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GIELIS TRANSFORMATION OF THE ARCHIMEDEAN SPIRAL

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Abstract. The article shows that the Archimedean spiral, usually described as a smooth spiral, can be transformed in many different shapes. The main part of the article concentrates on the curvature of the transformed spirals. It will also be shown that the shape some of them is an approximation of spiral antennas.

1. Introduction

Gielis transformations of curves were originally introduced in connection with the modelling of shapes of various biological objects, e.g., flowers, fruits, an arrangement of leaves, shapes of shells, and so on [1–4]. Gradually, studies have appeared pointing to the possibility of using transformed curves also in technical applications, e.g., [7–12].

This article aims to continue in theoretical studies in the area of the so-called Gielis' superformula and Gielis curves. In the early 19th century, a French mathematician Gabriel Lamé introduced a generalized equation of the ellipse

$$\left|\frac{x}{a}\right|^n + \left|\frac{y}{b}\right|^n = 1, \tag{1.1}$$

where $a, b, n \in \mathbb{Q}^+$. The equation (1.1) can generate different types of curves, such as asteroids ($n = 2/3$), parallelograms ($n = 1$), circles and ellipses ($n = 2$), squares and rectangles ($n \to \infty$). All these curves are called Lamé curves or superellipses (Figure 1), e.g., [1–4].

Key words and phrases. Gielis transformation, Archimedean spiral, Gielis curves, curvature, antennas.

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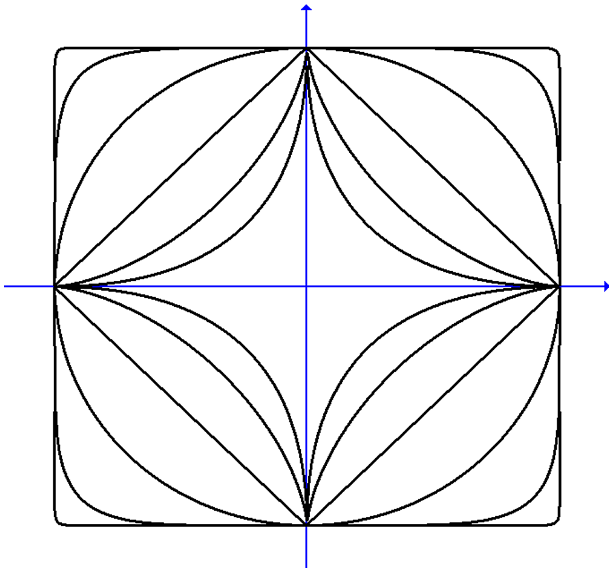

FIGURE 1. Lamé curves for $n = \frac{1}{2}, \frac{2}{3}, 1, 2, 6, 50$, whereas $a = b$ (curves for other cases are obtained by changing the scale on the axes)

The curve (1.1) can also be expressed in polar coordinates (ρ, θ)

$$\rho = \left(\left| \frac{\cos\theta}{a} \right|^n + \left| \frac{\sin\theta}{b} \right|^n \right)^{-\frac{1}{n}}. \tag{1.2}$$

In the late 20th century, Belgium botanist Johan Gielis generalized (1.2) to the form

$$\rho = \left(\left| \frac{1}{a} \cos \frac{m\theta}{4} \right|^{n_1} + \left| \frac{1}{b} \sin \frac{m\theta}{4} \right|^{n_2} \right)^{-\frac{1}{q}}, \tag{1.3}$$

where $a, b, m, n_1, n_2, q \in \mathbb{R}^+$. As can be seen from the equation (1.3), Gielis replaced the exponent n by three independent exponents n_1, n_2, q and inserted an extra parameter $\frac{m}{4}$ into the argument of both trigonometric functions. The Gielis transformation consists in replacing the plane curve expressed in polar coordinates (ρ, θ) with a curve

$$\rho = f(\theta) \left(\left| \frac{1}{a} \cos \frac{m\theta}{4} \right|^{n_1} + \left| \frac{1}{b} \sin \frac{m\theta}{4} \right|^{n_2} \right)^{-\frac{1}{q}}. \tag{1.4}$$

Gielis called the transformation (1.3) and (1.4) as a superformula. Without loss of generality, in (1.3), we focus on the case $a = b = 1$ and $n_1 = n_2 = p$ and put [13]

$$g_{m,p,q}(\theta) = \left(\left| \cos \frac{m\theta}{4} \right|^p + \left| \sin \frac{m\theta}{4} \right|^p \right)^{-\frac{1}{q}}. \tag{1.5}$$

The curve defined by the equation $\rho = g_{m,p,q}(\theta)$ can be interpreted as the Gielis transformation of a unit circle centered at the origin for various choices of the parameters m, p, q. Figure 2 shows that Gielis curves can provide far more complicated shapes than Lamé curves. There are plenty of examples of natural shapes similar to Gielis curves [2, 3, 15, 16].

