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INTUITIONISTIC FUZZY GRAPH STRUCTURES

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Abstract. In this paper, we introduce the concept of an intuitionistic fuzzy graph structure (IFGS). We discuss certain notions, including intuitionistic fuzzy B_i-cycles, intuitionistic fuzzy B_i-trees and ϕ-complement of an intuitionistic fuzzy graph structure with several examples. We also present ϕ-complement of an intuitionistic fuzzy graph structure along with self-complementary and strong self-complementary intuitionistic fuzzy graph structures.

1. Introduction

Fuzzy set was introduced by Zadeh in 1965. A fuzzy set gives the degree of membership of an object in a given set. Kaufmann's initial definition of a fuzzy graph [10] was based on Zadeh's fuzzy relations [22]. The fuzzy relations between fuzzy sets were considered by Rosenfeld and he developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. Later on, Bhattacharya [7] gave some remarks on fuzzy graphs and some operations on fuzzy graphs were introduced by Mordeson and Peng [14]. In 1983, Atanassov [5] extended the idea of a fuzzy set and introduced the concept of an intuitionistic fuzzy set. He added a new component, degree of non-membership, in the definition of a fuzzy set with the condition that sum of two degrees must be less or equal to one. Atanassov [6] also introduced the concept of intuitionistic fuzzy graphs and intuitionistic fuzzy relations. Shannon and Atanassov investigated some properties of intuitionistic fuzzy relations and intuitionistic fuzzy graphs in [20]. Parvathi et al. defined operations on intuitionistic fuzzy graphs in [16]. Karunambigai et al. used intuitionistic fuzzy graphs to find shortest paths in networks [11]. Akram et al. [1–4] introduced many

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new concepts, including strong intuitionistic fuzzy graphs, intuitionistic fuzzy trees, intuitionistic fuzzy hypergraphs, and intuitionistic fuzzy digraphs in decision support systems.

Fuzzy graph theory is finding an increasing number of applications in modeling real time systems where the level of information inherent in the system varies with different levels of precision. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems. Intuitionistic fuzzy set has got an advantage over fuzzy set because of its additional component which explains the deficiency of knowledge in assigning the degree of membership to an object because there is a fair chance of the existence of a non-zero hesitation part at each moment of evaluation of anything. The advantages of intuitionistic fuzzy sets and graphs are that they give more accuracy into the problems, reduce the cost of implementation and improve efficiency. Intuitionistic fuzzy sets are very useful in providing a flexible model to describe uncertainty and vagueness involved in decision making, so intuitionistic fuzzy graphs are playing a substantial role in chemistry, economics, computer sciences, engineering, medicine and decision making problems, now a days. Graph structures or generalized graph structures introduced by Sampathkumar in 2006 [19], are a generalization of graphs which is quite useful in studying signed graphs and graphs in which every edge is labeled or colored because they help to study various relations and corresponding edges simultaneously. Dinesh and Ramakrishnan [9] introduced fuzzy graph structures. In this paper, we have worked on intuitionistic fuzzy graph structures, some of their fundamental concepts and properties due to the improved influence of intuitionistic fuzzy sets and particular use of graph structures. In this paper, we introduce the concept of an intuitionistic fuzzy graph structure (IFGS). We discuss certain notions, including intuitionistic fuzzy B_i-cycles, intuitionistic fuzzy B_i-trees and ϕ-complement of an intuitionistic fuzzy graph structure with several examples. We also present ϕ-complement of an intuitionistic fuzzy graph structure along with self-complementary and strong self-complementary intuitionistic fuzzy graph structures.

2. Preliminaries

We first review some definitions from [19] that are necessary for this paper.

A *graph structure* $G^* = (U, E_1, E_2, \ldots, E_k)$, consists of a non-empty set U together with relations $E_1, E_2, \ldots, E_k$ on U, which are mutually disjoint such that each E_i is irreflexive and symmetric. If $(u, v) \in E_i$ for some $i, 1 \leq i \leq k$, we call it an E_i-*edge* and write it as "uv". A graph structure $G^* = (U, E_1, E_2, \ldots, E_k)$ is *complete*, if

(i) each edge E_i, $1 \leq i \leq k$ appears at least once in G^*;
(ii) between each pair of vertices uv in U, uv is an E_i-edge for some i, $1 \leq i \leq k$.

A graph structure $G^* = (U, E_1, E_2, \ldots, E_k)$ is *connected*, if the underlying graph is connected. In a graph structure, E_i-*path* between two vertices u and v, is the path which consists of only E_i-edges for some i, and similarly, E_i*-cycle* is the cycle, which consists of only E_i-edges for some i. A graph structure is a *tree*, if it is connected and contains no cycle or equivalently the underlying graph of G^* is a tree. G^* is an E_i*-tree*, if the subgraph structure induced by Ei-edges is a tree.

Similarly, G^* is a $E_1E_2 \dots E_k$*-tree*, if G^* is a E_j-tree for each j, $1 \leq j \leq k$.

A graph structure is an E_i*-forest*, if the subgraph structure induced by E_i-edges is a forest, i.e., if it has no E_i-cycles. Let $S \subseteq U$, then the subgraph structure $\langle S \rangle$ induced by S, has vertex set S, where two vertices u and v in $\langle S \rangle$ are joined by an E_i-edge if, and only if, they are joined by an E_i-edge in G^* for $1 \leq i \leq k$ For some i, $1 \leq i \leq k$, the E_i-subgraph induced by S, is denoted by E_i-$\langle S \rangle$ and it has only E_i-edges joining the vertices in S. If T is a subset of edge set in G^*, then subgraph structure $\langle T \rangle$ induced by T has the vertex set, the end vertices in T, and whose edges are those in T. Let $G^* = (U_1, E_1, E_2, \dots, E_m)$ and $H^* = (U_2, E'_1, E'_2, \dots, E'_n)$ be graph structures then G^* and H^* are *isomorphic*, if $m = n$ and there exists a bijection $f : U_1 \to U_2$ and a permutation $\phi : \{E_1, E_2, \dots, E_n\} \to \{E'_1, E'_2, \dots, E'_n\}$, say $E_i \to E'_j$, $1 \leq i, j \leq n$, such that for all $u, v \in U_1$, $uv \in E_i$ implies $f(u)f(v) \in E'_j$.

Two graph structures $G^* = (U, E_1, E_2, \dots, E_k)$ and $H^* = (U, E'_1, E'_2, \dots, E'_k)$, on the same vertex set U, are *identical*, if there exists a bijection $f : U \to U$, such that for all u and v in U and an E_i-edge uv in G^*, $f(u)f(v)$ is an E'_i-edge in H^*, where $1 \leq i \leq k$ and $E_i \simeq E'_i$ for all i. Let ϕ be a permutation on $\{E_1, E_2, \dots, E_k\}$ then the ϕ*-cyclic complement* of G^* denoted by $(G^*)^{\phi c}$ is obtained by replacing E_i with $\phi(E_i)$ for $1 \leq i \leq k$. Let $G^* = (U, E_1, E_2, \dots, E_k)$ be a graph structure and ϕ be a permutation on $\{E_1, E_2, \dots, E_k\}$, then

- G^* is ϕ*-self complementary*, if G^* is isomorphic to $(G^*)^{\phi c}$, the ϕ-cyclic complement of G^* and G^* is *self-complementary*, if $\phi \neq$ identity permutation;
- G^* is *strong* ϕ*-self complementary*, if G^* is identical to $(G^*)^{\phi c}$, the ϕ-complement of G^* and G^* is *strong self-complementary*, if $\phi \neq$ identity permutation.

Definition 2.1 ([6])**.** An *intuitionistic fuzzy set* (IFS) on an universe X is an object of the form

$$A = \{\langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X\},$$

where $\mu_A(x)(\in [0,1])$ is called degree of membership of $x \in A$, $\nu_A(x)(\in [0,1])$ is called degree of nonmembership of $x \in A$, and μ_A and ν_A satisfy the following condition: for all $x \in X$, $\mu_A(x) + \nu_A(x) \leq 1$.

Definition 2.2 ([6])**.** An intuitionistic fuzzy relation $R = (\mu_R(x,y), \nu_R(x,y))$ in an universe $X \times Y(R(X \to Y))$ is an intuitionistic fuzzy set of the form

$$R = \{\langle (x,y), \mu_R(x,y), \nu_R(x,y) \rangle \mid (x,y) \in X \times Y\},$$

where $\mu_R : X \times Y \to [0,1]$ and $\nu_R : X \times Y \to [0,1]$. The intuitionistic fuzzy relation R satisfies $\mu_R(x,y) + \nu_R(x,y) \leq 1$ for all $x, y \in X$.

