

ON LEFT-RIGHT $\bar{\gamma}$ - ε -CONDITION PSEUDOSPECTRUM

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ABSTRACT. In this paper, we introduce new pseudo-spectral sets called the left (right) $\bar{\gamma}$ -pseudospectrum and left (right) $\bar{\gamma}$ - ε -condition pseudospectrum of bounded linear operators on Banach spaces as a part of the left (resp. right) condition ε -pseudospectrum. Moreover, we provide a sufficient condition for a bounded linear operator to be invertible using the measure of noncompactness of Kuratowski. This sufficient condition was then used to present some properties and results of the left (right) $\bar{\gamma}$ -pseudospectrum and left (right) $\bar{\gamma}$ - ε -condition pseudo-spectral sets. Finally, we establish some characterizations of the left and right $\bar{\gamma}$ - ε -condition pseudospectrum.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, let $\mathcal{B}(X)$ denote the space of bounded linear operators and $\mathcal{C}(X)$ the set of all closed, densely defined linear operators on a Banach space X . For $T \in \mathcal{C}(X)$, $\sigma(T)$ and $\rho(T)$ denote the spectrum and resolvent of T , respectively. The left spectrum and right spectrum of T are defined as:

$$\begin{aligned}\sigma^l(T) &:= \{\lambda \in \mathbb{C} : \lambda - T \text{ is not left invertible}\}, \\ \sigma^r(T) &:= \{\lambda \in \mathbb{C} : \lambda - T \text{ is not right invertible}\}.\end{aligned}$$

In [17], L. N. Trefethen introduced the pseudospectrum of $T \in \mathcal{C}(X)$ and explored its applications to operators and matrices in various significant problems. Following his work, several other mathematicians, including J. M. Varah [18], E. B. Davies [7] and F. Abdmouleh et al. [1], further developed research in this area. First let us recall some definitions about this set.

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Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. The ε -pseudospectrum of T is defined by:

$$\Lambda_\varepsilon(T) := \sigma(T) \cup \left\{ \lambda \in \mathbb{C} : \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon} \right\},$$

or by

$$\Gamma_\varepsilon(T) := \sigma(T) \cup \left\{ \lambda \in \mathbb{C} : \|(\lambda - T)^{-1}\| \geq \frac{1}{\varepsilon} \right\}.$$

By convention, we write $\|(\lambda - T)^{-1}\| = +\infty$ if $(\lambda - T)^{-1}$ is unbounded or nonexistent. Further, the left and right pseudospectrum of $T \in \mathcal{C}(X)$ are respectively defined in [2] as follows:

$$\begin{aligned} \sigma_\varepsilon^l(T) &:= \sigma^l(T) \cup \left\{ \lambda \in \mathbb{C} : \inf\{\|S_l\| : S_l \text{ is a left inverse of } \lambda - T\} > \frac{1}{\varepsilon} \right\}, \\ \sigma_\varepsilon^r(T) &:= \sigma^r(T) \cup \left\{ \lambda \in \mathbb{C} : \inf\{\|S_r\| : S_r \text{ is a right inverse of } \lambda - T\} > \frac{1}{\varepsilon} \right\}, \end{aligned}$$

with the convention $\inf\{\|S_l\| : S_l \text{ is a left inverse of } \lambda - T\} = +\infty$ if $\lambda - T$ is not left invertible (resp. $\inf\{\|S_r\| : S_r \text{ is a right inverse of } \lambda - T\} = +\infty$ if $\lambda - T$ is not right invertible).

Note that several generalizations of the spectrum have been introduced in the literature, including the Ransford spectrum [15], the pseudospectrum [9, 16], and the condition spectrum [11]. A natural question arises as to whether similar results hold for operators in infinite-dimensional Banach spaces. The notion of condition pseudospectra is particularly interesting, as it provides more information than both the spectrum and the pseudospectrum. For a detailed study of the properties of the condition spectrum in finite dimensional space or in Banach algebras, we may refer to M. Karow in [10] and S. H. Lui in [12].

In this paper, we will deepen the study of these sets by considering an additional perspective based on the condition number. Note that the condition number of an invertible operator is defined as $\|T\| \cdot \|T^{-1}\|$. For convenience, we adopt the convention that $\|T\| \cdot \|T^{-1}\| = +\infty$, when T is not invertible.

The condition ε -pseudospectrum or condition spectrum was introduced in [11] as below.

Definition 1.1. Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. The ε -condition pseudospectrum of T is denoted by $C_\varepsilon(T)$ and is defined as:

$$C_\varepsilon(T) := \sigma(T) \cup \left\{ \lambda \in \mathbb{C} : \|(\lambda - T)\| \cdot \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon} \right\},$$

with the convention $\|(\lambda - T)\| \cdot \|(\lambda - T)^{-1}\| = +\infty$ if $\lambda - T$ is not invertible.

Similarly, the left and right condition ε -pseudospectrum of $T \in \mathcal{C}(X)$ are respectively defined in [2] as follows:

$$C_\varepsilon^l(T) := \sigma^l(T) \cup \left\{ \lambda \in \mathbb{C} : \inf\{\|(\lambda - T)\| \cdot \|S_l\| > \frac{1}{\varepsilon}\} \right\},$$

$$C_\varepsilon^r(T) := \sigma^r(T) \cup \left\{ \lambda \in \mathbb{C} : \inf\{\|(\lambda - T)\| \cdot \|S_r\| > \frac{1}{\varepsilon}\} \right\},$$

where S_l (resp. S_r) is a left (resp. right) inverse of $\lambda - T$ with the convention $\inf\{\|(\lambda - T)\| \cdot \|S_l\|\} = +\infty$ and $\inf\{\|(\lambda - T)\| \cdot \|S_r\|\} = +\infty$ if $\lambda - T$ is not left invertible and is not right invertible, respectively.