In this article, the properties of the curves generated by the Gielis transformation of the Archimedean spirals will be investigated. There are two available approaches to what the Archimedean spirals are. The first one considers the general equation in polar coordinates (ρ, θ) of the form

$$\rho = a\theta^{1/n} + b, \tag{1.6}$$

where a, b and n are real constants. Several special cases can be described, depending on the value of n: the arithmetic spiral ($n = 1$), the hyperbolic spiral ($n = -1$), the Fermat spiral ($n = 2$), and lituus ($n = -2$) [4, 5, 14]. The second approach considers the terms the arithmetic spiral and the Archimedean spiral as synonyms (Archimedean spiral, Wikipedia, The Free Encyclopedia, Available from: `https://en.wikipedia.org/w/index.php?title=Archimedean_spiral&oldid=949421005`). In the next parts of this article, the second approach will be followed, and the equation (1.6) will be of the form

$$\rho = a\theta + b. \tag{1.7}$$

The equation (1.7) describes the trajectory of a point moving at a constant speed along a ray spinning around the origin at a constant angular velocity. Changing the parameter b moves the center of the spiral outward from the origin (for the option $b > 0$ toward $\theta = 0$ and for the option $b < 0$ toward $\theta = \pi$). The parameter a changes the distance between loops of the spiral.

Without loss of generality, in the equation (1.7), we focus on the case $b = 0$ and put

$$\rho = a\theta. \tag{1.8}$$

The Archimedean spirals have a variety of real-world applications. Scroll compressors, made from two members (one of them fixed and the other rotating), each of them in the shape of an Archimedean spiral, are used for compressing gases (H. Sakata, O. Masayuki, Fluid compressing device having coaxial spiral members, United States Patent 5603614. `http://www.freepatentsonline.com/5603614.html`). The Archimedean spirals have a constant distance between successive coils and they appear naturally in such systems as a roll of paper, the grooves of a gramophone record, and so on [4, 5]. In food microbiology, the Archimedean spirals are used to quantify bacterial concentration through a spiral platter [6].

There are also plenty of types of Archimedean spiral shaped antennas. Some of them are in the shape of the smooth Archimedean spiral [11] and the others, as it will be shown latter, are in the shape of transformed Archimedean spirals, e.g., [7–10].

In the article [13], Matsuura discusses the mathematical structure of the curves given by the equation $\rho = g_{m,p,q}(\theta)$. Matsuura also introduces the concept of Gielis regular polygons, which he further compares with regular polygons. The substantial part of the article deals with the curvature of Gielis curves. In the article [15], the