Definition 2.3 ([9])**.** Let $G^* = (U, E_1, E_2, \dots, E_k)$ be a graph structure and $\nu, \rho_1, \rho_2, \dots, \rho_k$ be the fuzzy subsets of $U, E_1, E_2, \dots, E_k$, respectively such that

$$0 \leq \rho_i(xy) \leq \nu(x) \wedge \nu(y), \quad \text{for all } x, y \in U \text{ and } i = 1, 2, \dots, k.$$

Then $G = (\nu, \rho_1, \rho_2, \dots, \rho_k)$ is a fuzzy graph structure of G^*.

Definition 2.4 ([9])**.** Let $G = (\nu, \rho_1, \rho_2, \dots, \rho_k)$ be a fuzzy graph structure of a graph structure $G^* = (U, E_1, E_2, \dots, E_k)$. Then $F = (\nu, \tau_1, \tau_2, \dots, \tau_k)$ is a partial fuzzy spanning subgraph structure of G if, $\tau_i \subseteq \rho_i$ for $i = 1, 2, \dots, k$.

Definition 2.5 ([9])**.** Let G^* be a graph structure and G be a fuzzy graph structure of G^*. If $xy \in \text{supp}(\rho_i)$, then xy is said to be a ρ_i-edge of G.

Definition 2.6 ([9])**.** The strength of a ρ_i-path $x_0 x_1 \dots x_n$ of a fuzzy graph structure G is $\bigwedge_{j=1}^{n} \rho_i(x_{j-1}x_j)$ for $i = 1, 2, \dots, k$.

Definition 2.7 ([9])**.** In any fuzzy graph structure G,

$$\rho_i^2(xy) = \rho_i \circ \rho_i(xy) = \bigvee_z \{\rho_i(xz) \wedge \rho_i(zy)\},$$

$$\rho_i^j(xy) = (\rho_i^{j-1} \circ \rho_i)(xy) = \bigvee_z \left\{\rho_i^{j-1}(xz) \wedge \rho_i(zy)\right\},$$

$j = 2, 3, \dots, m$, for any $m \geq 2$. Also $\rho_i^{\infty}(xy) = \vee \left\{\rho_i^j(xy),\ j = 1, 2, \dots\right\}$.

Definition 2.8 ([9])**.** $G = (\nu, \rho_1, \rho_2, \dots, \rho_k)$ is a ρ_i-cycle iff $(\text{supp}(\nu), \text{supp}(\rho_1), \text{supp}(\rho_2), \dots, \text{supp}(\rho_k))$ is a E_i-cycle.

Definition 2.9 ([9])**.** $G = (\nu, \rho_1, \rho_2, \dots, \rho_k)$ is a fuzzy ρ_i-cycle iff $(\text{supp}(\nu), \text{supp}(\rho_1), \text{supp}(\rho_2), \dots, \text{supp}(\rho_k))$ is a E_i-cycle and there exists no unique xy in $\text{supp}(\rho_i)$ such that

$$\rho_i(xy) = \wedge \{\rho_i)(uv) | uv \in \text{supp}(\rho_i)\}.$$

Definition 2.10 ([9])**.** $G = (\nu, \rho_1, \rho_2, \dots, \rho_k)$ is a fuzzy ρ_i-tree if it has a partial fuzzy spanning subgraph structure, $\breve{F}_i = (\nu, \tau_1, \tau_2, \dots, \tau_k)$ which is a τ_i-tree where for all ρ_i-edges not in $\breve{F}_i$, $\rho_i(xy) < \tau_i^{\infty}(xy)$.

3. Intuitionistic Fuzzy Graph Structures

Definition 3.1. Let $\{E_i : i = 1, 2, \dots, n\}$ be a set of irreflexive, symmetric and mutually disjoint relations on a non-empty set U. An *intuitionistic fuzzy graph structure (IFGS)* with underlying vertex set U is denoted by $\breve{G}_s = (A, B_1, B_2, \dots, B_n)$, where

(i) A is an intuitionistic fuzzy set of U with $\mu_A : U \to [0, 1]$ and $\nu_A : U \to [0, 1]$, namely the degree of membership and the degree of nonmembership of $x \in U$, respectively, such that

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \quad \text{for all } x \in U.$$

(ii) Each B_i is an intuitionistic fuzzy set of E_i such that the functions $\mu_{B_i} : E_i \to [0,1]$ and $\nu_{B_i} : E_i \to [0,1]$ are defined by

$$\mu_{B_i}(xy) \leq \mu_A(x) \wedge \mu_A(y), \quad \nu_{B_i}(xy) \leq \nu_A(x) \vee \nu_A(y)$$

and

$$0 \leq \mu_{B_i}(xy) + \nu_{B_i}(xy) \leq 1, \qquad \text{for all } xy \in\subset U \times U,\ i = 1, 2, \dots, n.$$

Equivalently, an IFGS of a graph structure may be defined in the following way.

Let $G^* = (U, E_1, E_2, \dots, E_n)$ be a graph structure and let $A, B_1, B_2, \dots, B_{n-1}$ andB_n be intuitionistic fuzzy subsets of $U, E_1, E_2, \dots, E_{n-1}$ and E_n, respectively. Then $\breve{G}_s = (A, B_1, B_2, \dots, B_n)$ is called an *IFGS* of G^*, if

$$\mu_{B_i}(xy) \leq \mu_A(x) \wedge \mu_A(y), \quad \nu_{B_i}(xy) \leq \nu_A(x) \vee \nu_A(y),$$

for all $xy \in E_i$, $i = 1, 2, \dots, n$, and

$$\mu_{B_i}(xy) + \nu_{B_i}(xy) \leq 1, \quad \text{for all } xy \in U \times U.$$

Example 3.1. Let $G^* = (U, E_1, E_2)$ be a graph structure such that $U = \{a_1, a_2, a_3, a_4\}$, $E_1 = \{a_1a_2, a_2a_3\}$ and $E_2 = \{a_3a_4, a_1a_4\}$. Let A, B_1 andB_2 be intuitionistic fuzzy subsets of U, E_1 and E_2, respectively, such that

$$\begin{aligned} A =& \{(a_1, 0.5, 0.2), (a_2, 0.7, 0.3), (a_3, 0.4, 0.3), (a_4, 0.7, 0.3)\}, \\ B_1 =& \{(a_1a_2, 0.5, 0.3), (a_2a_3, 0.4, 0.3)\}, \\ \text{and } B_2 =& \{(a_3a_4, 0.4, 0.3), (a_1a_4, 0.1, 0.2)\}. \end{aligned}$$

Then $\breve{G}_s = (A, B_1, B_2)$ is an IFGS of G^* as shown in Fig. 1.

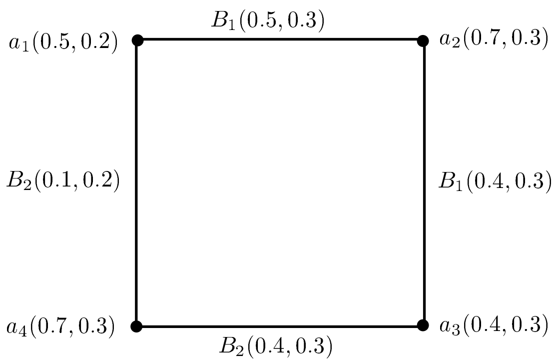


FIGURE 1. IFGS $\breve{G}_s = (A, B_1, B_2)$

Definition 3.2. An IFGS $\breve{H}_s = (C, D_1, D_2, \dots, D_n)$ is said to be an intuitionistic fuzzy subgraph structure of an IFGS $\breve{G}_s = (A, B_1, B_2, \dots, B_n)$ with underlying vertex set U, if $C \subseteq A$ and $D_i \subseteq C_i$ for all i, that is

