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SOLUTION TO $B = AXB$ AND $B = BXA$ UNDER CONSTRAINT $\operatorname{rank}(A) = \operatorname{rank}(B)$

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Abstract. The first goal of the current paper is to characterize consistency and solutions of the constrained linear matrix equation (CLME) $B = AXB$ under the rank constraint $\operatorname{rank}(A) = \operatorname{rank}(B)$, where $X \in \mathbb{C}^{n\times m}$, $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times k}$, and to present the general solution form of this CLME. The CLME $B = BXA$ under the constraint $\operatorname{rank}(A) = \operatorname{rank}(B)$, where $X \in \mathbb{C}^{n\times m}$, $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{k\times n}$, is considered in the dual case. System of the previously mentioned two constrained equations is studied too. The results obtained in this research show that solutions to each of considered systems are certain inner inverses of A. As special cases of these systems, we investigate new systems with the stronger condition $\operatorname{rank}(X) = \operatorname{rank}(B) = \operatorname{rank}(A)$ instead of $\operatorname{rank}(B) = \operatorname{rank}(A)$ and express solutions of new systems in terms of inner and outer inverses of A. Examples based on symbolic and exact computation are presented.

1. Introduction

Following the standard nomenclature from linear algebra, the set of $m \times n$ complex matrices is denoted by $\mathbb{C}^{m\times n}$, while the rank, conjugate-transpose, null space (kernel) and image (range, column space) of $A \in \mathbb{C}^{m\times n}$ are marked as $\operatorname{rank}(A)$, A^*, $\operatorname{Ker}(A)$ and $\operatorname{Im}(A)$, respectively.

The Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$ is unique $X \in \mathbb{C}^{n\times m}$ (denoted by $A^\dagger$) [1] which is satisfying $XAX = X$, $AXA = A$, $(XA)^* = XA$, $(AX)^* = AX$. In the case if only $AXA = A$ (resp. $XAX = X$) holds, X is an inner (resp. outer) inverse of A, which is designated as $A^{(1)}$ (resp. $A^{(2)}$). The set of all inner inverses corresponding to A is standardly marked as $A\{1\}$.

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Matrix and operator equations have been applied frequently in linear system theory, information theory, control theory, sampling and other scientific disciplines. The main application of inner inverses is in solving systems of linear matrix or vector equations, where they are used analogously as the usual matrix inverse in the nonsingular environment [1, 5].

The equations $BAX = B = XAB$ for linear bounded operators amongst Hilbert spaces were studied in [2]. Separately, solvability of matrix equations $XAB = B$ or $BAX = B$ with the same constraint $\mathrm{rank}(X) = \mathrm{rank}(B)$ is investigated in [6, 9]. Exchanging positions of A and X in the above mentioned equations, characterizations as well as the existence of solutions to the matrix equations $AXB = B = BXA$ were presented in [3, 11, 12].

Motivated by the importance of solving matrix equations and previously mentioned results, we continue to study this topic. Investigating characterizations of solutions of equation $B = AXB$ (or $B = BXA$) with restriction $\mathrm{rank}(B) = \mathrm{rank}(A)$, it is interesting that we can use only inner inverses of A. If we consider a constraint $\mathrm{rank}(X) = \mathrm{rank}(B) = \mathrm{rank}(A)$, the solutions are not only inner inverse and they are also outer inverses of A.

In details, our research streams are described in the following highlights, assuming that $\wedge$ denotes conjunction.

(1) If $X \in \mathbb{C}^{n\times m}$, $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times k}$, we examine properties and the general form of solutions to the constrained linear matrix equation (CLME)

$$B = AXB \wedge \mathrm{rank}(B) = \mathrm{rank}(A), \tag{1.1}$$

with respect to X. It is verified that arbitrary solution of (1.1) is an inner inverse of A and the set of all solutions to (1.1) is characterized.

(2) Some particular case of the CLME (1.1) are studied too. Under the assumption $\mathrm{rank}(X) = \mathrm{rank}(B) = \mathrm{rank}(A)$ in (1.1), the solution of this new system is both outer and inner inverse of A.