Obviously, $C_\varepsilon(T) = C_\varepsilon^l(T) \cup C_\varepsilon^r(T)$ and $\Lambda_\varepsilon(T) = \sigma_\varepsilon^l(T) \cup \sigma_\varepsilon^r(T)$.

We now recall some definitions and results that will be essential for the subsequent discussion. We begin by introducing the Kuratowski measure of noncompactness (abbreviated as MNC) for a bounded subset in a normed space. Let X be a normed space. We denote by \mathcal{M}_X (respectively, \mathcal{Q}_X) the family of all bounded (respectively, relatively compact) subsets of X . For any $D \in \mathcal{M}_X$, the MNC of D , denoted by $\gamma(D)$, is defined as follows:

$$\gamma(D) = \inf \left\{ \varepsilon > 0 : D \subset \bigcup_{i=1}^n S_i, S_i \subset X, \text{diam}(S_i) \leq \varepsilon, i = 1, 2, \dots, n \right\}.$$

The following proposition outlines some key properties of the Kuratowski MNC.

Proposition 1.1 ([13]). *Let X be a Banach space and let D_1 and $D_2 \in \mathcal{M}_X$. Then, we have the following properties.*

- (i) $\gamma(D_1) = 0$ if, and only if, D_1 is relatively compact.
- (ii) If $D_1 \subset D_2$, then $\gamma(D_1) \leq \gamma(D_2)$.
- (iii) $\gamma(D_1 + D_2) \leq \gamma(D_1) + \gamma(D_2)$.
- (iv) $\gamma(\lambda D_1) = |\lambda| \gamma(D_1)$, for every $\lambda \in \mathbb{C}$.

Definition 1.2 ([14]). Let X be a Banach space and let $T : \mathcal{D}(T) \subset X \rightarrow X$ be a continuous operator. Let $k \geq 0$, then T is said to be k -set-contraction if for any bounded subset D of $\mathcal{D}(T)$, $T(D)$ is a bounded subset of X and $\gamma(T(D)) \leq k\gamma(D)$.

We now give the definition of the Kuratowski MNC of $T \in \mathcal{B}(X)$:

$$\bar{\gamma}(T) := \inf\{k : T \text{ is } k\text{-set-contraction}\}.$$

The Kuratowski MNC of a bounded operator T can be also defined by the following formula.

Definition 1.3. Let X be a Banach space and let $T \in \mathcal{B}(X)$. Then,

$$\bar{\gamma}(T) = \sup_{D \in \mathcal{M}_X \setminus \mathcal{Q}_X} \frac{\gamma(T(D))}{\gamma(D)}.$$

Proposition 1.2 ([8]). *Let X be a Banach space and let $T_1, T_2 \in \mathcal{B}(X)$. Then, the following properties are satisfied.*

- (i) $\bar{\gamma}(T_1) = 0$ if, and only if, T_1 is compact.

- (ii) $\bar{\gamma}(T_1 T_2) \leq \bar{\gamma}(T_1) \bar{\gamma}(T_2)$.
- (iii) If $K \in \mathcal{K}(X)$, then $\bar{\gamma}(T_1 + K) = \bar{\gamma}(T_1)$.
- (iv) If $D \in \mathcal{M}_X$, then $\gamma(T_1(D)) \leq \bar{\gamma}(T_1) \gamma(D)$.

In this work, motivated by the works done in [3, 4], we will continue by introducing the so-called left and right $\bar{\gamma}$ -pseudospectrum and the left and right $\bar{\gamma}$ - ε -condition pseudospectrum of a bounded linear operator. Our primary objective is to investigate the properties of the left and right $\bar{\gamma}$ - ε -condition pseudospectrum of a bounded linear operator T in Banach spaces and to explore the relationship between the $\bar{\gamma}$ - ε -condition pseudospectrum and the $\bar{\gamma}$ -pseudospectrum. A central focus of this work is to characterize the left and right $\bar{\gamma}$ - ε -condition pseudospectra for all bounded linear operators acting on a Banach space.

The paper is organized as follows. In Section 2, we characterize the left and right $\bar{\gamma}$ -pseudospectrum and analyze the sensitivity of the set $\sigma^l(T)$ under additive perturbations of T by an operator $D \in \mathcal{B}(X)$ with $\bar{\gamma}(D) < \varepsilon$. In Section 3, we summarize key properties and useful results related to the left and right $\bar{\gamma}$ - ε -condition pseudospectrum. In particular, we extend and refine several recent results from [11, 12] in the context of infinite-dimensional Banach spaces.

2. CHARACTERIZATION OF LEFT-RIGHT $\bar{\gamma}$ -PSEUDOSPECTRUM OF BOUNDED LINEAR OPERATORS ON BANACH SPACES

In this section, we establish some results on the left and right $\bar{\gamma}$ -pseudospectrum.

Note that for $K \in \mathcal{K}(X)$, we can choose $\alpha \in \mathbb{C}$ such that $K_1 = \alpha K$, where $\|K_1\| > 1$ but $\bar{\gamma}(K_1) = 0$. This motivates the following definition which generalizes the definition of the left and right pseudospectrum.