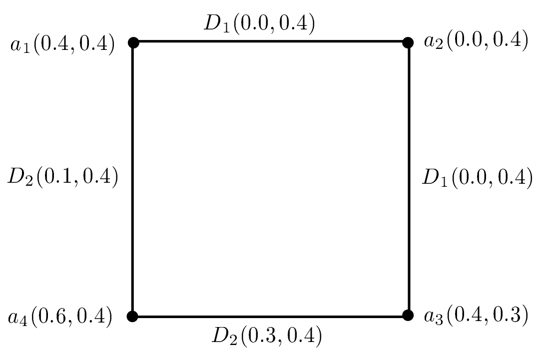
$$\mu_C(x) \leq \mu_A(x), \quad \nu_C(x) \geq \nu_A(x), \quad \text{for all } x \in U,$$

and for $i = 1, 2, \dots, n$

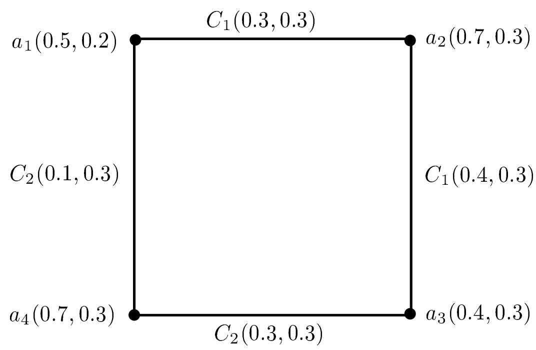
$$\mu_{D_i}(xy) \le \mu_{B_i}(xy), \quad \nu_{D_i}(xy) \ge \nu_{B_i}(xy), \quad \text{for all } xy \in U \times U.$$

$\breve{H}_s$ is called an intuitionistic fuzzy spanning subgraph structure of an IFGS $\breve{G}_s$, if $C = A$.

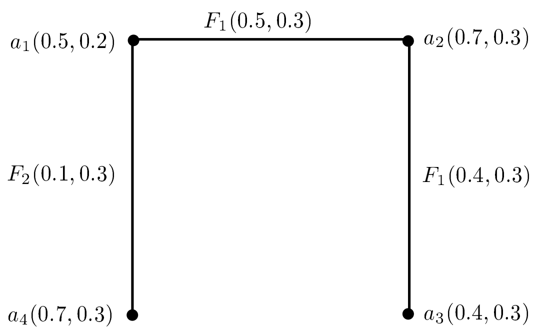
$\breve{H}_s$ is called an intuitionistic fuzzy partial spanning subgraph structure of an IFGS $\breve{G}_s$, if it excludes some edges of $\breve{G}_s$.



Intuitionistic Fuzzy Subgraph Structure
$\breve{H}_s = (C, D_1, D_2)$



Intuitionistic Fuzzy Spanning Subgraph Structure $\breve{J}_s = (A, C_1, C_2)$



Intuitionistic Fuzzy Partial Spanning Subgraph Structure
$\breve{K}_s = (A, F_1, F_2)$

FIGURE 2. Intuitionistic Fuzzy Subgraph Structures

Example 3.2. Consider an IFGS $\breve{G}_s = (A, B_1, B_2)$, as shown in Fig. 1. Let

$$\begin{aligned} C &= \{(a_1, 0.4, 0.4), (a_2, 0.0, 0.4), (a_3, 0.4, 0.3), (a_4, 0.6, 0.4)\}, \\ D_1 &= \{(a_1a_2, 0, 0.4), (a_2a_3, 0, 0.4)\}, \\ D_2 &= \{(a_3a_4, 0.3, 0.4), (a_1a_4, 0.1, 0.4)\}, \end{aligned}$$

$$\begin{aligned} C_1 &= \{(a_1a_2, 0.3, 0.3), (a_2a_3, 0.4, 0.3)\}, \\ C_2 &= \{(a_3a_4, 0.3, 0.3), (a_1a_4, 0.1, 0.3)\}, \\ F_1 &= \{(a_1a_2, 0.5, 0.3), (a_2a_3, 0.4, 0.3)\}, \\ \text{and } F_2 &= \{(a_1a_4, 0.1, 0.3)\}. \end{aligned}$$

By routine calculations, it is easy to see that $\breve{H}_s = (C, D_1, D_2)$, $\breve{J}_s = (A, C_1, C_2)$ and $\breve{K}_s = (A, F_1, F_2)$ are respectively the intuitionistic fuzzy subgraph structure, intuitionistic fuzzy spanning subgraph structure and intuitionistic fuzzy partial spanning subgraph structure of $\breve{G}_s$. Their respective drawings are shown in Fig. 2.

Definition 3.3. Let $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ be an IFGS with underlying vertex set U. Then there is a B_i-*edge* between two vertices x and y of U, if one of the following is true:

(i) $\mu_{B_i}(xy) > 0$ and $\nu_{B_i}(xy) > 0$,
(ii) $\mu_{B_i}(xy) > 0$ and $\nu_{B_i}(xy) = 0$,
(iii) $\mu_{B_i}(xy) = 0$ and $\nu_{B_i}(xy) > 0$,

for some i.

Definition 3.4. For an intuitionistic fuzzy graph structure $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ with vertex set U, *support* of B_i is given by:

$$\mathrm{supp}(B_i) = \{xy \in U \times U : \mu_{B_i}(xy) \neq 0 \text{ or } \nu_{B_i}(xy) \neq 0\}, \quad i = 1, 2, \ldots, n.$$

Definition 3.5. B_i-path of an IFGS $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ with underlying vertex set U, is a sequence of distinct vertices $v_1, v_2, \ldots, v_m \in U$ (except the choice $v_m = v_1$), such that $v_{j-1}v_j$ is a B_i-edge for all $j = 2, 3, \ldots, m$.

Definition 3.6. In an IFGS $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ with underlying vertex set U, two vertices x and y of U are said to be B_i-connected, if they are joined by a B_i-Path, for some $i \in \{1, 2, 3, \ldots, n\}$.

Definition 3.7. An IFGS $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ with underlying vertex set U, is said to be B_i-strong, if for all B_i-edges xy

$$\mu_{B_i}(xy) = \mu_A(x) \wedge \mu_A(y), \quad \nu_{B_i}(xy) = \nu_A(x) \vee \nu_A(y),$$

for some $i \in \{1, 2, 3, \ldots, n\}$.

Example 3.3. Consider the IFGS $\breve{G}_s = (A, B_1, B_2)$, as shown in Fig. 1. Then

(i) a_1a_2, a_2a_3 are B_1-edges and a_3a_4, a_1a_4 are B_2-edges;
(ii) $a_1a_2a_3$ and $a_3a_4a_1$ are B_1- and B_2-paths, respectively;
(iii) a_1 and a_3 are B_1-connected vertices of U;
(iv) $\breve{G}_s$ is B_1-strong, since $\mathrm{supp}(B_1) = \{a_1a_2, a_2a_3\}$ and

$$\begin{aligned} \mu_{B_1}(a_1a_2) &= 0.5 = (\mu_A(a_1) \wedge \mu_A(a_2)), \\ \nu_{B_1}(a_1a_2) &= 0.3 = (\nu_A(a_1) \vee \nu_A(a_2)), \end{aligned}$$

$$\mu_{B_1}(a_2a_3) = 0.4 = (\mu_A(a_2) \wedge \mu_A(a_3)),$$
$$\text{and } \nu_{B_1}(a_2a_3) = 0.3 = (\nu_A(a_2) \vee \nu_A(a_3)).$$

Definition 3.8. An IFGS $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ is said to be strong, if it is B_i-strong for all $i \in \{1, 2, 3, \ldots, n\}$.

Definition 3.9. An IFGS $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ with underlying vertex set U, is called *complete* or $B_1B_2 \ldots B_n$*-complete* if
(i) $\breve{G}_s$ is a strong IFGS;
(ii) $\text{supp}(B_i) \neq \emptyset$ for all $i = 1, 2, 3, \ldots, n$;
(iii) For each pair of vertices $x, y \in U$, xy is a B_i-edge for some i.

Example 3.4. Let $\breve{G}_s = (A, B_1, B_2)$ shown in Fig. 3, be IFGS of the graph structure $G^* = (U, E_1, E_2)$ where $U = \{a_1, a_2, a_3, a_4\}$, $E_1 = \{a_1a_3, a_3a_4, a_1a_4\}$ and $E_2 = \{a_1a_2, a_2a_3, a_2a_4\}$. Then $\breve{G}_s$ is a strong IFGS since it is both B_1-strong and B_2-strong.