(3) When $X \in \mathbb{C}^{n\times m}$ is unknown and $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{k\times n}$ are given, we investigate solvability of the CLME

$$B = BXA \wedge \mathrm{rank}(B) = \mathrm{rank}(A). \tag{1.2}$$

(4) The general solution to conjunction of (1.1) and (1.2) is also presented. Precisely, for $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times n}$, we investigate solutions to the CLMEs

$$B = AXB = BXA \wedge \mathrm{rank}(B) = \mathrm{rank}(A). \tag{1.3}$$

The results are organized in the next sections. Solutions of the constrained system (1.1) are characterized in Section 2. The system (1.2) and its particular case are studied in Section 3. We consider solutions of system (1.3) in Section 4. The last part presents final overview and remarks.

2. Solution to (1.1)

Main results of Theorem 2.1 is firstly to present equivalent conditions to (1.1) and then to give the general form for solutions to (1.1). Third part of Theorem 2.1 shows that solutions to the system (1.1) are a subset of inner inverses of A.

Theorem 2.1. *The subsequent statements are true for given matrices* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ *and unknown matrix* $X \in \mathbb{C}^{n\times m}$.

(a) *The subsequent are equivalent statements:*

(i) $B = AXB \wedge \operatorname{rank}(B) = \operatorname{rank}(A)$, *i.e.* X *is a solution to* (1.1);
(ii) $B = AXB \wedge \operatorname{Im}(B) = \operatorname{Im}(A)$;
(iii) $A = AXA \wedge \operatorname{Im}(B) = \operatorname{Im}(A)$.

(b) *If the conditions in* **(a)** *are satisfied, the general solution* X *to* (1.1) *is equal to*

$$X = A^\dagger + \Upsilon - A^\dagger A\Upsilon BB^\dagger = A^\dagger + \Upsilon - A^\dagger A\Upsilon AA^\dagger, \quad \Upsilon \in \mathbb{C}^{n\times m}. \tag{2.1}$$

(c) *If the conditions in* **(a)** *are satisfied, then the set of solutions to* (1.1) *coincides with the set of inner inverses:*

$$\{X \mid B = AXB \wedge \operatorname{rank}(B) = \operatorname{rank}(A)\} = A\{1\}. \tag{2.2}$$

(d) *If the conditions in* **(a)** *are satisfied, the general solution* X *to* (1.1) *is equal to*

$$X = \mho + \Upsilon - A^\dagger A\Upsilon AA^\dagger, \quad \Upsilon \in \mathbb{C}^{n\times m}, \tag{2.3}$$

where $\mho$ *is any particular solution to* $AXA = A$ *and* $\Upsilon = X - \mho$.

Proof. **(a)** (i) $\Rightarrow$ (ii): The supposition $B = AXB$ implies $\operatorname{Im}(B) \subseteq \operatorname{Im}(A)$. Now, $\operatorname{rank}(B) = \operatorname{rank}(A)$ implies $\operatorname{Im}(B) = \operatorname{Im}(A)$.

(ii) $\Rightarrow$ (iii): The assumption $\operatorname{Im}(B) = \operatorname{Im}(A)$ leads to $A = BZ$, for some $Z \in \mathbb{C}^{k\times n}$. In addition, applying $B = AXB$, it can be obtained

$$AX\underbrace{A}_{=BZ} = (\underbrace{AXB}_{=B})Z = BZ = A.$$

(iii) $\Rightarrow$ (i): Since $\operatorname{Im}(B) = \operatorname{Im}(A)$, one can conclude $\operatorname{rank}(B) = \operatorname{rank}(A)$ and $B = AZ_1$, for some $Z_1 \in \mathbb{C}^{n\times k}$. The condition $AXA = A$ gives

$$B = AZ_1 = (AXA)Z_1 = AX(AZ_1) = AXB.$$

(b) Furthermore, $\operatorname{Im}(BB^\dagger) = \operatorname{Im}(B) = \operatorname{Im}(A) = \operatorname{Im}(AA^\dagger)$ implies $BB^\dagger = P_{\operatorname{Im}(B)} = P_{\operatorname{Im}(A)} = AA^\dagger$. Hence,

$$\underbrace{B}_{=BB^\dagger B} = (\underbrace{BB^\dagger}_{=AA^\dagger})B = AA^\dagger \underbrace{B}_{=BB^\dagger B} = AA^\dagger BB^\dagger B. \tag{2.4}$$

