ASYMPTOTIC PROPERTIES OF THE SOLUTION TO NEUTRAL STOCHASTIC DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY

TEODORA LJUJIĆ[1], MILJANA JOVANOVIĆ[2], AND MARIJA MILOŠEVIĆ[2]

ABSTRACT. The existence and uniqueness of the solution of the n-dimensional neutral stochastic differential equation with state-dependent finite or infinite delay is proved, so the focus of this paper is on the investigation of the asymptotic properties of this solution. The whole consideration is influenced by characteristics of the neutral term and the state-dependent delay. Theoretical results are complemented by an example.

1. INTRODUCTION AND PRELIMINARIES

Describing the physical systems with differential equations whose delay depends both on time and on the current state of the system is of great importance, since it is known that these systems depend on some features associated with the state of the system itself. Adding a noise to these systems leads to more realistic models, so studying stochastic differential equations with state-dependent delay (SDEwSDD) is significant. In the paper [15] more general type of stochastic differential equation (SDE) - the neutral stochastic differential equation with state-dependent delay (NSDEwSDD), is considered, where the existence and uniqueness of the solution is proven and some asymptotic properties of the solution are given. This type of equations caught our attention for several reasons. First, the state-dependent delay was not the subject of many papers involving stochastic differential equations [1,6,14]. Second, none of these papers handles equations of neutral type. Our attention was addressed to NSDEwSDD

as we wanted to consider a more general type of equations, since SDEwSDD are a special case of NSDEwSDD. The Khasminskii approach plays a significant role when SDEs with highly nonlinear coefficients are considered and such equations have been applied for the description of many real-life phenomena. Although the Khasminskii approach is used in the existing literature (see e.g. [9,10]) for neutral SDEs with time dependent delay, this approach could not be applied in the present paper because of the dependence of the delay function on the state of the system under consideration. In that sense, it should be emphasized that the techniques used in the present paper, as well as the conditions on the coefficients of the equations, are influenced by the nature of the SDEs observed, mainly by the presence of the neutral term and the state-dependent delay. Our objective in this paper is to continue the previous work by weakening the conditions for the existence and uniqueness of the solution to Eq. (1.1), which we established in [15], as well as introducing new asymptotic properties including the asymptotic p-moment estimates.

From now on, $\mathbb{R}_+$ and $\mathbb{R}_-$ will stand for intervals $[0, +\infty)$ and $(-\infty, 0]$, respectively, D^T will denote the transpose of a vector or matrix D and $|\cdot|$ will signify the Euclidean norm in $\mathbb{R}^n$. Also, the following notation will be used:

- $C(A; B)$ for the family of continuous functions from set A to set B,
- $C^1(A; B)$ for the family of continuously differentiable functions from set A to set B,
- $C^\beta(\mathbb{R}_-; \mathbb{R}^n)$ for the family of functions $\alpha : \mathbb{R}_- \to \mathbb{R}^n$, such that $\frac{\alpha}{\beta}$ is uniformly continuous on $\mathbb{R}_-$ and $\sup_{\theta \in \mathbb{R}_-} \frac{|\alpha(\theta)|}{\beta(\theta)} < +\infty$, where $\beta : \mathbb{R}_- \to [1, +\infty)$ is a continuous, non-increasing function with $\beta(\theta) \to +\infty$, as $\theta \to -\infty$ and $\beta(0) = 1$, with the norm $\|\alpha\| = \sup_{\theta \in \mathbb{R}_-} \frac{|\alpha(\theta)|}{\beta(\theta)}$,
- $C(J; \mathbb{R}^n) = \{\alpha_1 \mid \alpha_1 : J \to \mathbb{R}^n\}$, where $J = [-\eta, 0]$, with the norm $\|\alpha_1\| = \sup_{\theta \in J} |\alpha_1(\theta)|$,
- $C^b_{\mathcal{F}_0}(J; \mathbb{R}^n)$ for the family of $\mathcal{F}_0$-measurable $\mathbb{R}^n$-valued random variables from $\mathcal{M}^p(J; \mathbb{R}^n)$, where $\mathcal{M}^p(J; \mathbb{R}^n)$ is the family of $\mathcal{F}_0$-measurable, $\mathbb{R}^n$-valued processes $\kappa(t) = \kappa(t, \omega)$, $t \in J$, with $E \int_J |\kappa(t)|^p dt < +\infty$ and
- $BC(\mathbb{R}_-; \mathbb{R}^n)$ - bounded functions $\alpha_2 : \mathbb{R}_- \to \mathbb{R}^n$, with the norm $\|\alpha_2\| = \sup_{\theta \in \mathbb{R}_-} |\alpha_2(\theta)|$.