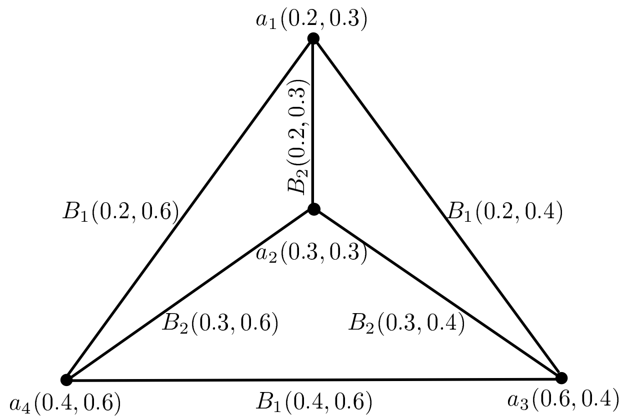


FIGURE 3. IFGS $\breve{G}_s = (A, B_1, B_2)$

Moreover $\text{supp}(B_1) \neq \emptyset$, $\text{supp}(B_2) \neq \emptyset$, every pair of vertices belonging to U, is either a B_1-edge or a B_2-edge, so $\breve{G}_s$ is a complete or B_1B_2-complete IFGS as well.

Definition 3.10. In an IFGS $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ with underlying vertex set U, μ_{B_i}- and ν_{B_i}*-strengths* of a B_i-path "$\underline{P}_{B_i} = v_1v_2 \ldots v_m$", are denoted by $\delta.\underline{P}_{B_i}$ and $\Delta.\underline{P}_{B_i}$, respectively, such that

$$\delta.\underline{P}_{B_i} = \bigwedge_{j=2}^{m} [\mu_{B_i}(v_{j-1}v_j)] \text{ and } \Delta.\underline{P}_{B_i} = \bigvee_{j=2}^{m} [\nu_{B_i}(v_{j-1}v_j)].$$

Then we write, strength of the path $\underline{P}_{B_i} = (\delta.\underline{P}_{B_i}, \Delta.\underline{P}_{B_i})$.

Example 3.5. In $\breve{G}_s = (A, B_1, B_2)$ shown in Fig. 3, $\underline{P}_1 = a_1a_3a_4a_1$ is a B_1-path and $\underline{P}_2 = a_3a_2a_4$ is a B_2-path and

$$\delta.\underline{P}_1 = \mu_{B_1}(a_1a_3) \wedge \mu_{B_1}(a_3a_4) \wedge \mu_{B_1}(a_4a_1) = 0.2 \wedge 0.4 \wedge 0.2 = 0.2,$$

$$\Delta.\underset{\sim}{P}_1 = \nu_{B_1}(a_1a_3) \vee \nu_{B_1}(a_3a_4) \vee \nu_{B_1}(a_4a_1) = 0.4 \vee 0.6 \vee 0.6 = 0.6,$$
$$\delta.\underset{\sim}{P}_2 = \mu_{B_2}(a_3a_2) \wedge \mu_{B_2}(a_2a_4) = 0.3 \wedge 0.3 = 0.3,$$
$$\Delta.\underset{\sim}{P}_2 = \nu_{B_2}(a_3a_2) \vee \nu_{B_2}(a_2a_4) = 0.4 \vee 0.6 = 0.6.$$

Thus strength of B_1-path $\underset{\sim}{P}_1 = (\delta.\underset{\sim}{P}_1, \Delta.\underset{\sim}{P}_1) = (0.2, 0.6)$, strength of B_2-path $\underset{\sim}{P}_2 = (\delta.\underset{\sim}{P}_2, \Delta.\underset{\sim}{P}_2) = (0.3, 0.6)$.

Definition 3.11. In an IFGS $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ with underlying vertex set U:

(i) μ_{B_i}*-strength of connectedness* between x and y, is defined by $\mu^{\infty}_{B_i}(xy) = \bigvee_{j\geq 1}\{\mu^{j}_{B_i}(xy)\}$, where $\mu^{j}_{B_i}(xy) = (\mu^{j-1}_{B_i} o\ \mu_{B_i})(xy)$ for $j \geq 2$ and $\mu^{2}_{B_i}(xy) = (\mu_{B_i} o\ \mu_{B_i})(xy) = \bigvee_z\{\mu_{B_i}(xz) \wedge \mu_{B_i}(zy)\}$;

(ii) ν_{B_i}*-strength of connectedness* between x and y, is defined by $\nu^{\infty}_{B_i}(xy) = \bigvee_{j\geq 1}\{\nu^{j}_{B_i}(xy)\}$, where $\nu^{j}_{B_i}(xy) = (\nu^{j-1}_{B_i} o\ \nu_{B_i})(xy)$ for $j \geq 2$ and $\nu^{2}_{B_i}(xy) = (\nu_{B_i} o\ \nu_{B_i})(xy) = \bigwedge_z\{\nu_{B_i}(xz) \vee \nu_{B_i}(zy)\}$.

Example 3.6. Let $\breve{G}_s = (A, B_1, B_2)$, as shown in Fig. 4, be IFGS of graph structure $G^* = (U, E_1, E_2)$, such that $U = \{a_1, a_2, a_3\}$, $E_1 = \{a_1a_2, a_1a_3\}$ and $E_2 = \{a_2a_3\}$. Since $\mu_{B_1}(a_1a_2) = 0.3, \mu_{B_1}(a_1a_3) = 0.3, \mu_{B_1}(a_2a_3) = 0$, therefore

$$\mu^2_{B_1}(a_1a_2) = (\mu_{B_1} o \mu_{B_1})(a_1a_2) = \mu_{B_1}(a_1a_3) \wedge \mu_{B_1}(a_3a_2) = 0.3 \wedge 0.0 = 0,$$
$$\mu^2_{B_1}(a_2a_3) = (\mu_{B_1} o \mu_{B_1})(a_2a_3) = \mu_{B_1}(a_2a_1) \wedge \mu_{B_1}(a_1a_3) = 0.3 \wedge 0.3 = 0.3,$$
$$\mu^2_{B_1}(a_1a_3) = (\mu_{B_1} o \mu_{B_1})(a_1a_3) = \mu_{B_1}(a_1a_2) \wedge \mu_{B_1}(a_2a_3) = 0.3 \wedge 0.0 = 0,$$
$$\mu^3_{B_1}(a_1a_2) = (\mu^2_{B_1} o \mu_{B_1})(a_1a_2) = \mu^2_{B_1}(a_1a_3) \wedge \mu_{B_1}(a_3a_2) = 0.0 \wedge 0.0 = 0,$$
$$\mu^3_{B_1}(a_2a_3) = (\mu^2_{B_1} o \mu_{B_1})(a_2a_3) = \mu^2_{B_1}(a_2a_1) \wedge \mu_{B_1}(a_1a_3) = 0.0 \wedge 0.3 = 0,$$
$$\mu^3_{B_1}(a_1a_3) = (\mu^2_{B_1} o \mu_{B_1})(a_1a_3) = \mu^2_{B_1}(a_1a_2) \wedge \mu_{B_1}(a_2a_3) = 0.0 \wedge 0.0 = 0.$$

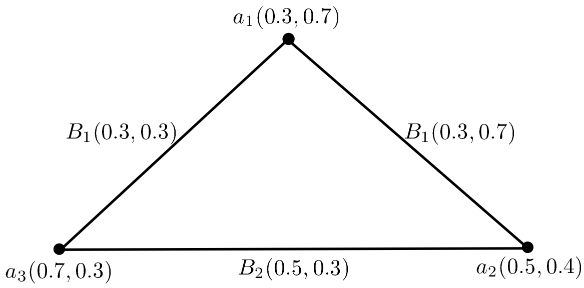


FIGURE 4. IFGS $\breve{G}_s = (A, B_1, B_2)$

Thus, we have

$$\mu^{\infty}_{B_1}(a_1a_2) = \vee\{0.3, 0.0, 0.0\} = 0.3,$$
$$\mu^{\infty}_{B_1}(a_2a_3) = \vee\{0.0, 0.3, 0.0\} = 0.3,$$
$$\mu^{\infty}_{B_1}(a_1a_3) = \vee\{0.3, 0.0, 0.0\} = 0.3.$$