According to [1, p. 52], [7] and (2.4), the LME $B = AXB$ is consistent, and further, for arbitrary $\Upsilon \in \mathbb{C}^{n\times m}$, its general solution is

$$
\begin{aligned}
X &= A^\dagger \underbrace{BB^\dagger}_{=AA^\dagger} + \Upsilon - A^\dagger A\Upsilon \underbrace{BB^\dagger}_{=AA^\dagger} \\
&= \underbrace{A^\dagger AA^\dagger}_{=A^\dagger} + \Upsilon - A^\dagger A\Upsilon AA^\dagger \qquad (2.5)\\
&= A^\dagger + \Upsilon - A^\dagger A\Upsilon AA^\dagger \in A\{1\}.
\end{aligned}
$$

(c) The results obtained in part **(b)** and [1, p. 52, Corollary 1], [7] imply

$$
\begin{aligned}
\{X |\ B = AXB \wedge \operatorname{rank}(B) = \operatorname{rank}(A)\} &= \left\{A^\dagger + \Upsilon - A^\dagger A\Upsilon AA^\dagger |\ \Upsilon \in \mathbb{C}^{n\times m}\right\} \\
&= A\{1\},
\end{aligned}
$$

which confirms this part of the proof.

(d) This statement follows from the part **(b)** and [8, Theorem 2.9]. □

Several additional characterizations of the CLME (1.1) are given in Corollary 2.1.

Corollary 2.1. *For given $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and unknown $X \in \mathbb{C}^{n\times m}$, the subsequent statements are equivalent:*

(i) $B = AXB \wedge \operatorname{rank}(B) = \operatorname{rank}(A)$, *i.e. X is a solution to* (1.1)*;*
(ii) $A^\dagger = A^\dagger AXAA^\dagger \wedge \operatorname{Im}(B) = \operatorname{Im}(A)$*;*
(iii) $A^\dagger = A^\dagger AXBB^\dagger \wedge \operatorname{Im}(B) = \operatorname{Im}(A)$*.*

Proof. (i) ⇒ (ii): By Theorem 2.1, $A = AXA$ and $\operatorname{Im}(B) = \operatorname{Im}(A)$. So, $A^\dagger AXAA^\dagger = A^\dagger AA^\dagger = A^\dagger$.

(ii) ⇒ (i): Multiplying $A^\dagger = A^\dagger AXAA^\dagger$ by A from both left and right side, we get the identity $A = AXA$.

(ii) ⇔ (iii): Because $\operatorname{Im}(B) = \operatorname{Im}(A)$ gives $BB^\dagger = AA^\dagger$, this equivalence is evident. □

Recall that, by [1, Theorem 2], if $A = AXA$, then $X \in A\{1, 2\}$ if and only if $\operatorname{rank}(X) = \operatorname{rank}(A)$. According to results obtained in [4, 10], the minimal rank of X is equal to

$$\min_{B=AXB} \operatorname{rank}(X) = \operatorname{rank}(B).$$

Our goal is to investigate behaviour of (1.1) in this extreme case satisfying $\operatorname{rank}(X) = \operatorname{rank}(B) = \operatorname{rank}(A)$ in (1.1). According to Corollary 2.2, the solution of the considered system is both inner and outer inverse of A.