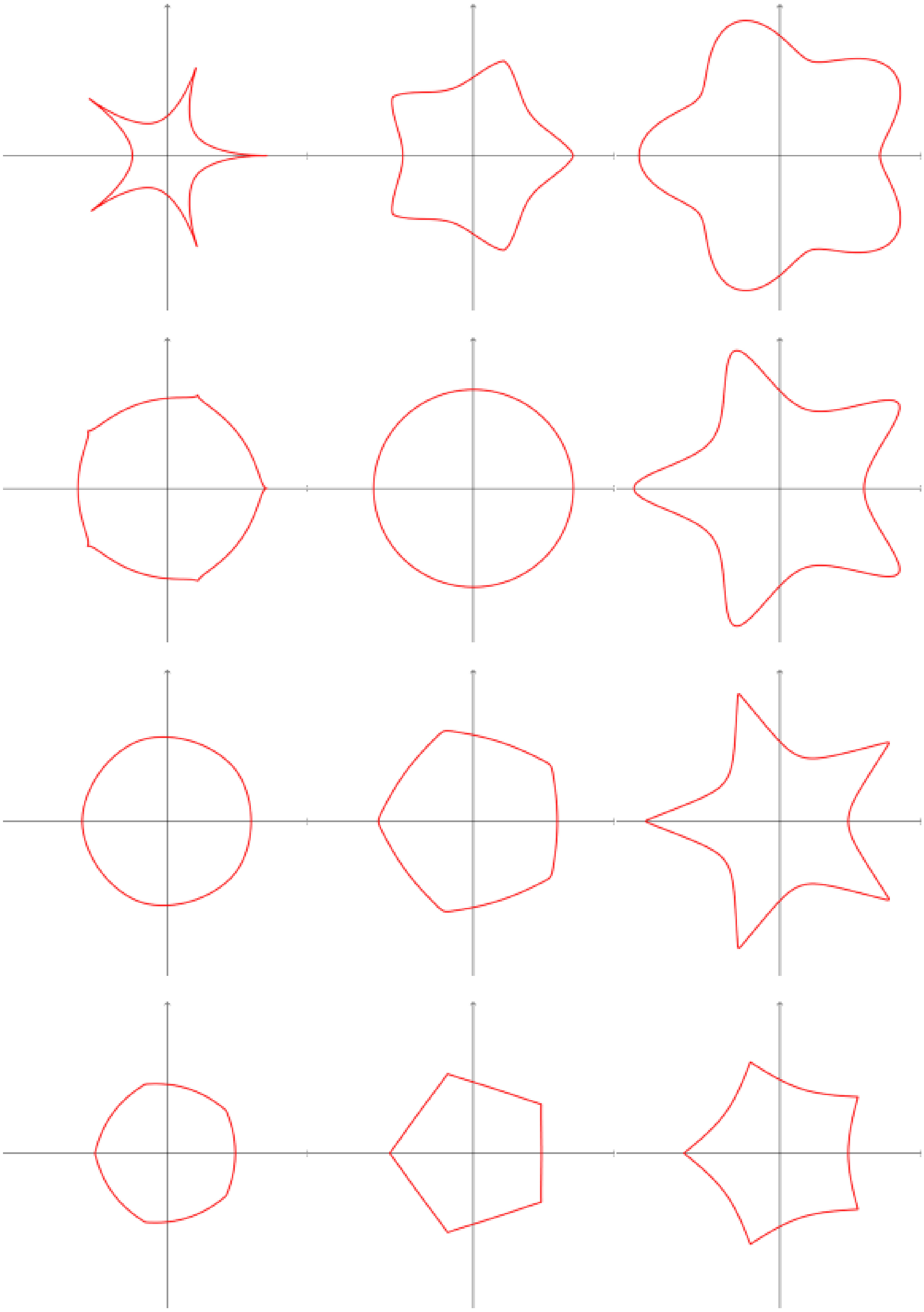

FIGURE 2. Gielis curves defined by the equation $\rho = g_{m,p,q}(\theta)$ $(m = 5)$: first row $q = 0{,}5$ $(p = 0{,}5,\ p = 1{,}5,\ p = 2{,}5)$; second row $q = 5$ $(p = 0{,}5,\ p = 2,\ p = 10)$; third row $q = 50$ $(p = 5,\ p = 20,\ p = 100)$; fourth row $q = 500$ $(p = 100,\ p = 300,\ p = 500)$

properties of the transformed logarithmic spirals were investigated and compared with similarly shaped objects.

The rest of this paper is organized as follows. Firstly, we summarize the known facts about the transformations of Gielis curves [13] and the logarithmic spirals [15] and compare them with the Gielis transformation of the Archimedean spiral. Subsequently, we investigate the curvature of the subspiral ($p < 2$) and superspiral ($p > 2$) at the anchor points and the vertices of the curves. We also discuss the influence of the value of the parameter m (integer or non-integer) on the shape of spirals. Finally, we point out objects and shapes, which could be modelled with transformed spirals.

2. Gielis Transformation of the Archimedean Spiral

Using equations (1.5) and (1.8) we obtain the equation

$$g_{a,m,p,q}(\theta) = a\theta\left(\left|\cos\frac{m\theta}{4}\right|^p + \left|\sin\frac{m\theta}{4}\right|^p\right)^{-\frac{1}{q}}, \tag{2.1}$$

which determines Gielis transformation of the Archimedean spiral. Throughout the rest of this paper we will be using the following notation and terms.

(i) We denote the planar curves obtain according to the equation (2.1) by the symbol $G_{a,m,p,q}$, i.e. $G_{a,m,p,q}(\theta) = g_{a,m,p,q}(\theta)(\cos\theta, \sin\theta)$, the Archimedean spiral by the symbol G_a, i.e., $G_a(\theta) = a\theta(\cos\theta, \sin\theta)$. Figures 3, 5 and 6 show some examples of transformations of the Archimedean spiral. In Figure 3 one can see that the coils of the spiral intersect only for rational values of m.

(ii) The pole of the spiral is the point which spiral approaches for $\theta \to -\infty$. In the case of the non-shifted spiral, this point lies at the origin of the Cartesian coordinate system.

(iii) The anchor point of $G_{a,m,p,q}$ means such a point of G_a, whose position does not change during the transformation, i.e., $G_{a,m,p,q}(\theta) = G_a(\theta)$.

(iv) The vertex of $G_{a,m,p,q}$ means the point of $G_{a,m,p,q}$ corresponding to the value of θ (Fig. 4), where $g_{m,p,q}$ has a local maximum (later we will show that for $p < 2$ the vertices are identical with anchor points).

(v) The coil of the spiral means the part of the curve where $\theta \in [2k\pi, 2(k+1)\pi)$ for given $k \in \mathbb{Z}$.