Since $\nu_{B_1}(a_1a_2) = 0.7$, $\nu_{B_1}(a_1a_3) = 0.3$, $\nu_{B_1}(a_2a_3) = 0$, therefore

$$\begin{aligned}
\nu^2_{B_1}(a_1a_2) &= (\nu_{B_1} o \nu_{B_1})(a_1a_2) = \nu_{B_1}(a_1a_3) \vee \nu_{B_1}(a_3a_2) = 0.3 \vee 0.0 = 0.3,\\
\nu^2_{B_1}(a_2a_3) &= (\nu_{B_1} o \nu_{B_1})(a_2a_3) = \nu_{B_1}(a_2a_1) \vee \nu_{B_1}(a_1a_3) = 0.7 \vee 0.3 = 0.7,\\
\nu^2_{B_1}(a_1a_3) &= (\nu_{B_1} o \nu_{B_1})(a_1a_3) = \nu_{B_1}(a_1a_2) \vee \nu_{B_1}(a_2a_3) = 0.7 \vee 0.0 = 0.7,\\
\nu^3_{B_1}(a_1a_2) &= (\nu^2_{B_1} o \nu_{B_1})(a_1a_2) = \nu^2_{B_1}(a_1a_3) \vee \nu_{B_1}(a_3a_2) = 0.7 \vee 0.0 = 0.7,\\
\nu^3_{B_1}(a_2a_3) &= (\nu^2_{B_1} o \nu_{B_1})(a_2a_3) = \nu^2_{B_1}(a_2a_1) \vee \nu_{B_1}(a_1a_3) = 0.3 \vee 0.3 = 0.3,\\
\nu^3_{B_1}(a_1a_3) &= (\nu^2_{B_1} o \nu_{B_1})(a_1a_3) = \nu^2_{B_1}(a_1a_2) \vee \nu_{B_1}(a_2a_3) = 0.3 \vee 0.0 = 0.3,
\end{aligned}$$

and

$$\begin{aligned}
\nu^4_{B_1}(a_1a_2) &= (\nu^3_{B_1} o \nu_{B_1})(a_1a_2) = \nu^3_{B_1}(a_1a_3) \vee \nu_{B_1}(a_3a_2) = 0.3 \vee 0.0 = 0.3,\\
\nu^4_{B_1}(a_2a_3) &= (\nu^3_{B_1} o \nu_{B_1})(a_2a_3) = \nu^3_{B_1}(a_2a_1) \vee \nu_{B_1}(a_1a_3) = 0.7 \vee 0.3 = 0.7,\\
\nu^4_{B_1}(a_1a_3) &= (\nu^3_{B_1} o \nu_{B_1})(a_1a_3) = \nu^3_{B_1}(a_1a_2) \vee \nu_{B_i}(a_2a_3) = 0.7 \vee 0.0 = 0.7.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\nu^\infty_{B_1}(a_1a_2) &= \vee\{0.7, 0.3, 0.7, 0.3\} = 0.7,\\
\nu^\infty_{B_1}(a_2a_3) &= \vee\{0.0, 0.7, 0.3, 0.7\} = 0.7,\\
\nu^\infty_{B_1}(a_1a_3) &= \vee\{0.3, 0.7, 0.3, 0.7\} = 0.7.
\end{aligned}$$

By similar calculations, it can be easily checked that

$$\begin{aligned}
\mu^\infty_{B_2}(a_1a_2) &= 0, \quad \mu^\infty_{B_2}(a_2a_3) = 0.5, \quad \mu^\infty_{B_2}(a_1a_3) = 0,\\
\nu^\infty_{B_2}(a_1a_2) &= 0.3, \quad \nu^\infty_{B_2}(a_2a_3) = 0.3, \quad \nu^\infty_{B_2}(a_1a_3) = 0.3.
\end{aligned}$$

Definition 3.12. An IFGS $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ of a graph structure $G^* = (U, E_1, E_2, \ldots, E_n)$ is a *B_i-cycle*, if G^* is an E_i-cycle.

Definition 3.13. An IFGS $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ of a graph structure $G^* = (U, E_1, E_2, \ldots, E_n)$ is an *intuitionistic fuzzy B_i-cycle* for some i, if following conditions hold:
(i) $\breve{G}_s$ is a B_i-cycle;
(ii) There is no unique B_i-edge uv in $\breve{G}_s$, such that $\mu_{B_i}(uv) = \min\{\mu_{B_i}(xy) : xy \in E_i = \text{supp}(B_i)\}$ or $\nu_{B_i}(uv) = \max\{\nu_{B_i}(xy) : xy \in E_i = \text{supp}(B_i)\}$.

Example 3.7. IFGS $\breve{G}_s = (A, B_1, B_2)$ shown in Fig. 3, is a B_1-cycle as well as intuitionistic fuzzy B_1-cycle, since $(\text{supp}(A), \text{supp}(B_1), \text{supp}(B_2))$ is an E_1-cycle and there are two B_1-edges with minimum degree of membership and two B_1-edges with maximum degree of nonmembership of all B_1-edges.

Definition 3.14. An IFGS $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ of a graph structure $G^* = (U, E_1, E_2, \ldots, E_n)$ is a *B_i-tree*, if $(\text{supp}(A), \text{supp}(B_1), \text{supp}(B_2), \ldots, \text{supp}(B_n))$ is an E_i-tree. In other words, $\breve{G}_s$ is a B_i-tree if the subgraph of $\breve{G}_s$, induced by $\text{supp}(B_i)$, forms a tree.

Definition 3.15. An IFGS $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ of a graph structure $G^* = (U, E_1, E_2, \ldots, E_n)$ is an *intuitionistic fuzzy B_i-tree* (*intuitionistic fuzzy B_i-forest*), if $\breve{G}_s$ has an intuitionistic fuzzy partial spanning subgraph structure $\breve{H}_s = (A, C_1, C_2, \ldots, C_n)$, such that $\breve{H}_s$ is a C_i-tree (C_i-forest) and $\mu_{B_i}(xy) < \mu^{\infty}_{C_i}(xy)$ and $\nu_{B_i}(xy) < \nu^{\infty}_{C_i}(xy)$ for all $B_i$edges not in $\breve{H}_s$.

Example 3.8. The IFGS, shown in Fig. 3, is a B_2-tree but not an intuitionistic fuzzy B_2-tree. While IFGS $\breve{G}_s = (A, B_1, B_2)$, shown in Fig. 5, is not a B_1-tree but an intuitionistic fuzzy B_1-tree, since it has an intuitionistic fuzzy partial spanning subgraph structure (A, B'_1, B'_2) as a B_1-tree, which is obtained by deleting B_1-edge a_1a_4 from $\breve{G}_s$, with $\mu_{B_1}(a_1a_4) = 0.3 < 0.4 = \mu^{\infty}_{B'_1}(a_1a_4)$ and $\nu_{B_1}(a_1a_4) = 0.5 < 0.6 = \nu^{\infty}_{B'_1}(a_1a_4)$.

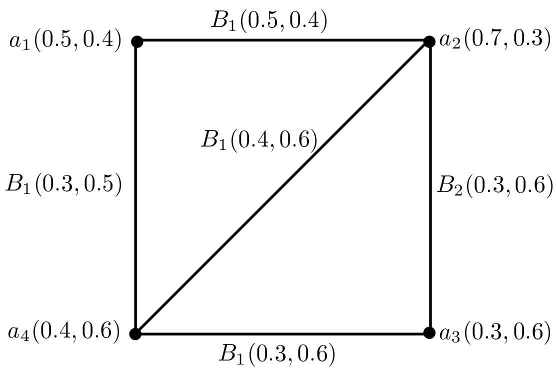


FIGURE 5. IFGS $\breve{G}_s = (A, B_1, B_2)$

Definition 3.16. An IFGS $\breve{G}_{s1} = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ of GS $G^*_1 = (U_1, E_{11}, E_{12}, \ldots, E_{1n})$ is *isomorphic* to an IFGS $\breve{G}_{s2} = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ of $G^*_2 = (U_2, E_{21}, E_{22}, \ldots, E_{2n})$, if there exist a bijection $f : U_1 \to U_2$ and a permutation ϕ on the set $\{1, 2, \ldots, n\}$, such that:

$$\mu_{A_1}(u_1) = \mu_{A_2}(f(u_1)),\ \nu_{A_1}(u_1) = \nu_{A_2}(f(u_1)), \quad \text{for all } u_1 \in U_1$$

and for $\phi(i) = j$

$$\mu_{B_{1i}}(u_1u_2) = \mu_{B_{2j}}(f(u_1)f(u_2)), \quad \nu_{B_{1i}}(u_1u_2) = \nu_{B_{2j}}(f(u_1)f(u_2)),$$

for all $u_1u_2 \in E_{1i}$, $i = 1, 2, \ldots, n$.