Corollary 2.2. *For $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times k}$, the following statements (i) and (ii) are equivalent:*

(i) $B = AXB \wedge \operatorname{rank}(X) = \operatorname{rank}(B) = \operatorname{rank}(A)$*;*
(ii) $A = AXA \wedge X = XAX \wedge \operatorname{Im}(B) = \operatorname{Im}(A)$*.*

Proof. (i) $\Rightarrow$ (ii): Theorem 2.1 implies $A = AXA$ and $\mathrm{Im}(B) = \mathrm{Im}(A)$. Applying [1, Theorem 2] and the equalities $A = AXA$ and $\mathrm{rank}(X) = \mathrm{rank}(A)$, we deduce $X = XAX$.

(ii) $\Rightarrow$ (i): By Theorem 2.1, we conclude $B = AXB$ and $\mathrm{rank}(B) = \mathrm{rank}(A)$. Based on [1, Theorem 2] and the assumptions $A = AXA$ and $X = XAX$, it can be concluded $\mathrm{rank}(X) = \mathrm{rank}(A)$. □

Example 2.1. Choose

$$A = \begin{pmatrix} 6 & 2 & 7 \\ 4 & 2 & 3 \\ 12 & 4 & 14 \\ 10 & 2 & 15 \end{pmatrix}.$$

For arbitrary generic 3×2 matrix

$$Z = \begin{pmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \\ z_{3,1} & z_{3,2} \end{pmatrix}, \quad z_{ij} \in \mathbb{R},$$

the matrix

$$B = AZ = \begin{pmatrix} 6z_{1,1} + 2z_{2,1} + 7z_{3,1} & 6z_{1,2} + 2z_{2,2} + 7z_{3,2} \\ 4z_{1,1} + 2z_{2,1} + 3z_{3,1} & 4z_{1,2} + 2z_{2,2} + 3z_{3,2} \\ 12z_{1,1} + 4z_{2,1} + 14z_{3,1} & 12z_{1,2} + 4z_{2,2} + 14z_{3,2} \\ 10z_{1,1} + 2z_{2,1} + 15z_{3,1} & 10z_{1,2} + 2z_{2,2} + 15z_{3,2} \end{pmatrix}$$

satisfies $\mathrm{Im}(B) \subseteq \mathrm{Im}(A)$, which implies consistency of $B = AXB$. Also, $\mathrm{rank}(B) = \mathrm{rank}(A) = 2$ is satisfied. In this case, the solution of $B = AXB$ is equal to

$$\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & \frac{1}{32}(-20x_{1,1} - 30x_{1,2} - 40x_{1,3} - 16x_{2,1} - 24x_{2,2} + 15) & \frac{1}{16}(10x_{1,2} - 20x_{1,4} + 8x_{2,2} - 7) \\ x_{3,1} & x_{3,2} & \frac{1}{16}(-4x_{1,1} - 6x_{1,2} - 8x_{1,3} - 8x_{3,1} - 12x_{3,2} - 1) & \frac{1}{8}(2x_{1,2} - 4x_{1,4} + 4x_{3,2} + 1) \end{pmatrix}.$$

In addition, symbolic calculations confirms that the matrix X coincides with the generic solution of the LME $AXA = A$ in symbolic form. It means that arbitrary matrix X satisfying (i) of Theorem 2.1 also satisfies (iii) of Theorem 2.1.

Note that it is valid $\mathrm{rank}(X) = 3$, so that the matrix equation $XAX = X$ is not valid.

As a confirmation of Corollary 2.1, it can be verified by symbolic calculus

$$A^\dagger = A^\dagger AXAA^\dagger = \begin{pmatrix} \frac{131}{3060} & \frac{92}{765} & \frac{131}{1530} & -\frac{343}{3060} \\ \frac{7}{153} & \frac{22}{153} & \frac{14}{153} & -\frac{23}{153} \\ -\frac{22}{765} & -\frac{91}{765} & -\frac{44}{765} & \frac{116}{765} \end{pmatrix}$$

and

$$AA^\dagger = BB^\dagger = \begin{pmatrix} \frac{5}{34} & \frac{3}{17} & \frac{5}{17} & \frac{3}{34} \\ \frac{3}{17} & \frac{7}{17} & \frac{6}{17} & -\frac{5}{17} \\ \frac{5}{17} & \frac{6}{17} & \frac{10}{17} & \frac{3}{17} \\ \frac{3}{34} & -\frac{5}{17} & \frac{3}{17} & \frac{29}{34} \end{pmatrix}.$$