The following statements summarize some properties of transformed spirals, the proofs are routine.

Lemma 2.1. *The parameter m determines the number of anchor points in one spiral coil of $G_{a,m,p,q}$ as follows.*

(i) For $m \in N$, the spiral has exactly m anchor points in one coil.

(ii) For $m \notin N$, the number of anchor points in one coil corresponds to $\lceil m \rceil$, i.e., the next higher integer.

Lemma 2.2. *The function $g_{m,p,q}$ satisfies the following properties ($k \in \mathbb{Z}$).*

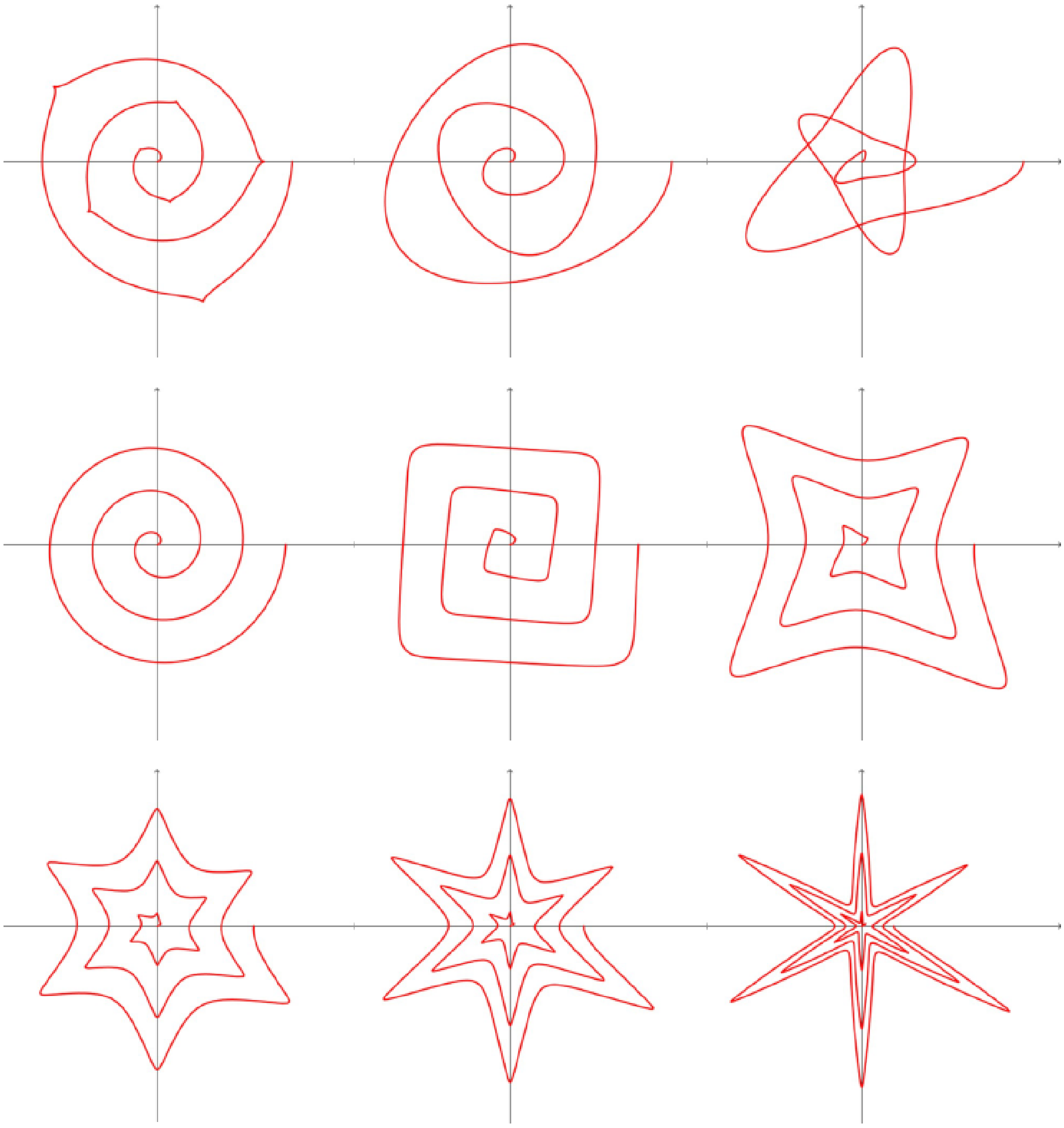

FIGURE 3. Gielis transformation of the Archimedean spiral ($\theta \in [0, 6\pi]$): first row $m = 2{,}5, q = 3$ ($p = 0{,}5$, $p = 4$, $p = 10$); second row $m = 4, q = 10$ ($p = 2$, $p = 10$, $p = 20$); third row $m = 6, q = 20$ ($p = 30$, $p = 50$, $p = 100$)