Definition 3.17. An IFGS $\breve{G}_{s1} = (A_1, B_{11}, B_{12}, \ldots, B_{1n})$ of GS $G^*_1 = (U, E_{11}, E_{12}, \ldots, E_{1n})$ is *identical* to an IFGS $\breve{G}_{s2} = (A_2, B_{21}, B_{22}, \ldots, B_{2n})$ of $G^*_2 = (U, E_{21}, E_{22}, \ldots, E_{2n})$, if there exist a bijection $f : U \to U$, such that:

$$\mu_{A_1}(u) = \mu_{A_2}(f(u)), \quad \nu_{A_1}(u) = \nu_{A_2}(f(u)), \quad \text{for all } u \in U$$

and

$$\mu_{B_{1i}}(u_1u_2) = \mu_{B_{2i}}(f(u_1)f(u_2)), \quad \nu_{B_{1i}}(u_1u_2) = \nu_{B_{2i}}(f(u_1)f(u_2)),$$

for all $u_1u_2 \in E_{1i}$, $i = 1, 2, \ldots, n$.

Example 3.9. $\breve{G}_{s1}$ and $\breve{G}_{s2}$, as shown in Fig. 6 and Fig. 7, are IFGSs of graph structures $G^*{}_1 = (U_1, E_1, E_2, E_3, E_4)$ and $G^*{}_2 = (U_2, E_1', E_2', E_3', E_4')$, respectively, where

$$U_1 = \{a_1, a_2, a_3, a_4, a_5\}, \quad E_1 = \{a_1a_2, a_2a_5\},$$
$$E_2 = \{a_2a_3, a_2a_4\}, \quad E_3 = \{a_1a_3, a_4a_5\},$$
$$E_4 = \{a_1a_5, a_3a_4\}, \quad U_2 = \{b_1, b_2, b_3, b_4, b_5\},$$
$$E_1' = \{b_2b_4, b_3b_4\}, \quad E_2' = \{b_1b_4, b_4b_5\},$$
$$E_3' = \{b_1b_2, b_3b_5\}, \quad E_4' = \{b_1b_5, b_2b_3\}.$$

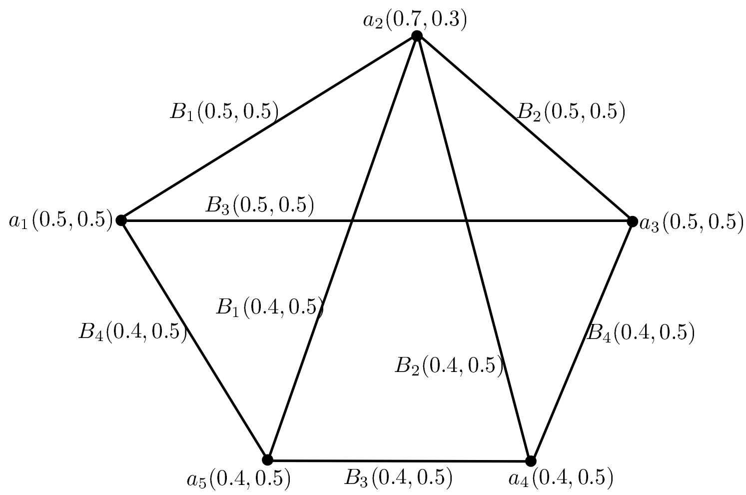


FIGURE 6. IFGS $\breve{G}_{s1} = (A_1, B_1, B_2, B_3, B_4)$

Then $\breve{G}_{s1}$ is isomorphic to $\breve{G}_{s2}$ under the mapping $f : U_1 \to U_2$, given by

$$f(a_1) = b_5, \quad f(a_2) = b_4, \quad f(a_3) = b_3, \quad f(a_4) = b_2, \quad f(a_5) = b_1,$$

and a permutation ϕ given by

$$\phi(1) = 2, \quad \phi(2) = 1, \quad \phi(3) = 3, \quad \phi(4) = 4,$$

such that

$$\mu_{A_1}(a_i) = \mu_{A_2}(f(a_i)), \quad \nu_{A_1}(a_i) = \nu_{A_2}(f(a_i))$$

for all $a_i \in U_1$, and

$$\mu_{B_k}(a_ia_j) = \mu_{B_{\phi(k)}}(f(a_i)f(a_j)), \quad \nu_{B_k}(a_ia_j) = \nu_{B_{\phi(k)}}(f(a_i)f(a_j)),$$

for all $a_ia_j \in E_k$, $k = 1, 2, 3, 4$.

Also, $\breve{G}_{s1}$ is identical with $\breve{G}_{s2}$ under the mapping $f : U_1 \to U_2$, given by

$$f(a_1) = b_3, \quad f(a_2) = b_4, \quad f(a_3) = b_5, \quad f(a_4) = b_1, \quad f(a_5) = b_2,$$

such that

$$\mu_{A_1}(a_i) = \mu_{A_2}(f(a_i)), \quad \nu_{A_1}(a_i) = \nu_{A_2}(f(a_i)),$$

for all $a_i \in U_1$, and

$$\mu_{B_k}(a_ia_j) = \mu_{B'_k}(f(a_i)f(a_j)), \quad \nu_{B_k}(a_ia_j) = \nu_{B'_k}(f(a_i)f(a_j)),$$

for all $a_ia_j \in E_k$, $k = 1, 2, 3, 4$.

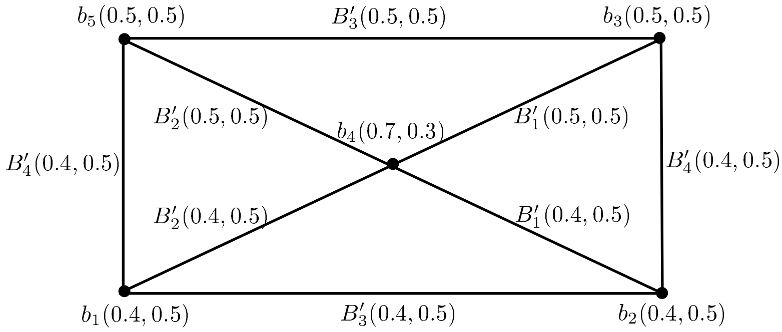


FIGURE 7. IFGS $\breve{G}_{s2} = (A_2, B'_1, B'_2, B'_3, B'_4)$

Remark 3.1. Identical IFGSs are always isomorphic but the converse is not necessarily true. As IFGS shown in Fig. 3 is isomorphic to IFGS shown in Fig. 8 but they are not identical.

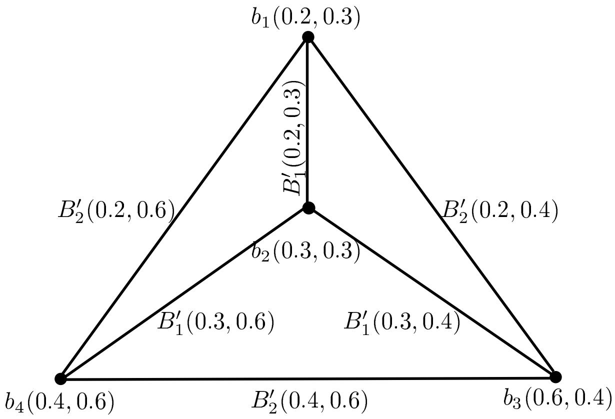


FIGURE 8. IFGS $\breve{G}_{s1} = (A_1, B'_1, B'_2)$

Definition 3.18. Let $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ be an intuitionistic fuzzy graph structure of a graph structure $G^* = (U, E_1, E_2, \ldots, E_n)$. Let ϕ denote a permutation on the set $\{E_1, E_2, \ldots, E_n\}$ and the corresponding permutation on $\{B_1, B_2, \ldots, B_n\}$, i.e., $\phi(B_i) = B_j$ iff $\phi(E_i) = E_j$ for all i.

If $xy \in B_r$ for some r and

$$\mu_{B_i^\phi}(xy) = \mu_A(x) \wedge \mu_A(y) - \bigvee_{j \neq i} \mu_{\phi B_j}(xy),$$

$$\nu_{B_i^\phi}(xy) = \nu_A(x) \vee \nu_A(y) - \bigvee_{j \neq i} \nu_{\phi B_j}(xy), \quad i = 1, 2, \ldots, n,$$

then $xy \in B_m^\phi$, while m is chosen such that $\mu_{B_m^\phi}(xy) \geq \mu_{B_i^\phi}(xy)$ and $\nu_{B_m^\phi}(xy) \geq \nu_{B_i^\phi}(xy)$ for all i.

Then IFGS $(A, B_1^\phi, B_2^\phi, \ldots, B_n^\phi)$ denoted by $\breve{G}_s^{\phi c}$, is called the *ϕ-complement* of IFGS $\breve{G}_s$.