For a complete probability space $(\Omega, \mathcal{F}, P)$, with a filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$ that satisfies the usual conditions, $W(t) = (W_1(t), \ldots, W_k(t))^T$, $t \geqslant 0$, is a k-dimensional Brownian motion and we consider the n-dimensional neutral stochastic differential equation with state-dependent delay

$$(1.1) \qquad d[y(t) - D(t, y(t - \rho(t, y(t))))]$$
$$= a(t, y(t), y(t - \rho(t, y(t))))dt + b(t, y(t), y(t - \rho(t, y(t))))dW(t), \quad t \geqslant 0,$$

with the initial condition

$$(1.2) \qquad\qquad y_0 = \psi = \{\psi(\theta) : \theta \in J\} \in C^b_{\mathcal{F}_0}(J; \mathbb{R}^n),$$

where $J = \mathbb{R}_-$ if

$$\sup_{(t,y) \in \mathbb{R}_+ \times \mathbb{R}^n} (\rho(t, y) - t) = +\infty,$$

or $J = [-\eta, 0]$ if

$$\sup_{(t,y)\in\mathbb{R}_+\times\mathbb{R}^n} (\rho(t, y) - t) = \eta.$$

The functions $D : \mathbb{R}_+\times\mathbb{R}^n \to \mathbb{R}^n$, $a : \mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ and $b : \mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^{n\times k}$ are Borel-measurable, as well as the state-dependent delay function ρ, defined on the set $\mathbb{R}_+ \times \mathbb{R}^n$, which is nonnegative, bounded or unbounded. It is important to accentuate the fact that we do not propose any additional condition on the delay function. For example, in many papers involving time dependent delay the delay function is differentiable, with a bounded first derivative (see [5, 11, 12]). In [1] delay function is state dependent and Lipschitz continuous in both variables.

For $\mathcal{V} \in C^{1,2}(J \cup \mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$, where $C^{1,2}(J \cup \mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$ is the family of functions $\mathcal{V} : J\cup\mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, which are continuously once differentiable in t and twice in y, an operator $\mathcal{LV} : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\mathcal{LV}(t, y, z) = \mathcal{V}_t(t, y - D(t, z)) + \mathcal{V}_y(t, y - D(t, z))a(t, y, z)$$

(1.3)
$$+ \frac{1}{2}\text{trace}\left[b^T(t, y, z)\mathcal{V}_{yy}(t, y - D(t, z))b(t, y, z)\right].$$

With regard to simplify the notation in the following assertions we take

$$\widetilde{\mathcal{V}} = \mathcal{V}\Big(0, \psi(0) - D(0, \psi(-\rho(0, \psi(0))))\Big).$$

In order to prove some of the following results, we give a Gronwall-type inequality and a generalization of the Bellman-Gronwall inequality, which can be found in [4] and [13], respectively.

Theorem 1.1 (Bellman 1943). *If $f, g \in C(\mathbb{R}_+, \mathbb{R}_+)$ and*

$$f(t) \leqslant C + \int_0^t g(s)f(s)ds, \quad t \geqslant 0,$$

where $C > 0$ is a constant, then

$$f(t) \leqslant Ce^{\int_0^t g(s)ds}, \quad t \geqslant 0.$$

Theorem 1.2 (Dhongade-Deo). *Let $f_1, g_1, n \in C((0, +\infty), (0, +\infty))$, where $n(t)$ is monotonic non-decreasing function, and $p \in C((0, +\infty), [1, +\infty))$. If*

(1.4)
$$f_1(t) \leqslant n(t) + p(t) \int_0^t g_1(s)f_1(s)ds, \quad t > 0,$$

then

$$f_1(t) \leqslant n(t)p(t) \exp\left(\int_0^t g_1(s)p(s)ds\right), \quad t > 0.$$

The paper is structured as follows. In Section 2, the theorem that ensures the existence and uniqueness of the solution to Eq. (1.1) is introduced together with the necessary assumptions. Under these assumptions, Section 3 presents the main results on the asymptotic properties of the solution to Eq. (1.1). Section 4 contains an example to illustrate the assertions from the previous section. In Section 5, we conclude this paper by giving topics of our further research.