(*i*) *For* $p < 2$ *it is increasing on* $\left[\frac{(2k-1)\pi}{m}, \frac{2k\pi}{m}\right]$ *and decreasing on* $\left[\frac{2k\pi}{m}, \frac{(2k+1)\pi}{m}\right]$.
(*ii*) *For* $p = 2$ *it is constant on the whole real axis.*
(*iii*) *For* $p > 2$ *it is increasing on* $\left[\frac{2k\pi}{m}, \frac{(2k+1)\pi}{m}\right]$ *and decreasing on* $\left[\frac{(2k+1)\pi}{m}, \frac{2(k+1)\pi}{m}\right]$.
(*iv*) *For all* $\theta = \frac{2k\pi}{m}$ $(k \in \mathbb{Z})$ *it is* $g_{m,p,q}(\theta) = \theta$.

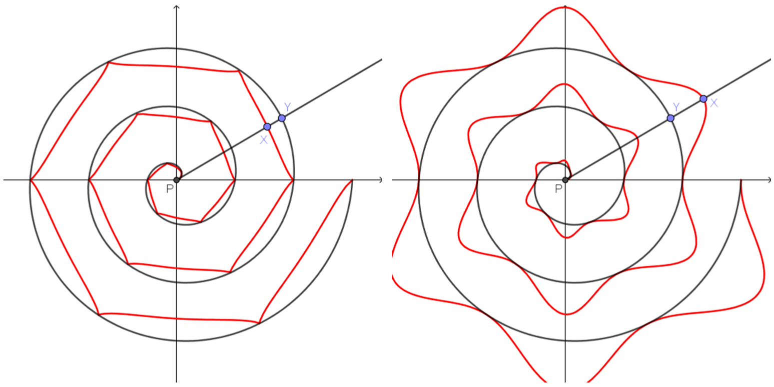

FIGURE 4. Gielis transformation of the Archimedean spiral ($\theta \geq 0$, left $p < 2$, right $p > 2$)

Corollary 2.1. (i) If $p = 2$, then $G_{a,m,p,q}$ is the Archimedean spiral G_a. Let the points X, Y lie on the same coils of $G_{a,m,p,q}$ and G_a, and at the same time on the same half-line starting from the pole P of the spiral. If

- $p < 2$, then $|PX| \leq |PY|$;
- $p > 2$, then $|PX| \geq |PY|$.

(ii) If $p < 2$, then the anchor points and vertices of $G_{a,m,p,q}$ correspond to the choice $\theta = \frac{2k\pi}{m}$ $(k \in \mathbb{Z})$. If $p > 2$, then the anchor points of $G_{a,m,p,q}$ correspond to the choice $\theta = \frac{2k\pi}{m}$ $(k \in \mathbb{Z})$, and the vertices to the choice $\theta = \frac{(2k+1)\pi}{m}$ $(k \in \mathbb{Z})$.

Theorem 2.2. *The function $g_{a,m,p,q}$ satisfies the following properties*

(i) $g_{a,m,p,q}(\theta + \frac{2\pi}{m}) = \frac{2\pi a}{m} g_{a,m,p,q}(\theta)$;
(ii) $\lim_{q\to\infty} g_{a,m,p,q}(\theta) = a\theta$.

Proof. The claims (i) and (ii) follow directly from the definition of the function $g_{a,m,p,q}$. $\square$

Remark 2.1. For $p < 2$ we call the curve $G_{a,m,p,q}$ a subspiral of the Archimedean spiral, for $p > 2$ is the curve $G_{a,m,p,q}$ a superspiral of the Archimedean spiral.

3. CURVATURE OF SUBSPIRAL AND SUPERSPIRAL

The aim of this section is to examine the curvature of subspiral and superspiral. The curvature can generally be characterized as an amount by which a curve deviates from being a straight line whose curvature is zero. If we consider, that spirals are given with (2.1), and we use the relation for the curvature of the curve given in polar coordinates, then we obtain

$$\kappa_{a,m,p,q}(\theta) = \frac{g_{a,m,p,q}(\theta)^2 + 2g'_{a,m,p,q}(\theta)^2 - g_{a,m,p,q}(\theta)g''_{a,m,p,q}(\theta)}{\left\{g_{a,m,p,q}(\theta)^2 + g'_{a,m,p,q}(\theta)^2\right\}^{\frac{3}{2}}}, \tag{3.1}$$