Definition 2.1. For $T \in \mathcal{B}(X)$, we define the left-right $\bar{\gamma}$ -pseudospectrum of T as:

$$\sigma_\varepsilon^{l,\bar{\gamma}}(T) := \sigma^l(T) \cup \left\{ \lambda \in \mathbb{C} : \inf\{\bar{\gamma}(S_l) : S_l \text{ is a left inverse of } \lambda - T\} > \frac{1}{\varepsilon} \right\},$$

$$\sigma_\varepsilon^{r,\bar{\gamma}}(T) := \sigma^r(T) \cup \left\{ \lambda \in \mathbb{C} : \inf\{\bar{\gamma}(S_r) : S_r \text{ is a right inverse of } \lambda - T\} > \frac{1}{\varepsilon} \right\},$$

with the convention $\inf\{\bar{\gamma}(S_l) : S_l \text{ is a left inverse of } \lambda - T\} = +\infty$ if $\lambda - T$ is not left invertible (resp. $\inf\{\bar{\gamma}(S_r) : S_r \text{ is a right inverse of } \lambda - T\} = +\infty$ if $\lambda - T$ is not right invertible).

The set of left-right $\bar{\gamma}$ -pseudospectrum of T is given by:

$$\sigma_\varepsilon^{\bar{\gamma}}(T) = \sigma_\varepsilon^{l,\bar{\gamma}}(T) \cup \sigma_\varepsilon^{r,\bar{\gamma}}(T).$$

In general, for all $T \in \mathcal{B}(X)$ we have

$$\sigma^l(T) \subseteq \sigma_\varepsilon^{l,\bar{\gamma}}(T) \subseteq \sigma_\varepsilon^{\bar{\gamma}}(T) \quad \text{and} \quad \sigma^r(T) \subseteq \sigma_\varepsilon^{r,\bar{\gamma}}(T) \subseteq \sigma_\varepsilon^{\bar{\gamma}}(T).$$

In particular $\sigma_\varepsilon^{l,\bar{\gamma}}(T)$ and $\sigma_\varepsilon^{r,\bar{\gamma}}(T)$ are nonempty sets.

Remark 2.1. In general, we have

$$\sigma_\varepsilon^{l,\bar{\gamma}}(T) \subset \sigma_\varepsilon^l(T) \text{ and } \sigma_\varepsilon^{r,\bar{\gamma}}(T) \subset \sigma_\varepsilon^r(T).$$

Proposition 2.1. Let X be a Banach space and let $T \in \mathcal{B}(X)$ and $\varepsilon > 0$. Then, the following hold.

$$(i) \sigma^l(T) = \bigcap_{\varepsilon>0} \sigma_\varepsilon^{l,\bar{\gamma}}(T) \text{ and } \sigma^r(T) = \bigcap_{\varepsilon>0} \sigma_\varepsilon^{r,\bar{\gamma}}(T).$$

(ii) For all $\varepsilon_1, \varepsilon_2$ such that $0 < \varepsilon_1 < \varepsilon_2$ we have $\sigma^l(T) \subset \sigma_{\varepsilon_1}^{l,\bar{\gamma}}(T) \subset \sigma_{\varepsilon_2}^{l,\bar{\gamma}}(T)$ and $\sigma^r(T) \subset \sigma_{\varepsilon_1}^{r,\bar{\gamma}}(T) \subset \sigma_{\varepsilon_2}^{r,\bar{\gamma}}(T)$.

Proof. (i) The direct inclusion is trivial. For the converse, if $\lambda \in \bigcap_{\varepsilon>0} \sigma_\varepsilon^{l,\bar{\gamma}}(T)$ and $\lambda \notin \sigma^l(T)$, then for all $\varepsilon > 0$, we have $\inf\{\bar{\gamma}(S_l) : S_l \text{ a left inverse of } \lambda - T\} > \frac{1}{\varepsilon}$. Taking limits as $\varepsilon \rightarrow 0^+$, we get $\inf\{\bar{\gamma}(S_l) : S_l \text{ a left inverse of } \lambda - T\} = +\infty$, which contradicts the fact that $\lambda - T$ is left invertible. Similarly, we prove the second equality.

(ii) Let $\varepsilon_1, \varepsilon_2$ such that $0 < \varepsilon_1 < \varepsilon_2$ and $\lambda \in \sigma_{\varepsilon_1}^{l,\bar{\gamma}}(T)$. Then,

$$\inf\{\bar{\gamma}(S_l) : S_l \text{ a left inverse of } \lambda - T\} > \frac{1}{\varepsilon_1} > \frac{1}{\varepsilon_2}.$$

Thus, $\lambda \in \sigma_{\varepsilon_2}^{l,\bar{\gamma}}(T)$. Using a similar argument, we show the second inclusion. \square

Next, we present the following useful lemma.

Lemma 2.1. Let X be a Banach space and let $T \in \mathcal{B}(X)$ such that $\bar{\gamma}(T) < 1$. Then, $I - T$ is invertible.

Proof. First, let us prove the surjectivity of $I - T$. We aim to show that for every $y \in X$, there exists $x \in X$ such that: $(I - T)x = y$.

The equation $(I - T)x = y$ can be rewritten as: $x = Tx + y$. This suggests that x is a fixed point of the mapping $\Psi(x) = Tx + y$. Obviously, for any bounded set $A \subset X$, we have $\Psi(A) = T(A) + \{y\}$. Using the subadditivity and monotonicity of γ given in Proposition 1.1, we get

$$\gamma(\Psi(A)) = \gamma(T(A) + \{y\}) \leq \gamma(T(A)) + \gamma(\{y\}).$$

Since $\{y\}$ is a singleton, $\gamma(\{y\}) = 0$. Thus, $\gamma(\Psi(A)) \leq \gamma(T(A))$. By Proposition 1.2, we have $\gamma(T(A)) \leq \bar{\gamma}(T)\gamma(A)$. Since $\bar{\gamma}(T) < 1$, it follows that Ψ is $\bar{\gamma}(T)$ -set-contractive.

Now, by Darbo's Fixed point theorem [6], Ψ has a fixed point $x \in X$, satisfying:

$$x = Tx + y.$$

This implies that $I - T$ is surjective.