Example 2.2. Let

$$A = \begin{pmatrix} 14 & 20 \\ 16 & 24 \\ 17 & 26 \\ 14 & 20 \end{pmatrix}.$$

For arbitrary generic 2×2 matrix

$$Z = \begin{pmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{pmatrix}, \quad z_{ij} \in \mathbb{R},$$

the matrix of the form

$$B = AZ = \begin{pmatrix} 14z_{1,1} + 20z_{2,1} & 14z_{1,2} + 20z_{2,2} \\ 16z_{1,1} + 24z_{2,1} & 16z_{1,2} + 24z_{2,2} \\ 17z_{1,1} + 26z_{2,1} & 17z_{1,2} + 26z_{2,2} \\ 14z_{1,1} + 20z_{2,1} & 14z_{1,2} + 20z_{2,2} \end{pmatrix}$$

satisfies $\mathrm{Im}(B) \subseteq \mathrm{Im}(A)$, which implies consistency of $B = AXB$. In addition, $\mathrm{rank}(B) = \mathrm{rank}(A) = 2$ is satisfied. In this case, the general solution of $B = BXA$ is

$$X = \begin{pmatrix} x_{1,1} & x_{1,2} & \frac{1}{6}(-4x_{1,2} - 5) & -x_{1,1} - \frac{x_{1,2}}{3} + \frac{13}{12} \\ x_{2,1} & x_{2,2} & \frac{1}{12}(7 - 8x_{2,2}) & -x_{2,1} - \frac{x_{2,2}}{3} - \frac{17}{24} \end{pmatrix}.$$

Symbolic calculations reveals that the matrix X coincides with the general solution of $AXA = A$. It means that arbitrary matrix X satisfying (i) of Theorem 2.1 also satisfies (iii) of Theorem 2.1.

Since $\mathrm{rank}(X) = 2$, the equation $XAX = X$ is satisfied.

3. Solution to (1.2)

The system (1.2) is considered in this section. Conditions for consistency are investigated as well as the pattern of the general solution. An illustrative example is presented.

Theorem 3.1. *The next assertions hold for given* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{k\times n}$ *and unknown* $X \in \mathbb{C}^{n\times m}$.

(a) *The subsequent are equivalent assertions:*

(i) $B = BXA \wedge \mathrm{rank}(B) = \mathrm{rank}(A)$, *i.e.,* X *is a solution to* (1.2)*;*

(ii) $B = BXA \wedge \mathrm{Ker}(B) = \mathrm{Ker}(A)$*;*

(iii) $A = AXA \wedge \mathrm{Ker}(B) = \mathrm{Ker}(A)$.

(b) *If the conditions in* **(a)** *are satisfied, the general solution to* (1.2) *is given by*

$$X = A^\dagger + \Upsilon - B^\dagger B\Upsilon AA^\dagger = A^\dagger + \Upsilon - A^\dagger A\Upsilon AA^\dagger, \quad \Upsilon \in \mathbb{C}^{n\times m}. \tag{3.1}$$

(c) *If the conditions in* **(a)** *are satisfied, then*

$$\{X |\ B = BXA \wedge \mathrm{rank}(B) = \mathrm{rank}(A)\} = A\{1\}. \tag{3.2}$$

(d) *If the conditions in* **(a)** *are satisfied, the general solution X to* (1.2) *is given by* (2.3).

Proof. **(a)** (i) $\Rightarrow$ (ii): We observe, by $B = BXA$, that $\mathrm{Ker}(A) \subseteq \mathrm{Ker}(B)$. The assumption $\mathrm{rank}(B) = \mathrm{rank}(A)$ gives $\mathrm{Ker}(B) = \mathrm{Ker}(A)$.

(ii) $\Rightarrow$ (iii): Because $\mathrm{Ker}(B) = \mathrm{Ker}(A)$, it follows $A = WB$, for some $W \in \mathbb{C}^{m\times k}$. Utilizing $B = BXA$, we get $AXA = W(BXA) = WB = A$.