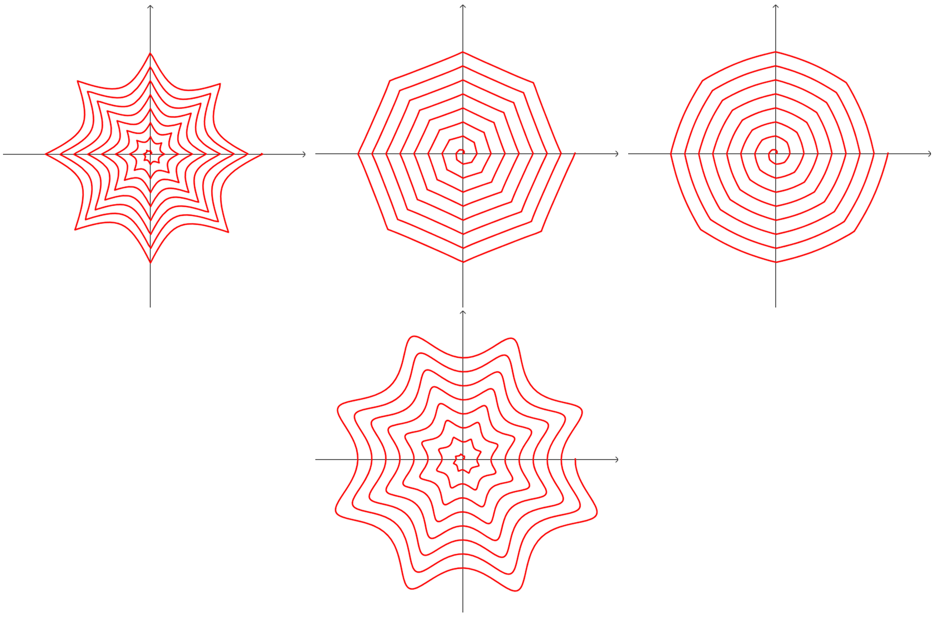

FIGURE 5. Curvature at the anchor points (vertices) of the subspiral (upper row: $p = 1$ and successively $q = 1$, $q = (m/4)^2 p$, $q = 8$), and at the vertices of the superspiral (lower row: $p = q = 8$), $m = 8$

where $\kappa_{a,m,p,q}(\theta)$ denotes the curvatute of $G_{a,m,p,q}$. For $p < 2$ the function $x \mapsto |x|^p$ does not have the second derivative in zero, therefore $g(\theta)$ does not have the second derivative at the points $\frac{2k\pi}{m}$ $(k \in \mathbb{Z})$ and the curvature is not defined there. If we substitute in (3.1) the formula $a\theta g_{m,p,q}(\theta)$ for $g_{a,m,p,q}(\theta)$, then after simplifying we obtain

$$\kappa_{a,m,p,q}(\theta) = \frac{1}{a} \cdot \frac{\theta^2 g(\theta)^2 + 2\Big(g(\theta) + \theta g'(\theta)\Big)^2 - \theta g(\theta)\Big(2g'(\theta) + \theta g''(\theta)\Big)}{\Big\{\theta^2 g(\theta)^2 + \Big(g(\theta) + \theta g'(\theta)\Big)^2\Big\}^{\frac{3}{2}}}, \tag{3.2}$$

where $g(\theta)$ is a shortcut for $g_{m,p,q}(\theta)$.

Since the second fraction in formula (3.2) represents $\frac{2\pi}{m}$-periodic function is sufficient to examine the curvature on the interval $[0, \frac{2\pi}{m})$. When investigating the curvature of transformed spirals we focus on the anchor points, i.e., we determine the curvature for $\theta = \frac{(2k+1)\pi}{m}$ $(k \in \mathbb{Z})$, and the vertices of the spirals, i.e., we determine the curvature for $\theta = \frac{2k\pi}{m}$ $(k \in \mathbb{Z})$. Because of the above, it will be sufficient to do the calculations in case of anchor points for $\theta = \frac{\pi}{m}$, and in the case of vertices for $\theta = 0$.

Theorem 3.1. *The curvature $\kappa_{a,m,p,q}(\frac{\pi}{m})$ satisfies the following properties.*

(i) $\kappa_{a,m,p,q}(\frac{\pi}{m}) = \frac{1}{a g_{m,p,q}(\frac{\pi}{m})} \cdot \frac{1}{\{1+(\frac{\pi}{m})^2\}^{\frac{3}{2}}} \Big\{2 + (\frac{\pi}{m})^2 + \frac{\pi^2 p(p-2)}{16q}\Big\}$

(ii) *Let $p < 2$. Then*

- $\kappa_{a,m,p,q}(\frac{\pi}{m}) < 0$ *if* $q < \frac{\pi^2 (m/4)^2 p(2-p)}{2m^2+\pi^2}$,