Moreover, since $\bar{\gamma}(T) < 1$, then by applying Lemma 2.11 and Theorem 3.2 in [5], we infer that $I - T$ is a Fredholm operator of index zero. This result combined with the fact that $I - T$ is surjective enables us to deduce that $I - T$ is injective. Consequently, $I - T$ is invertible. \square

Question. What is the formula of the inverse of an operator T , when its norm is greater than one and its measure $\bar{\gamma}$ is less than one?

Proposition 2.2. Let X be a Banach space, $T \in \mathcal{B}(X)$ and $\varepsilon > 0$. Then,

$$\bigcup_{D \in \mathcal{B}(X): \bar{\gamma}(D) < \varepsilon} \sigma^l(T + D) \subset \sigma_\varepsilon^{l, \bar{\gamma}}(T).$$

Proof. Let $\lambda \notin \sigma_\varepsilon^{l, \bar{\gamma}}(T)$, then $\lambda \notin \sigma^l(T)$ and $\inf\{\bar{\gamma}(S_l) : S_l \text{ is a left inverse of } \lambda - T\} \leq \frac{1}{\varepsilon}$. This implies that $\bar{\gamma}(DS_l) < 1$. By applying Lemma 2.1 we deduce that $I - DS_l$ is invertible. Now, let us consider the operator $B = S_l(I + DS_l)^{-1}$. So, for all $y \in X$, we have $B(I + DS_l)y = S_ly$. Set $y = (\lambda - T)x$, where $x \in X$. Then,

$$x = B(I + DS_l)(\lambda - T)x = B(\lambda - T + DS_l(\lambda - T))x = B(\lambda - (T + D))x.$$

This shows that $\lambda - (T + D)$ is left invertible. Hence, $\lambda \notin \bigcup_{D \in \mathcal{B}(X): \bar{\gamma}(D) < \varepsilon} \sigma^l(T + D)$. \square

3. CHARACTERIZATION OF LEFT-RIGHT $\bar{\gamma}$ - ε -CONDITION PSEUDOSPECTRUM

In this section, we provide some results on the characterization of left-right $\bar{\gamma}$ - ε -condition pseudospectrum of bounded linear operator. We start by introducing the following definitions.

Definition 3.1. For $T \in \mathcal{B}(X)$, we define the left-right $\bar{\gamma}$ - ε -condition pseudospectrum of T as:

$$C_\varepsilon^{l, \bar{\gamma}}(T) := \sigma^l(T) \cup \left\{ \lambda \in \mathbb{C} : \inf\{\bar{\gamma}(\lambda - T)\bar{\gamma}(S_l)\} > \frac{1}{\varepsilon} \right\},$$

$$C_\varepsilon^{r, \bar{\gamma}}(T) := \sigma^r(T) \cup \left\{ \lambda \in \mathbb{C} : \inf\{\bar{\gamma}(\lambda - T)\bar{\gamma}(S_r)\} > \frac{1}{\varepsilon} \right\},$$

where S_l (resp. S_r) is a left (resp. right) inverse of $\lambda - T$ with the convention $\inf\{\bar{\gamma}(\lambda - T)\bar{\gamma}(S_l)\} = +\infty$ if $\lambda - T$ is not left invertible (resp. $\inf\{\bar{\gamma}(\lambda - T)\bar{\gamma}(S_r)\} = +\infty$ if $\lambda - T$ is not right invertible).

The set of left and right $\bar{\gamma}$ - ε -condition pseudospectrum of T is given by:

$$C_\varepsilon^{\bar{\gamma}}(T) = C_\varepsilon^{l, \bar{\gamma}}(T) \cup C_\varepsilon^{r, \bar{\gamma}}(T).$$

Remark 3.1. (i) It is not difficult to see the following inclusions:

$$C_\varepsilon^{l, \bar{\gamma}}(T) \subset C_\varepsilon^l(T) \quad \text{and} \quad C_\varepsilon^{r, \bar{\gamma}}(T) \subset C_\varepsilon^r(T).$$

(ii) In general, we have

$$C_\varepsilon^{l, \bar{\gamma}}(T) \subset C_\varepsilon^{\bar{\gamma}}(T) \subset C_\varepsilon(T) \quad \text{and} \quad C_\varepsilon^{r, \bar{\gamma}}(T) \subset C_\varepsilon^{\bar{\gamma}}(T) \subset C_\varepsilon(T).$$

Proposition 3.1. Let X be a Banach space and $T \in \mathcal{B}(X)$ and $\varepsilon > 0$. Then, the following hold.

(i) $\sigma^l(T) = \bigcap_{\varepsilon > 0} C_\varepsilon^{l, \bar{\gamma}}(T)$ and $\sigma^r(T) = \bigcap_{\varepsilon > 0} C_\varepsilon^{r, \bar{\gamma}}(T)$.

(ii) If ε_1 and ε_2 are such that $0 < \varepsilon_1 < \varepsilon_2$, then $\sigma^l(T) \subset C_{\varepsilon_1}^{l, \bar{\gamma}}(T) \subset C_{\varepsilon_2}^{l, \bar{\gamma}}(T)$ and $\sigma^r(T) \subset C_{\varepsilon_1}^{r, \bar{\gamma}}(T) \subset C_{\varepsilon_2}^{r, \bar{\gamma}}(T)$.

Proof. (i) It is clear that for any $\varepsilon > 0$, that $\sigma^l(T) \subset C_\varepsilon^{l, \bar{\gamma}}(T)$. Conversely, if $\lambda \in \bigcap_{\varepsilon > 0} C_\varepsilon^{l, \bar{\gamma}}(T)$, then for all $\varepsilon > 0$, $\lambda \in C_\varepsilon^{l, \bar{\gamma}}(T)$. If $\lambda \notin \sigma^l(T)$, then $\lambda \in \{\lambda \in \mathbb{C} : \inf\{\bar{\gamma}(\lambda - T)\bar{\gamma}(S_l) : S_l \text{ a left inverse of } \lambda - T\} > \frac{1}{\varepsilon}\}$, taking limits as $\varepsilon \rightarrow 0^+$, we get $\inf\{\bar{\gamma}(\lambda - T)\bar{\gamma}(S_l) : S_l \text{ a left inverse of } \lambda - T\} = +\infty$. Hence, $\lambda \in \sigma^l(T)$. By using a similar way as previous, we can show the second equality.