(iii) $\Rightarrow$ (i): Notice that $\mathrm{Ker}(B) = \mathrm{Ker}(A)$ implies $\mathrm{rank}(B) = \mathrm{rank}(A)$ and $B = W_1A$, for some $W_1 \in \mathbb{C}^{k\times m}$. Now, $AXA = A$ yields $B = W_1A = (W_1A)XA = BXA$.

(b) Since $\mathrm{Ker}(B^\dagger B) = \mathrm{Ker}(B) = \mathrm{Ker}(A) = \mathrm{Ker}(A^\dagger A)$ gives $B^\dagger B = P_{\mathrm{Ker}(B)} = P_{\mathrm{Ker}(A)} = A^\dagger A$, we obtain $B = B(B^\dagger B) = BA^\dagger A = BB^\dagger BA^\dagger A$. By [1], the equation $B = BXA$ is consistent and, for arbitrary $\Upsilon \in \mathbb{C}^{n\times m}$,

$$\begin{aligned} X &= \underbrace{B^\dagger B}_{=A^\dagger A} A^\dagger + \Upsilon - \underbrace{B^\dagger B}_{=A^\dagger A} \Upsilon A A^\dagger \\ &= \underbrace{A^\dagger A A^\dagger}_{=A^\dagger} + \Upsilon - A^\dagger A \Upsilon A A^\dagger \\ &= A^\dagger + \Upsilon - A^\dagger A \Upsilon A A^\dagger \in A\{1\}. \end{aligned}$$

(c), (d) The proof of these statements is the same as in Theorem 3.1. □

Similarly as in Corollary 2.1, the next corollary is derived using Theorem 3.1.

Corollary 3.1. *For selected $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{k\times n}$ and unknown $X \in \mathbb{C}^{n\times m}$, the subsequent assertions are equivalent:*

(i) $B = BXA$ *and* $\mathrm{rank}(B) = \mathrm{rank}(A)$, *i.e., X represents a solution of* (1.2);
(ii) $A^\dagger = A^\dagger AXAA^\dagger \wedge \mathrm{Ker}(B) = \mathrm{Ker}(A)$;
(iii) $A^\dagger = B^\dagger BXAA^\dagger \wedge \mathrm{Ker}(B) = \mathrm{Ker}(A)$.

Replacing the condition $\mathrm{rank}(B) = \mathrm{rank}(A)$ in (1.2) with $\mathrm{rank}(X) = \mathrm{rank}(B) = \mathrm{rank}(A)$, we get the following result in Corollary 3.2.

Corollary 3.2. *For fixed $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{k\times n}$ and unknown $X \in \mathbb{C}^{n\times m}$, the below listed assertions are equivalent:*

(i) $B = BXA \wedge \mathrm{rank}(X) = \mathrm{rank}(B) = \mathrm{rank}(A)$;
(ii) $A = AXA \wedge X = XAX \wedge \mathrm{Ker}(B) = \mathrm{Ker}(A)$.

Example 3.1. Let

$$A = \begin{pmatrix} 8 & 16 & 24 & 4 \\ 10 & 20 & 30 & 5 \\ 8 & 16 & 24 & 4 \\ 2 & 4 & 6 & 1 \\ 4 & 10 & 10 & 3 \end{pmatrix}.$$

For arbitrary generic 3×5 matrix

$$Z = \begin{pmatrix} z_{1,1} & z_{1,2} & z_{1,3} & z_{1,4} & z_{1,5} \\ z_{2,1} & z_{2,2} & z_{2,3} & z_{2,4} & z_{2,5} \\ z_{3,1} & z_{3,2} & z_{3,3} & z_{3,4} & z_{3,5} \end{pmatrix}, \quad z_{ij} \in \mathbb{R},$$