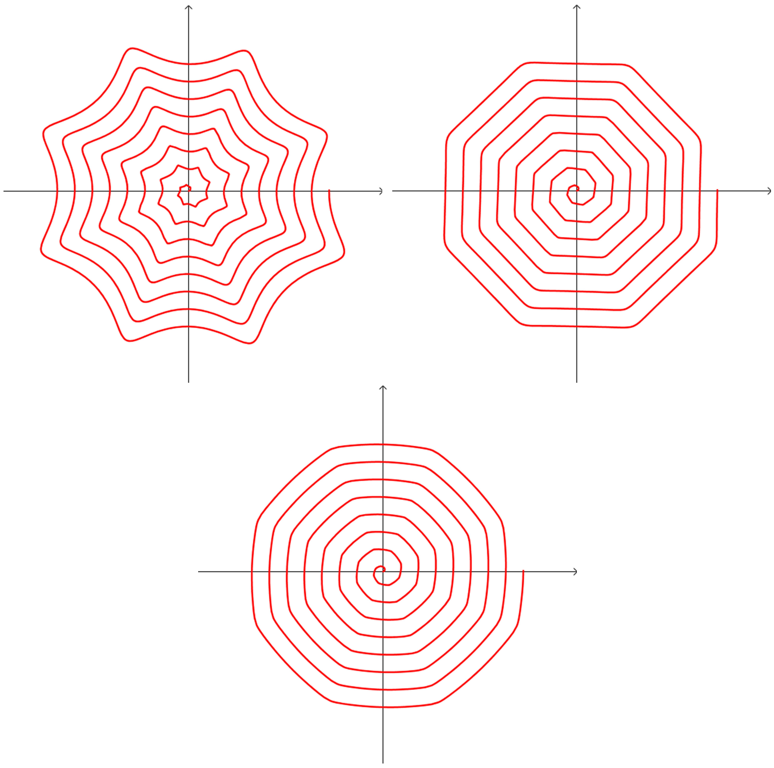

FIGURE 6. Curvature at the anchor points of the superspiral ($p = 10$ and successively $q = 15$, $q = (m/4)^2p$, $q = 75$), $m = 8$

- $\kappa_{a,m,p,q}(\frac{\pi}{m}) = 0$ *if* $q = \frac{\pi^2(m/4)^2p(2-p)}{2m^2+\pi^2}$,
- $\kappa_{a,m,p,q}(\frac{\pi}{m}) > 0$ *if* $q > \frac{\pi^2(m/4)^2p(2-p)}{2m^2+\pi^2}$.

(*iii*) *If* $p \geq 2$, *then* $\kappa_{a,m,p,q}(\frac{\pi}{m}) > 0$.

Proof. To prove (i), it is sufficient to substitute into formula (3.2)

$$g\left(\frac{\pi}{m}\right) = 2^{\frac{p-2}{2q}}, \quad g'\left(\frac{\pi}{m}\right) = 0, \quad g''\left(\frac{\pi}{m}\right) = \frac{(m/4)^2p(2-p)}{q}2^{\frac{p-2}{2q}}.$$

The claims (ii) and (iii) follow directly from (i). □

In the claim (ii) of the previous theorem is for the choice $p < 2$ mentioned the dependence of curvature on the value of the parameter q. Examples of the curves with the negative, zero and positive curvature at points that are "halfway" between the anchor points are shown in Figure 5.

Theorem 3.2. *Let* $p > 2$. *The curvature* $\kappa_{a,m,p,q}(0)$ *satisfies the following properties.*

(*i*) $\kappa_{a,m,p,q}(0) = \frac{1}{a} \cdot \frac{1}{\left\{1+(\frac{\pi}{m})^2\right\}^{\frac{3}{2}}} \left\{2 + (\frac{\pi}{m})^2 - \frac{\pi^2 p}{16q}\right\}$.

(*ii*) *If*

- $q < \frac{\pi^2(m/4)^2p}{2m^2+\pi^2}$, *then* $\kappa_{a,m,p,q}(0) < 0$,

- $q = \frac{\pi^2(m/4)^2 p}{2m^2+\pi^2}$, *then* $\kappa_{a,m,p,q}(0) = 0$,
- $q > \frac{\pi^2(m/4)^2 p}{2m^2+\pi^2}$, *then* $\kappa_{a,m,p,q}(0) > 0$.

Proof. To prove (i), it is sufficient to substitute into formula (3.2)

$$g(0) = 1, \quad g'(0) = 0, \quad g''(0) = \frac{(m/4)^2 p}{q}.$$

The claim (ii) follows directly from (i). □

Examples of the curves with the negative, zero and positive curvature at the anchor points of the superspirals are shown in Figure 6.

Remark 3.1. For $p = 0$ or $p = 2$ is

$$\kappa_{a,m,p,q}(\theta) = \frac{1}{g_a(\theta)} \cdot \frac{2+\theta^2}{(1+\theta^2)^{3/2}}$$

the curvature of the Archimedean spiral G_a.