(ii) Let ε_1 and ε_2 such that $0 < \varepsilon_1 < \varepsilon_2$. Let $\lambda \in C_{\varepsilon_1}^{r, \bar{\gamma}}(T)$. Then,

$$\inf\{\bar{\gamma}(\lambda - T)\bar{\gamma}(S_l) : S_l \text{ a left inverse of } \lambda - T\} > \frac{1}{\varepsilon_1} > \frac{1}{\varepsilon_2},$$

hence $\lambda \in C_{\varepsilon_2}^{r, \bar{\gamma}}(T)$. Similarly, we have $\sigma^r(T) \subset C_{\varepsilon_1}^{r, \bar{\gamma}}(T) \subset C_{\varepsilon_2}^{r, \bar{\gamma}}(T)$. \square

Next, we define the $\bar{\gamma}$ -condition pseudospectral radius $r_\varepsilon^\gamma(T)$ of $T \in \mathcal{B}(X)$ by:

$$r_\varepsilon^\gamma(T) := \sup\{|\lambda| : \lambda \in C_\varepsilon^{\bar{\gamma}}(T)\}.$$

Following Definition 3.1, we give the following property of the $\bar{\gamma}$ -condition pseudospectral radius. Before that, let us recall the following definition.

Definition 3.2. The spectral radius of $T \in \mathcal{B}(X)$ is defined by:

$$r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

Lemma 3.1. Let $T \in \mathcal{B}(X)$ and $0 < \varepsilon < 1$. Then,

$$r(T) \leq r_\varepsilon^\gamma(T) \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right) \bar{\gamma}(T).$$

Proof. Since $\sigma(T) \subseteq C_\varepsilon^{\bar{\gamma}}(T)$ for $0 < \varepsilon < 1$, then we have $r(T) \leq r_\varepsilon^\gamma(T)$. Now, we will show that

$$r_\varepsilon^\gamma(T) \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right) \bar{\gamma}(T).$$

Let $\lambda \in C_\varepsilon^{\bar{\gamma}}(T)$. Then, we consider two cases.

Case 1. If $|\lambda| \leq \bar{\gamma}(T)$, then $r_\varepsilon^\gamma(T) \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right) \bar{\gamma}(T)$.

Case 2. If $|\lambda| > \bar{\gamma}(T)$, then by Lemma 2.1 we conclude that $\lambda - T$ is invertible. Furthermore, we have

$$\bar{\gamma}((\lambda - T)^{-1}) \leq \frac{1}{|\lambda| - \bar{\gamma}(T)}.$$

Since

$$1 < \varepsilon \bar{\gamma}((\lambda - T)^{-1}) \bar{\gamma}(\lambda - T) \leq \varepsilon \left(\frac{|\lambda| + \bar{\gamma}(T)}{|\lambda| - \bar{\gamma}(T)}\right),$$

it yields that

$$\left(\frac{1-\varepsilon}{\varepsilon}\right) |\lambda| \leq \left(\frac{1+\varepsilon}{\varepsilon}\right) \bar{\gamma}(T).$$

Thus,

$$|\lambda| \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right) \bar{\gamma}(T).$$

Hence, $r_\varepsilon^\gamma(T) \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right) \bar{\gamma}(T)$. \square

Theorem 3.1. Let $T \in \mathcal{B}(X)$ such that $T - \lambda$ is not compact and let $0 < \varepsilon < 1$. Then,

$$C_\varepsilon^{l,\bar{\gamma}}(T) \subseteq \sigma_{\alpha_\varepsilon}^{l,\bar{\gamma}} \subseteq C_{\beta_\varepsilon}^{l,\bar{\gamma}}(T),$$

where

$$\alpha_\varepsilon = \frac{2\varepsilon\bar{\gamma}(T)}{1 - \varepsilon}, \quad 0 < \beta_\varepsilon = \frac{2\varepsilon\bar{\gamma}(T)}{(1 - \varepsilon)\theta_T} < 1 \quad \text{and} \quad \theta_T := \inf\{\bar{\gamma}(\lambda - T) : \lambda \in \mathbb{C}\}.$$

Proof. Let $\lambda \in C_\varepsilon^{l,\bar{\gamma}}(T)$, then for all S_l , a left inverse of $\lambda - T$, we have

$$\bar{\gamma}(\lambda - T)\bar{\gamma}(S_l) > \frac{1}{\varepsilon}.$$

This implies that

$$\bar{\gamma}(S_l) > \frac{1}{\varepsilon\bar{\gamma}(\lambda - T)} > \frac{1}{\varepsilon(|\lambda| + \bar{\gamma}(T))}.$$

Combining the fact that $\lambda \in C_\varepsilon^{l,\bar{\gamma}}(T)$ and the use of Lemma 3.1, then we obtain that

$$\bar{\gamma}(S_l) \geq \frac{1 - \varepsilon}{2\varepsilon\bar{\gamma}(T)}.$$

Hence, $\lambda \in \sigma_{\alpha_\varepsilon}^{l,\bar{\gamma}}(T)$.

