
= \frac{1}{2}(1 + kn)^{p+q} + \frac{1}{2}(1 - kn)^{p+q} + k^2 n^{p+q} < 1,
\]

for all \( n \geq n_0 \). According to Theorem 3.1, there exists a unique Drygas function \( D : X \to Y \) such that

\[
\|f(x) - D(x), g(z)\| \leq \theta \lambda_0 h_1(x, z) h_2(x, z),
\]
for all \( x, z \in X_0 \), where
\[
\lambda_0 = \frac{\lambda_2(n)}{2(1 - \alpha_m)},
\]
with
\[
\alpha_n = \frac{1}{2} \lambda_1(1 + kn) \lambda_2(1 + kn) + \frac{1}{2} \lambda_1(1 - kn) \lambda_2(1 - kn) + \frac{k^2}{2} \lambda_1(n) \lambda_2(n) + \frac{k^2}{2} \lambda_1(-n) \lambda_2(-n).
\]
Since \( p + q < 0 \), one of \( p \) and \( q \) must be negative. Assume that \( q < 0 \). Then
\[
\lim_{n \to \infty} \lambda_2(n) = \lim_{n \to \infty} n^q = 0, \\
\lim_{n \to \infty} \lambda_1(1 + kn) \lambda_2(1 + kn) = \lim_{n \to \infty} (1 + kn)^{p+q} = 0, \\
\lim_{n \to \infty} \lambda_1(1 - kn) \lambda_2(1 - kn) = \lim_{n \to \infty} (1 + kn)^{p+q} = 0, \\
\lim_{n \to \infty} \lambda_1(n) \lambda_2(n) = \lim_{n \to \infty} n^{p+q} = 0.
\]
Thus by Theorem 4.1, we get the desired results. \( \square \)

The next corollary prove the hyperstability results for the inhomogeneous Drygas functional equation
\[
f(x + ky) + f(x - ky) = 2f(x) + k^2f(y) + k^2f(-y) + G(x, y).
\]

**Corollary 4.2.** Let \( (X, \| \cdot \|) \) be a normed space, \( (Y, \| \cdot, \cdot \|) \) be a real 2-Banach space and \( \theta \geq 0, s \geq 0, p, q \in \mathbb{R} \) such that \( p + q < 0 \). Assume that \( G : X^2 \to Y \) and \( f : X \to Y \) such that \( f(0) = 0 \) and satisfies the inequality
\[
\|f(x + ky) + f(x - ky) - 2f(x) - k^2f(y) - k^2f(-y) - G(x, y), g(z)\| \leq \theta \|x\|^p \|y\|^q \|z\|^s,
\]
for all \( x, y, z \in X_0 \) such that \( x + ky \neq 0 \) and \( x - ky \neq 0 \), where \( g : X \to Y \) is a surjective mapping. If the functional equation
\[
f(x + ky) + f(x - ky) = 2f(x) + k^2f(y) + k^2f(-y) + G(x, y),
\]
for all \( x, y \in X_0 \) such that \( x + ky \neq 0 \) and \( x - ky \neq 0 \) has a solution \( f_0 : X \to Y \) on \( X_0 \), then \( f \) is a solution to (4.5) on \( X_0 \).

**Proof.** From (4.4) we get that the function \( K : X \to Y \) defined by \( K := f - f_0 \) satisfies (4.3). Consequently, Corollary 4.1 implies that \( K \) is a solution to Drygas functional equation (1.3) on \( X_0 \). Therefore,
\[
f(x + ky) + f(x - ky) - 2f(x) - k^2f(y) - k^2f(-y) - G(x, y)
\]
\[
= K(x + ky) + f_0(x + ky) + K(x - ky) + f_0(x - ky) - 2K(x) - 2f_0(x)
\]
\[
- k^2K(y) - k^2f_0(y) - k^2K(-y) - k^2f_0(-y) - G(x, y)
\]
\[
= 0,
\]
for all \( x, y \in X_0 \) such that \( x + ky \neq 0 \) and \( x - ky \neq 0 \) which means \( f \) is a solution to (4.5) on \( X_0 \). \( \square \)
References


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