Theorem 3.1. *A ϕ-complement of an intuitionistic fuzzy graph structure is always a strong IFGS. Moreover, if $\phi(i) = r$ for $r, i \in \{1, 2, \ldots, n\}$, then all B_r-edges in IFGS $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ become B_i^ϕ-edges in $\breve{G}_s^{\phi c} = (A, B_1^\phi, B_2^\phi, \ldots, B_n^\phi)$.*

Proof. First part is obvious from the definition of ϕ-complement $\breve{G}_s^{\phi c}$ of IFGS $\breve{G}_s$, since for any B_i^ϕ-edge xy, $\mu_{B_i}^\phi(xy)$ and $\nu_{B_i}^\phi(xy)$ respectively have the maximum values of

$$[\mu_A(x) \wedge \mu_A(y)] - \bigvee_{j \neq i} \mu_{\phi B_j}(xy) \quad \text{and} \quad [\nu_A(x) \vee \nu_A(y)] - \bigvee_{j \neq i} \nu_{\phi B_j}(xy). \tag{3.1}$$

That is,

$$\mu_{B_i}^\phi(xy) = \mu_A(x) \wedge \mu_A(y), \quad \nu_{B_i}^\phi(xy) = \nu_A(x) \vee \nu_A(y), \tag{3.2}$$

for all edges xy in $\breve{G}_s^{\phi c}$, hence $\breve{G}_s^{\phi c}$ is always a strong IFGS.

Now suppose on contrary that $\phi(i) = r$ but xy is a B_s-edge in $\breve{G}_s$ with $s \neq r$, which implies that $\phi B_i \neq B_s$. Comparing expressions (3.1) and (3.2), we get

$$\bigvee_{j \neq i} \mu_{\phi B_j}(xy) = 0, \quad \bigvee_{j \neq i} \nu_{\phi B_j}(xy) = 0,$$

which is not possible because $B_s = \phi B_j$ for some $j \in \{1, 2, \ldots, i-1, i+1, \ldots, n\}$. So our supposition is wrong and xy must be a B_r-edge. Hence we can conclude that if $\phi(i) = r$, then all B_r-edges in IFGS $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ become B_i^ϕ-edges in $\breve{G}_s^{\phi c} = (A, B_1^\phi, B_2^\phi, \ldots, B_n^\phi)$ for $r, i \in \{1, 2, \ldots, n\}$. $\square$

Example 3.10. Consider IFGS $\breve{G}_s = (A, B_1, B_2)$ shown in Fig. 4 and let ϕ be a permutation on the set $\{B_1, B_2\}$ such that $\phi(B_1) = B_2$ and $\phi(B_2) = B_1$.

Now for $a_1a_2 \in B_1$,

$$\begin{aligned}
\mu_{B_1}^\phi(a_1a_2) &= \mu_A(a_1) \wedge \mu_A(a_2) - \bigvee_{j \neq 1}[\mu_{\phi B_j}(a_1a_2)] = 0.3 \wedge 0.5 - [\mu_{\phi B_2}(a_1a_2)] \\
&= 0.3 - \mu_{B_1}(a_1a_2) = 0.3 - 0.3 = 0, \\
\nu_{B_1}^\phi(a_1a_2) &= \nu_A(a_1) \vee \nu_A(a_2) - \bigvee_{j \neq 1}[\nu_{\phi B_j}(a_1a_2)] = 0.7 \vee 0.4 - [\nu_{\phi B_2}(a_1a_2)] \\
&= 0.7 - \nu_{B_1}(a_1a_2) = 0.7 - 0.7 = 0, \\
\mu_{B_2}^\phi(a_1a_2) &= \mu_A(a_1) \wedge \mu_A(a_2) - \bigvee_{j \neq 2}[\mu_{\phi B_j}(a_1a_2)] = 0.3 \wedge 0.5 - [\mu_{\phi B_1}(a_1a_2)]
\end{aligned}$$

$$
\begin{aligned}
&= 0.3 - \mu_{B_2}(a_1a_2) = 0.3 - 0 = 0.3,\\
\nu^{\phi}_{B_2}(a_1a_2) &= \nu_A(a_1) \vee \nu_A(a_2) - \bigvee_{j\neq 2}[\nu_{\phi B_j}(a_1a_2)] = 0.7 \vee 0.4 - [\nu_{\phi B_1}(a_1a_2)]\\
&= 0.7 - \nu_{B_2}(a_1a_2) = 0.7 - 0 = 0.7.
\end{aligned}
$$

Clearly, $\mu^{\phi}_{B_2}(a_1a_2) = 0.3 > 0 = \mu^{\phi}_{B_1}(a_1a_2)$ and $\nu^{\phi}_{B_2}(a_1a_2) = 0.7 > 0 = \nu^{\phi}_{B_1}(a_1a_2)$, so $a_1a_2 \in B^{\phi}_2$.

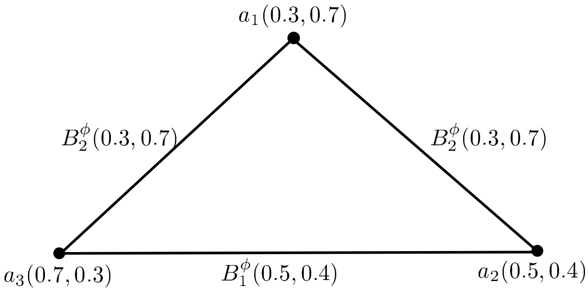


FIGURE 9. IFGS $\breve{G}^{\phi c}_s = (A, B^{\phi}_1, B^{\phi}_2)$

Similarly for $a_1a_3 \in B_1$,

$$\mu^{\phi}_{B_1}(a_1a_3) = 0, \quad \nu^{\phi}_{B_1}(a_1a_3) = 0.4, \quad \mu^{\phi}_{B_2}(a_1a_3) = 0.3, \quad \nu^{\phi}_{B_2}(a_1a_3) = 0.7.$$

Clearly, $\mu^{\phi}_{B_2}(a_1a_3) = 0.3 > 0 = \mu^{\phi}_{B_1}(a_1a_3)$ and $\nu^{\phi}_{B_2}(a_1a_3) = 0.7 > 0.4 = \nu^{\phi}_{B_1}(a_1a_3)$, so $a_1a_3 \in B^{\phi}_2$.

And for $a_2a_3 \in B_2$

$$\mu^{\phi}_{B_1}(a_2a_3) = 0.5, \quad \nu^{\phi}_{B_1}(a_2a_3) = 0.4, \quad \mu^{\phi}_{B_2}(a_2a_3) = 0, \quad \nu^{\phi}_{B_2}(a_2a_3) = 0.1,$$

that is, $\mu^{\phi}_{B_1}(a_2a_3) = 0.5 > 0 = \mu^{\phi}_{B_2}(a_2a_3)$ and $\nu^{\phi}_{B_1}(a_2a_3) = 0.4 > 0.1 = \nu^{\phi}_{B_2}(a_2a_3)$, so $a_2a_3 \in B^{\phi}_1$.

This implies that $B^{\phi}_1 = \{(a_2a_3, 0.5, 0.4)\}$, $B^{\phi}_2 = \{(a_1a_2, 0.3, 0.7), (a_1a_3, 0.3, 0.7)\}$ and $\breve{G}^{\phi c}_s = (A, B^{\phi}_1, B^{\phi}_2)$ shown in Fig. 9 is the ϕ-complement of $\breve{G}_s$.

Definition 3.19. Let $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ be an IFGS and ϕ be a permutation on the set $\{1, 2, \ldots, n\}$. Then

(i) $\breve{G}_s$ is *self-complementary*, if it is isomorphic to $\breve{G}^{\phi c}_s$, the ϕ-complement of $\breve{G}_s$.
(ii) $\breve{G}_s$ is *strong self-complementary*, if it is identical to $\breve{G}^{\phi c}_s$.
(iii) $\breve{G}_s$ is *totally self-complementary*, if it is isomorphic to $\breve{G}^{\phi c}_s$, the ϕ-complement of $\breve{G}_s$ for all permutations ϕ on the set $\{1, 2, \ldots, n\}$.
(iv) $\breve{G}_s$ is *totally strong self-complementary*, if it is identical to $\breve{G}^{\phi c}_s$, the ϕ-complement of $\breve{G}_s$ for all permutations ϕ on the set $\{1, 2, \ldots, n\}$.