For the second inclusion, let $\lambda \in \sigma_{\alpha_\varepsilon}^{l,\bar{\gamma}}(T)$. Then, for all S_l , a left inverse of $\lambda - T$, we have

$$\bar{\gamma}(S_l) \geq \frac{1 - \varepsilon}{2\varepsilon\bar{\gamma}(T)}.$$

Taking into account that $\bar{\gamma}(\lambda - T) \geq \inf\{\bar{\gamma}(\lambda - T) : \lambda \in \mathbb{C}\}$, then we infer that

$$\bar{\gamma}(\lambda - T)\bar{\gamma}(S_l) > \frac{\theta_T(1 - \varepsilon)}{2\varepsilon\bar{\gamma}(T)}.$$

Consequently, $\lambda \in C_{\beta_\varepsilon}^{l,\bar{\gamma}}(T)$. □

Remark 3.2. Theorem 2.2 remains true if we replace $C_\varepsilon^{l,\bar{\gamma}}(T)$ with $C_\varepsilon^{r,\bar{\gamma}}(T)$, $\sigma_{\alpha_\varepsilon}^{l,\bar{\gamma}}(T)$ with $\sigma_{\alpha_\varepsilon}^{r,\bar{\gamma}}(T)$, and $C_{\beta_\varepsilon}^{l,\bar{\gamma}}(T)$ with $C_{\beta_\varepsilon}^{r,\bar{\gamma}}(T)$.

Lemma 3.2. Let $T \in \mathcal{B}(X)$, $0 < \varepsilon < 1$, and $\lambda \notin \sigma^l(T)$. Then, $\lambda \in C_\varepsilon^{l,\bar{\gamma}}(T)$ if and only if there exists $M \in \mathcal{M}_X$, such that

$$\gamma((\lambda - T)(M)) < \varepsilon\bar{\gamma}(\lambda - T)\gamma(M).$$

Proof. Let $\lambda \in C_\varepsilon^{l,\bar{\gamma}}(T) \setminus \sigma^l(T)$, then for all S_l , a left inverse of $\lambda - T$, we have

$$\bar{\gamma}(S_l) > \frac{1}{\varepsilon\bar{\gamma}(\lambda - T)}.$$

This equivalently to

$$\sup_{B \in \mathcal{M}_X \setminus \mathcal{Q}_X} \frac{\gamma(S_l(B))}{\gamma(B)} > \frac{1}{\varepsilon\bar{\gamma}(\lambda - T)}.$$

Therefore,

$$\gamma(S_l(B)) > \frac{\gamma(B)}{\varepsilon\bar{\gamma}(\lambda - T)}.$$

By setting $B = (\lambda - T)M$, we deduce the result.

Conversely, assume that there exists $M \in \mathcal{M}_X \setminus \mathcal{Q}_X$ such that

$$\gamma((\lambda - T)(M)) < \varepsilon\bar{\gamma}(\lambda - T)\gamma(M).$$

Let $\lambda \notin \sigma^l(T)$, then $\lambda - T$ is left-invertible. Let S_l be any left inverse, then $M = S_l(\lambda - T)(M)$. Therefore, $\gamma(M) \leq \bar{\gamma}(S_l)\gamma((\lambda - T)(M))$. Moreover, $1 < \varepsilon\bar{\gamma}(\lambda - T)\bar{\gamma}(S_l)$. So, $\lambda \in C_\varepsilon^{l,\bar{\gamma}}(T) \setminus \sigma^l(T)$. \square

Theorem 3.2. Let $T \in \mathcal{B}(X)$, $\lambda \in \mathbb{C}$, and $0 < \varepsilon < 1$. If there exists $D \in \mathcal{B}(X)$ such that $\bar{\gamma}(D) < \varepsilon\bar{\gamma}(\lambda - T)$ and $\lambda \in \sigma^l(T + D)$, then $\lambda \in C_\varepsilon^{l,\bar{\gamma}}(T)$.

Proof. Suppose that there exists a bounded linear operator D on X such that $\bar{\gamma}(D) < \varepsilon\bar{\gamma}(\lambda - T)$ and $\lambda \in \sigma^l(T + D)$. Let $\lambda \notin C_\varepsilon^{l,\bar{\gamma}}(T)$, then for all S_l , a left inverse of $\lambda - T$, we have

$$\bar{\gamma}(S_l)\bar{\gamma}(\lambda - T) \leq \frac{1}{\varepsilon}.$$

Now, let us consider the operator $S = S_l(I - DS_l)^{-1}$. Then, for all $y \in X$, we obtain $S(I - DS_l)y = S_ly$. Putting $y = (\lambda - T)x$. Then,

$$S(\lambda - T - D)x = x,$$

for every $x \in X$. Hence, $\lambda - T - D$ is left-invertible, which is a contradiction. Therefore, $\lambda \in C_\varepsilon^{l,\bar{\gamma}}(T)$. \square

Proposition 3.2. Let $T \in \mathcal{B}(X)$, $\varepsilon > 0$ and $\bar{\gamma}(\lambda - T) \neq 0$. Then,

- (i) $\lambda \in C_\varepsilon^{l,\bar{\gamma}}(T)$ if, and only if, $\lambda \in \sigma_{\varepsilon\bar{\gamma}(\lambda - T)}^l(T)$;
- (ii) $\lambda \in \sigma_\varepsilon^{l,\bar{\gamma}}(T)$ if and only if $\lambda \in C_{\frac{\varepsilon}{\bar{\gamma}(\lambda - T)}}^{l,\bar{\gamma}}(T)$.

Proof. (i) Let $\lambda \in C_\varepsilon^{l,\bar{\gamma}}(T)$, then $\lambda \in \sigma^l(T)$ or

$$\inf\{\bar{\gamma}(\lambda - T)\bar{\gamma}(S_l) : S_l \text{ a left inverse of } \lambda - T\} > \frac{1}{\varepsilon}.$$

Hence, $\lambda \in \sigma^l(A)$ or for all S_l a left invertible of $\lambda - T$, $\bar{\gamma}(S_l) > \frac{1}{\varepsilon\bar{\gamma}(\lambda - T)}$. Consequently, $\lambda \in \sigma_{\varepsilon\bar{\gamma}(\lambda - T)}^l(T)$. The converse is similar.