the matrix $B = ZA$ is equal to

$$B = \left(\begin{matrix} 2(4z_{1,1} + 5z_{1,2} + 4z_{1,3} + z_{1,4} + 2z_{1,5}) & 2(8z_{1,1} + 10z_{1,2} + 8z_{1,3} + 2z_{1,4} + 5z_{1,5}) \\ 2(4z_{2,1} + 5z_{2,2} + 4z_{2,3} + z_{2,4} + 2z_{2,5}) & 2(8z_{2,1} + 10z_{2,2} + 8z_{2,3} + 2z_{2,4} + 5z_{2,5}) \\ 2(4z_{3,1} + 5z_{3,2} + 4z_{3,3} + z_{3,4} + 2z_{3,5}) & 2(8z_{3,1} + 10z_{3,2} + 8z_{3,3} + 2z_{3,4} + 5z_{3,5}) \end{matrix}\right.$$

$$\left.\begin{matrix} 24z_{1,1} + 30z_{1,2} + 24z_{1,3} + 6z_{1,4} + 10z_{1,5} & 4z_{1,1} + 5z_{1,2} + 4z_{1,3} + z_{1,4} + 3z_{1,5} \\ 24z_{2,1} + 30z_{2,2} + 24z_{2,3} + 6z_{2,4} + 10z_{2,5} & 4z_{2,1} + 5z_{2,2} + 4z_{2,3} + z_{2,4} + 3z_{2,5} \\ 24z_{3,1} + 30z_{3,2} + 24z_{3,3} + 6z_{3,4} + 10z_{3,5} & 4z_{3,1} + 5z_{3,2} + 4z_{3,3} + z_{3,4} + 3z_{3,5} \end{matrix}\right)$$

and satisfies $\mathrm{Ker}(A) \subseteq \mathrm{Ker}(B)$. Consequently, the equation $B = BXA$ is solvable. Also, $\mathrm{rank}(B) = \mathrm{rank}(A) = 2$ is satisfied. In this case, the general solution of $B = BXA$ is equal to

$$\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} \\ x_{3,1} & x_{3,2} & x_{3,3} & \frac{1}{8}y_1 & \frac{1}{8}(-2x_{1,5} - 2x_{2,5} - 1) \\ x_{4,1} & x_{4,2} & x_{4,3} & \frac{1}{4}y_2 & \frac{1}{4}(-2x_{1,5} - 10x_{2,5} + 3) \end{pmatrix},$$

where $y_1 = -8x_{1,1} - 10x_{1,2} - 8x_{1,3} - 2x_{1,4} - 8x_{2,1} - 10x_{2,2} - 8x_{2,3} - 2x_{2,4} - 32x_{3,1} - 40x_{3,2} - 32x_{3,3} + 3$ and $y_2 = -8x_{1,1} - 10x_{1,2} - 8x_{1,3} - 2x_{1,4} - 40x_{2,1} - 50x_{2,2} - 40x_{2,3} - 10x_{2,4} - 16x_{4,1} - 20x_{4,2} - 16x_{4,3} - 5$. Symbolic calculations reveals that the matrix X coincides with the generic solution of $AXA = A$. It means that arbitrary matrix X satisfying (i) of Theorem 3.1 also satisfies (iii) of Theorem 3.1.

As a confirmation of Corollary 3.1, it can be verified by symbolic calculus

$$A^\dagger = A^\dagger AXAA^\dagger = \begin{pmatrix} \frac{1}{1218} & \frac{5}{4872} & \frac{1}{1218} & \frac{1}{4872} & \frac{1}{84} \\ -\frac{5}{174} & -\frac{25}{696} & -\frac{5}{174} & -\frac{5}{696} & \frac{1}{4} \\ \frac{20}{609} & \frac{25}{609} & \frac{20}{609} & \frac{5}{609} & -\frac{4}{21} \\ -\frac{3}{203} & -\frac{15}{812} & -\frac{3}{203} & -\frac{3}{812} & \frac{5}{42} \end{pmatrix}$$

and

$$A^\dagger A = B^\dagger B = \begin{pmatrix} \frac{1}{14} & \frac{1}{6} & \frac{4}{21} & \frac{1}{21} \\ \frac{1}{6} & \frac{5}{6} & 0 & \frac{1}{3} \\ \frac{4}{21} & 0 & \frac{20}{21} & -\frac{2}{21} \\ \frac{1}{21} & \frac{1}{3} & -\frac{2}{21} & \frac{1}{7} \end{pmatrix}.$$

4. Solution to (1.3)

Applying Theorem 2.1 and Theorem 3.1, we characterize solutions for the system (1.3), which means that both the systems of constrained linear matrix equations (1.1) and (1.2) are satisfied.