Theorem 3.2. *An IFGS $\breve{G}_s$ is strong if and only if $\breve{G}_s$ is totally self-complementary.*

Proof. Let $\breve{G}_s$ be a strong IFGS and ϕ be any permutation on the set $\{1, 2, \ldots, n\}$.

If $\phi^{-1}(i) = j$, then by Theorem 3.1, all B_i-edges in $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ become B_j^ϕ-edges in $\breve{G}_s^{\phi c} = (A, B_1^\phi, B_2^\phi, \ldots, B_n^\phi)$. Also $\breve{G}_s^{\phi c}$ is strong, so

$$\begin{aligned}\mu_{B_i}(a_1a_2) =& \mu_A(a_1) \wedge \mu_A(a_2) = \mu_{B_j^\phi}(a_1a_2),\\ \nu_{B_i}(a_1a_2) =& \nu_A(a_1) \vee \nu_A(a_2) = \nu_{B_j^\phi}(a_1a_2).\end{aligned}$$

Then $\breve{G}_s$ is isomorphic to $\breve{G}_s^{\phi c}$, under the identity mapping $f : U \to U$ and a permutation ϕ $[\phi^{-1}(i) = j,\ i, j = 1, 2, \ldots, n]$, such that

$$\mu_A(a) = \mu_A(f(a)), \quad \nu_A(a) = \nu_A(f(a)), \quad \text{for all } a \in U$$

and

$$\begin{aligned}\mu_{B_i}(a_1a_2) =& \mu_{B_j^\phi}(a_1a_2) = \mu_{B_j^\phi}(f(a_1)f(a_2)),\\ \nu_{B_i}(a_1a_2) =& \nu_{B_j^\phi}(a_1a_2) = \nu_{B_j^\phi}(f(a_1)f(a_2)), \quad \text{for all } a_1a_2 \in E_i.\end{aligned}$$

This holds for all permutations on the set $\{1, 2, \ldots, n\}$. Hence $\breve{G}_s$ is totally self-complementary.

Conversely, let ϕ be any permutation on the set $\{1, 2, \ldots, n\}$ and $\breve{G}_s$ and $\breve{G}_s^{\phi c}$ be isomorphic. From the definition of ϕ-complement and isomorphism of IFGSs, we have

$$\begin{aligned}\mu_{B_i}(a_1a_2) =& \mu_{B_j^\phi}(f(a_1)f(a_2)) = \mu_A(f(a_1)) \wedge \mu_A(f(a_2)) = \mu_A(a_1) \wedge \mu_A(a_2),\\ \nu_{B_i}(a_1a_2) =& \nu_{B_j^\phi}(f(a_1)f(a_2)) = \nu_A(f(a_1)) \vee \mu_A(f(a_2)) = \nu_A(a_1) \vee \nu_A(a_2),\end{aligned}$$

for all $a_1a_2 \in E_i$, $i = 1, 2, \ldots, n$.

Hence, $\breve{G}_s$ is a strong IFGS. □

Remark 3.2. Every self-complementary IFGS is necessarily totally self-complementary.

Theorem 3.3. *If graph structure $G^* = (U, E_1, E_2, \ldots, E_n)$ is totally strong self-complementary and A is an IFS of U with constant fuzzy mappings μ_A and ν_A then a strong IFGS $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ of G^* is totally strong self-complementary.*

Proof. Consider a strong IFGS $\breve{G}_s = (A, B_1, B_2, \ldots, B_n)$ of a graph structure $G^* = (U, E_1, E_2, \ldots, E_n)$. Suppose that G^* is totally strong self-complementary and that for some constants $s, t \in [0, 1]$, $A = (\mu_A, \nu_A)$ is an IFS of U such that $\mu_A(u) = s$, $\nu_A(u) = t$, for all $u \in U$. Then we have to prove that $\breve{G}_s$ is totally strong self-complementary.

Let ϕ be an arbitrary permutation on the set $\{1, 2, \ldots, n\}$ and $\phi^{-1}(j) = i$. Since G^* is totally strong self-complementary, so there exists a bijection $f : U \to U$, such that for every E_i-edge a_1a_2 in G^*, $f(a_1)f(a_2)$ (an E_j-edge in G^*) is an E_i-edge in $(G^*)^{\phi^{-1}c}$. Consequently, for every B_i-edge a_1a_2 in $\breve{G}_s$, $f(a_1)f(a_2)$ (a B_j-edge in $\breve{G}_s$) is a B_i^ϕ-edge in $\breve{G}_s^{\phi c}$.

From the definition of A and the definition of strong IFGS $\breve{G}_s$

$$\mu_A(a) = s = \mu_A(f(a)), \quad \nu_A(a) = t = \nu_A(f(a)), \quad \text{for all } a, f(a) \in U,$$

and

$$\mu_{B_i}(a_1a_2) = \mu_A(a_1) \wedge \mu_A(a_2) = \mu_A(f(a_1)) \wedge \mu_A(f(a_2)) = \mu_{B_i^{\phi}}(f(a_1)f(a_2)),$$
$$\nu_{B_i}(a_1a_2) = \nu_A(a_1) \vee \nu_A(a_2) = \nu_A(f(a_1)) \vee \nu_A(f(a_2)) = \nu_{B_i^{\phi}}(f(a_1)f(a_2)),$$

for all $a_1a_2 \in B_i$, $i = 1, 2, \ldots, n$, which shows $\breve{G}_s$ is strong self-complementary. Hence $\breve{G}_s$ is totally strong self-complementary, since ϕ is arbitrary. □

Remark 3.3. Converse of Theorem 3.3 is not necessary, since a totally strong self-complementary and strong IFGS $\breve{G}_s = (A, B_1, B_2, B_3)$ as shown in Fig. 11, has a totally strong self-complementary underlying graph structure but μ_A and ν_A are not constant fuzzy functions.

Example 3.11. The IFGS shown in Fig. 8 is self-complementary, i.e., it is isomorphic to its ϕ-complement, where $\phi = (1\ 2)$. Also, it is totally self-complementary because ϕ is the only non-identity permutation on set $\{1, 2\}$.

Example 3.12. The IFGS $\breve{G}_s = (A_1, B_1, B_2, B_3, B_4)$ shown in Fig. 10, is strong self-complementary, i.e., it is identical to its ϕ-complement where the permutation ϕ is (1 2) (3 4). It is not totally strong self-complementary.

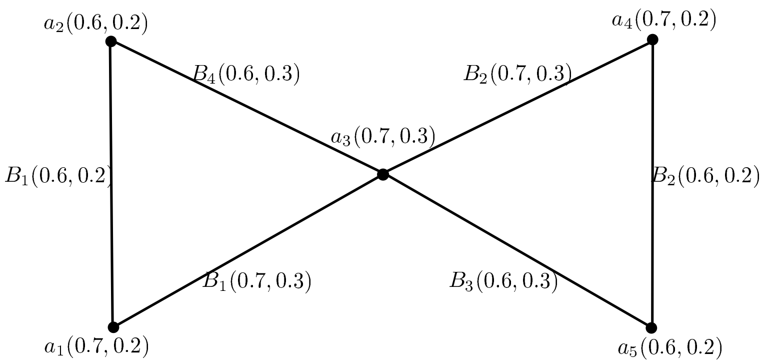


FIGURE 10. IFGS $\breve{G}_s = (A_1, B_1, B_2, B_3, B_4)$

Example 3.13. The IFGS $\breve{G}_s = (A_1, B_1, B_2, B_3)$, shown in Fig. 11, is totally strong self-complementary because it is identical to its ϕ-complement for all the permutations ϕ on the set $\{1, 2, 3\}$.

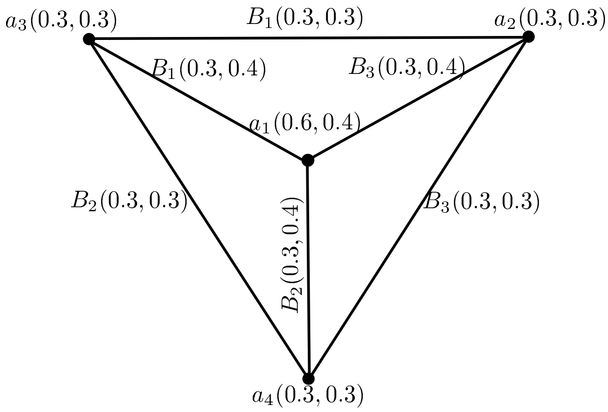


FIGURE 11. IFGS $\breve{G}_s = (A_1, B_1, B_2, B_3)$

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