(ii) Let $\lambda \in \sigma_\varepsilon^{l,\bar{\gamma}}(T)$, then $\lambda \in \sigma^l(A)$ or for all S_l a left inverse of $\lambda - T$, $\bar{\gamma}(S_l) > \frac{1}{\varepsilon}$. So, $\lambda \in C_{\frac{\varepsilon}{\bar{\gamma}(\lambda - T)}}^{l,\bar{\gamma}}(T)$. The converse implication follows by symmetry. \square

One can see the following corollary.

Corollary 3.1. Let X be a Banach space, $T \in \mathcal{B}(X)$ and $\varepsilon > 0$. If $\alpha, \beta \in \mathbb{C}$ with $\beta \neq 0$, then $C_\varepsilon^{l,\bar{\gamma}}(\beta T + \alpha I) = \alpha + \beta C_\varepsilon^{l,\bar{\gamma}}(T)$.

Proposition 3.3. Let X be a Banach space, $T \in \mathcal{B}(X)$ and $\varepsilon > 0$ such that $T - \lambda$ is not compact. Then, $\sigma_\varepsilon^{l,\bar{\gamma}}(T) \subset C_{\frac{\varepsilon}{\theta_T}}^{l,\bar{\gamma}}(T)$.

Proof. Let $\lambda \in \sigma_\varepsilon^{l,\bar{\gamma}}(T)$, then $\lambda \in \sigma^l(T)$ or for all S_l a left inverse of $\lambda - T$ we have $\bar{\gamma}(S_l) > \frac{1}{\varepsilon}$. It follows from the fact that $\bar{\gamma}(\lambda - T) \geq \theta_T > 0$ that $\bar{\gamma}(\lambda - T)\bar{\gamma}(S_l) > \frac{\theta_T}{\varepsilon}$. Thus, $\lambda \in C_{\frac{\varepsilon}{\theta_T}}^{l,\bar{\gamma}}(T)$. □

Theorem 3.3. *Let $T \in \mathcal{B}(X)$, $k = \bar{\gamma}(T)\bar{\gamma}(T^{-1})$ and $0 < \varepsilon < 1$. Then,*

- (i) $\lambda \in C_\varepsilon^{l,\bar{\gamma}}(T)$ if, and only if, $\bar{\lambda} \in C_\varepsilon^{l,\bar{\gamma}}(T^*)$;
- (ii) $\lambda \in C_\varepsilon^{l,\bar{\gamma}}(T^{-1}) \setminus \{0\}$ if and only if $\frac{1}{\lambda} \in C_{\varepsilon k}^{l,\bar{\gamma}}(T) \setminus \{0\}$.

Proof. (i) The result follows by using the following equality

$$\bar{\gamma}(\lambda - T)\bar{\gamma}((\lambda - T)^{-1}) = \bar{\gamma}(\bar{\lambda} - T^*)\bar{\gamma}((\bar{\lambda} - T^*)^{-1}).$$

(ii) Let $\lambda \in C_\varepsilon^{l,\bar{\gamma}}(T^{-1}) \setminus \{0\}$, then

$$\frac{1}{\varepsilon} < \bar{\gamma}(\lambda - T^{-1})\bar{\gamma}((\lambda - T^{-1})^{-1}).$$

Accordingly,

$$\begin{aligned} \bar{\gamma}(\lambda - T^{-1})\bar{\gamma}((\lambda - T^{-1})^{-1}) &= \bar{\gamma}\left(-\lambda\left(\frac{1}{\lambda} - T\right)T^{-1}\right)\bar{\gamma}\left(-\lambda^{-1}T\left(\frac{1}{\lambda} - T\right)^{-1}\right) \\ &\leq \bar{\gamma}(T)\bar{\gamma}(T^{-1})\bar{\gamma}\left(\frac{1}{\lambda} - T\right)\bar{\gamma}\left(\left(\frac{1}{\lambda} - T\right)^{-1}\right). \end{aligned}$$

Consequently,

$$\frac{1}{\lambda} \in C_{\varepsilon k}^{l,\bar{\gamma}}(T) \setminus \{0\}.$$

The converse implication can be checked by a similar way. □

Theorem 3.4. Let $T \in \mathcal{B}(X)$ and $V \in \mathcal{B}(X)$ be invertible. Let $S = V^{-1}TV$. Then, for all $0 < \varepsilon < 1$ and $k = \bar{\gamma}(V^{-1})\bar{\gamma}(V)$ such that $0 < k^2\varepsilon < 1$, we have

$$C_{\frac{\varepsilon}{k^2}}^{l,\bar{\gamma}}(T) \subseteq C_\varepsilon^{l,\bar{\gamma}}(S) \subseteq C_{k^2\varepsilon}^{l,\bar{\gamma}}(T).$$

Proof. We have

$$\begin{aligned} \bar{\gamma}(\lambda - S)\bar{\gamma}((\lambda - S)^{-1}) &= \bar{\gamma}V^{-1}(\lambda - T)V\bar{\gamma}(V^{-1}(\lambda - T)^{-1}V) \\ &\leq \left(\bar{\gamma}(V)\bar{\gamma}(V^{-1})\right)^2 \bar{\gamma}(\lambda - T)\bar{\gamma}((\lambda - T)^{-1}) \\ (3.1) \qquad \qquad \qquad &\leq k^2\bar{\gamma}(\lambda - T)\bar{\gamma}((\lambda - T)^{-1}) \end{aligned}$$

and

$$\begin{aligned} \bar{\gamma}(\lambda - T)\bar{\gamma}((\lambda - T)^{-1}) &= \bar{\gamma}(V(\lambda - S)V^{-1})\bar{\gamma}(V(\lambda - S)^{-1}V^{-1}) \\ &\leq \left(\bar{\gamma}(V)\bar{\gamma}(V^{-1})\right)^2 \bar{\gamma}(\lambda - S)\bar{\gamma}((\lambda - S)^{-1}) \\ (3.2) \qquad \qquad \qquad &\leq k^2\bar{\gamma}(\lambda - S)\bar{\gamma}((\lambda - S)^{-1}). \end{aligned}$$