Corollary 4.1. *Consider fixed $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times n}$.*

(a) *The below listed assertions with respect to unknown $X \in \mathbb{C}^{n\times m}$ are equivalent:*

(i) $B = AXB = BXA \wedge \text{rank}(B) = \text{rank}(A)$, *i.e., X is a solution to* (1.3)*;*

(ii) $B = AXB = BXA \wedge \text{Im}(B) = \text{Im}(A) \wedge \text{Ker}(B) = \text{Ker}(A)$*;*

(iii) $A = AXA \wedge \text{Im}(B) = \text{Im}(A) \wedge \text{Ker}(B) = \text{Ker}(A)$.

(b) *Under the conditions in* **(a)***, the general solution X to* (1.3) *is equal to*

$$X \;=\; A^\dagger + \Upsilon - A^\dagger A \Upsilon A A^\dagger,\ \Upsilon \in \mathbb{C}^{n\times m}.$$

(c) *If the conditions in* **(a)** *are satisfied, then*

$$\{X \mid B = BXA \wedge \text{rank}(B) = \text{rank}(A)\} = A\{1\}. \tag{4.1}$$

Corollary 4.2 follows from conjunction of Corollary 2.1 and Corollary 3.1.

Corollary 4.2. *For $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times n}$, the next assertions are equivalent:*

(i) $B = AXB = BXA \wedge \text{rank}(B) = \text{rank}(A)$, *i.e. X is a solution of* (1.3)*;*

(ii) $A^\dagger = A^\dagger AXAA^\dagger \wedge \text{Im}(B) = \text{Im}(A) \wedge \text{Ker}(B) = \text{Ker}(A)$*;*

(iii) $A^\dagger = B^\dagger BXBB^\dagger \wedge \text{Im}(B) = \text{Im}(A) \wedge \text{Ker}(B) = \text{Ker}(A)$.

Remark 4.1. The statements of Theorem 2.1, Theorem 3.1 and Corollary 4.1 provide two additional characterizations of the set of inner inverses. Namely, these statements characterize $A\{1\}$ as the set of all solutions to the constrained equations (1.1), equations (1.2), as well as the set of all solutions to the constrained linear matrix system (1.3).

Corollary 2.2 and Corollary 3.2 give a characterization for solutions to (1.3) under the constraints $\text{rank}(X) = \text{rank}(B) = \text{rank}(A)$.

Corollary 4.3. *For $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times n}$, the following statements are equivalent:*

(i) $B = AXB = BXA \wedge \text{rank}(X) = \text{rank}(B) = \text{rank}(A)$*;*

(ii) $A = AXA \wedge X = XAX \wedge \text{Im}(B) = \text{Im}(A) \wedge \text{Ker}(B) = \text{Ker}(A)$.

5. Conclusion

The main goal of this research is to characterize solutions of the matrix equation $B = AXB$ (or $B = BXA$) under the restriction $\text{rank}(B) = \text{rank}(A)$ on ranks of input matrices. It is proved that solutions of these equations belong to the set of inner inverses of A. In this way, the results obtained in the paper can be considered as further characterizations of inner inverses. In particular, we consider the system $B = AXB = BXA$ with constraint $\text{rank}(B) = \text{rank}(A)$. If we impose the stronger constraint $\text{rank}(X) = \text{rank}(B) = \text{rank}(A)$ instead of the restriction $\text{rank}(B) = \text{rank}(A)$, the solutions of the corresponding restricted systems not only qualify as inner inverses but also become outer inverses of A. Examples in symbolic form are presented.

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