4. Transformed Archimedean Spirals as Approximations of Spiral Antennas

The requirement for miniaturizing the antennas led to looking for specific transformed shapes. There are many types of planar spiral antennas whose design is based mainly on the use of the Archimedean or logarithmic geometry. The antennas operate in different configurations, e.g., the circular, the rectangular, the polygonal, sinuous meander or log-periodic. The mentioned configurations have their advantages and disadvantages but generally allow to reach frequency independent antennas.

Although there are different types of antennas with a different configuration, it can be shown that it is possible to approximate many of them in terms of Gielis transformation (Figure 7).

On the other hand, relative simplicity and flexibility of transformation might be used when looking for an advance or novel construction of the antennas.

5. Conclusion

In this paper, some properties of the Gielis transformation of the Archimedean spiral were analyzed. We focused in particular on the curvature in anchor points and the vertices of the transformed curves. In the end, we showed that the Gielis transformation might be handy when one looks for the appropriate shape of the Archimedean spiral-like antennas.

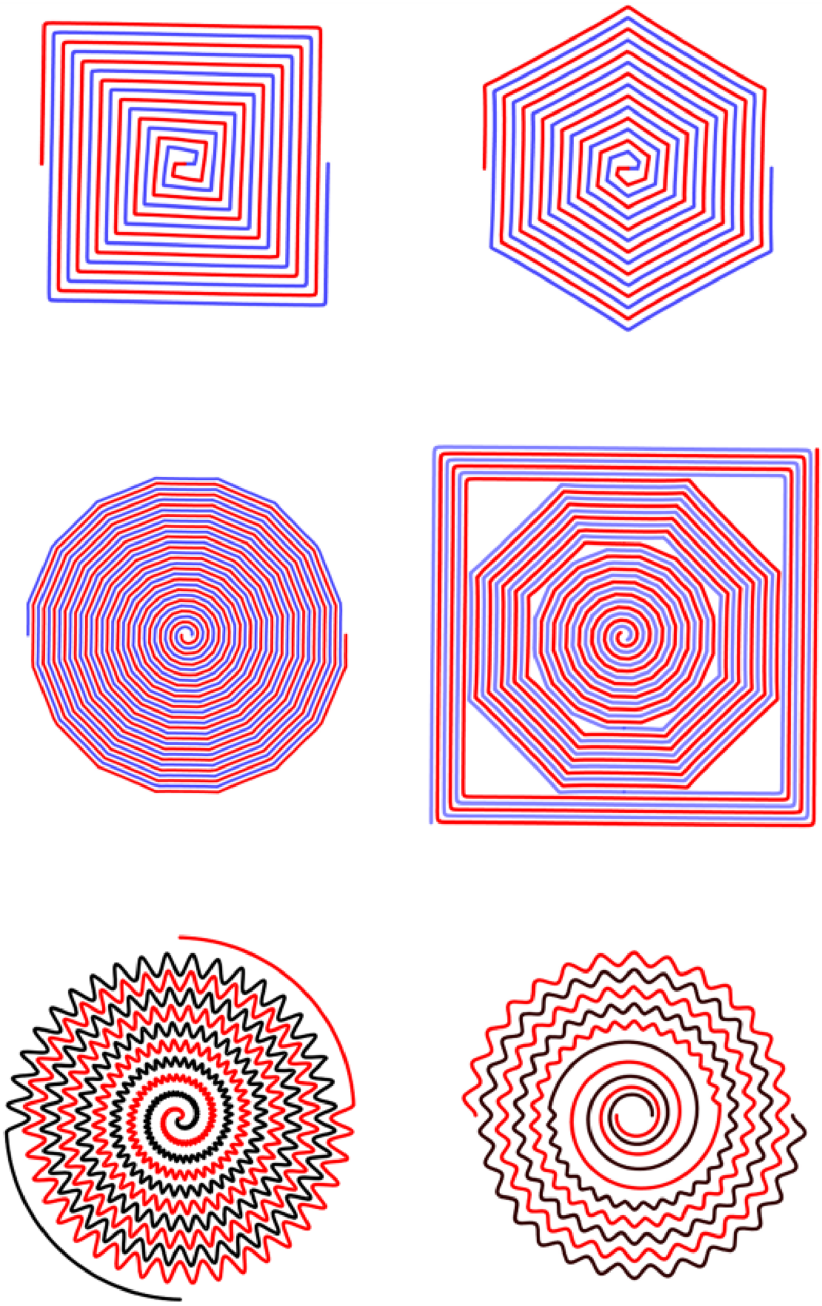

FIGURE 7. Models of spiral antennas with different configuration approximated via Gielis transformation of the Archimedean and logarithmic spiral

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