Let $\lambda \in C_{\frac{\varepsilon}{k^2}}^{l,\bar{\gamma}}(T)$. Then, by using (3.2), we deduce that $\lambda \in C_\varepsilon^{l,\bar{\gamma}}(S)$.

On the other hand, if $\lambda \in C_\varepsilon^{l,\bar{\gamma}}(S)$, then the use of (3.1), allows us to conclude that $\lambda \in C_{k^2\varepsilon}^{l,\bar{\gamma}}(T)$. □

In the sequel of this section, we consider the Hilbert space H , we use the notation $\text{conv}(C_\varepsilon^{l,\bar{\gamma}}(T))$, the convex hull in \mathbb{C} of a set $C_\varepsilon^{l,\bar{\gamma}}(T)$, $\mathbb{B}(a, r)$ the open ball with center at a and radius r and we define the distance between two nonempty subsets U and V by

$$\inf\{\|u - v\| : u \in U, v \in V\}.$$

The next theorem gives a relation between the the left $\bar{\gamma}$ - ε -condition pseudospectrum and the numerical range of T , given by

$$W(T) := \{\langle Ax, x \rangle : x \in S_H\},$$

where $S_H := \{x \in H : \|x\| = 1\}$ is the unit sphere in H . It is well known that $W(T)$ is a convex set whose closure contains the spectrum $\sigma(T)$ of T .

Theorem 3.5. Let $T \in \mathcal{B}(H)$ and $0 < \varepsilon < 1$. Then,

$$\text{conv}(C_\varepsilon^{l,\bar{\gamma}}(T)) \subseteq W(T) + \mathbb{B}\left(0, \frac{2\varepsilon}{1-\varepsilon}\bar{\gamma}(T)\right).$$

Proof. Let $\lambda \in \mathbb{C}$ such that $\text{dist}(\lambda, W(T)) > 0$, then $\lambda - T$ is left invertible, and so

$$\bar{\gamma}((\lambda - T)^{-1}) \leq \frac{1}{\text{dist}(\lambda, W(T))}.$$

Assume that $\lambda \in C_\varepsilon^{l,\bar{\gamma}}(T)$. There are two possible cases.

Case 1. If $\lambda \in \overline{W(T)}$, then the result is trivial.

Case 2. If $\lambda \in C_\varepsilon^{l,\bar{\gamma}}(T) \setminus \overline{W(T)}$, then

$$\frac{1}{\varepsilon} < \bar{\gamma}(\lambda - T)\bar{\gamma}((\lambda - T)^{-1}) \leq \frac{\bar{\gamma}(\lambda - T)}{\text{dist}(\lambda, W(T))}.$$

This shows that $\text{dist}(\lambda, W(T)) < \varepsilon\bar{\gamma}(\lambda - T) \leq \varepsilon(|\lambda| + \bar{\gamma}(T))$. Using Lemma 3.1, we get

$$\text{dist}(\lambda, W(T)) \leq \varepsilon \left(\frac{1+\varepsilon}{1-\varepsilon}\bar{\gamma}(T) + \bar{\gamma}(T) \right) = \left(\frac{2\varepsilon}{1-\varepsilon} \right) \bar{\gamma}(T).$$

Hence,

$$C_\varepsilon^{l,\bar{\gamma}}(T) \subseteq W(T) + \mathbb{B}\left(0, \frac{2\varepsilon}{1-\varepsilon}\bar{\gamma}(T)\right),$$

and so the result can deduced from the fact that $W(T)$ is a convex set. \square

Corollary 3.2. Let $T \in \mathcal{B}(H)$ and $0 < \varepsilon < 1$. Then,

$$D_{W,\lambda}^{\bar{\gamma}}(T) \subseteq C_\varepsilon^{l,\bar{\gamma}}(T),$$

where $D_{W,\lambda}^{\bar{\gamma}}(T) := \{\lambda \in \mathbb{C} : \text{dist}(\lambda, W(T)) < \varepsilon\bar{\gamma}(\lambda - T)\}$.

Proof. Let $\lambda \notin C_\varepsilon^{l,\bar{\gamma}}(T)$. By using Theorem 3.5, we have

$$\frac{1}{\text{dist}(\lambda, W(T))} \leq \bar{\gamma}((\lambda - T)^{-1}) \leq \frac{1}{\varepsilon\bar{\gamma}(\lambda - T)}.$$

Hence, $\text{dist}(\lambda, W(T)) \geq \varepsilon\bar{\gamma}(\lambda - T)$, and so the desired result is achieved. \square

CONCLUSION

This paper introduces and studies new pseudospectral sets for bounded linear operators on Banach spaces, namely the left and right $\bar{\gamma}$ -pseudospectrum and $\bar{\gamma}$ - ε -condition pseudospectrum, based on the Kuratowski measure of noncompactness. We establish fundamental properties, characterizations, and relationships between these sets, and provide perturbation results that connect them to traditional spectral theory. These constructions extend finite-dimensional condition pseudospectral concepts to infinite-dimensional settings, offering tools for analyzing operator stability and sensitivity under non-compact perturbations.

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