2. The Existence and Uniqueness of the Solution

Before we get to the asymptotic properties of the solution to Eq. (1.1), we need to make sure that there is such unique solution. Suppose that the coefficients of Eq. (1.1) with the initial condition (1.2) satisfy the following conditions.

C1. There exist a function $\mathcal{V} \in C^{1,2}(J \cup \mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$, functions $K_1 \in C^1(J \cup \mathbb{R}_+; \mathbb{R}_+ \setminus \{0\})$, $K_2 \in C(J \cup \mathbb{R}_+; \mathbb{R}_+ \setminus \{0\})$, $q \in C(J \cup \mathbb{R}_+; \mathbb{R}_+)$, with $\inf_{t \in J \cup \mathbb{R}_+} K_1(t) > 0$, such that, for every $(t, y) \in \mathbb{R}_+ \times \mathbb{R}^n$ and $p > 0$,

$$(2.1) \qquad K_1(t)|y|^p \leqslant \mathcal{V}(t, y) \leqslant q(t) + K_2(t)|y|^p.$$

Remark 2.1. It is obvious that we can simplify (2.1) by taking $K_1(t) = K_2(t)$, $t \geqslant 0$. But since these functions can be different constants, we maintain generality.

C2. (Contractivity condition) There exists a constant $\gamma \in (0, 1)$, such that, for all $y, z \in \mathbb{R}^n$ and $t \geqslant 0$,

$$|D(t, y) - D(t, z)| \leqslant \gamma |y - z|.$$

The previous inequality induces that $|D(t, y)| \leqslant \gamma |y| + F$, for every $(t, y) \in \mathbb{R}_+ \times \mathbb{R}^n$, where $\sup_{t \geqslant 0} |D(t, 0)| = F$.

In order to prove some of the asymptotic properties for a wide class of NSDEwSDD, let us discuss the existence of the unique global solution of Eq. (1.1). In paper [15] the existence and uniqueness of the global solution are provided if equation coefficients satisfy local Yamada-Watanabe condition or local Lipschitz condition in addition to contractivity condition and assumption similar to **C1**, where $K_1(t) \equiv K_1$, $K_1 > 0$, in (2.1), if for the function $\mathcal{V} \in C^{1,2}(J \cup \mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$ there exists a function $\overline{D} \in C(\mathbb{R}_+; \mathbb{R}_+)$, such that

$$(2.2) \qquad \mathcal{L}\mathcal{V}(t, y, z) \leqslant \overline{D}(t)\Big(1 + \mathcal{V}(t, y) + \mathcal{V}(t - \rho(t, y), z)\Big),$$

for every $(t, y, z) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$. Furthermore, there are many papers about the existence and uniqueness of the global solution to neutral stochastic functional differential equations (NSFDE), which can be applied to assure that there is unique global solution to Eq. (1.1), as NSDEwSDD are special case of NSFDE. For example, in [3] the existence and uniqueness of the global solution of NSFDE are obtained under uniform Lipschitz condition and linear growth condition, in [2] under the uniform Lipschitz condition and weakened linear growth condition, as well as under the linear growth condition and local Lipschitz condition. In [7], Kim considers Hölder condition together with weakened linear growth condition. Contractivity condition is imposed in all of the papers mentioned above.

As assumption **C1** is weaker than adequate assumption in [15], we present the assertion of existence and uniqueness of the solution of Eq. (1.1), which is proved in Appendix. In order to simplify the notation, we introduce following assumptions for the coefficients of Eq. (1.1).

YW. (Yamada-Watanabe condition) For any integer $m \geqslant 1$, there exists a function $K_m : \mathbb{R}_+ \to \mathbb{R}_+$, such that, for all $x_1, x_2 \in \mathbb{R}^n$, $y_1, y_2 \in C^\psi(\mathbb{R}_-; \mathbb{R}^n)$ (or $y_1, y_2 \in C(J; \mathbb{R}^n)$), with $|x_1| \vee |x_2| \vee \|y_1\| \vee \|y_2\| \leqslant m$ and all $t \geqslant 0$,

$$|a(t, x_1, y_1) - a(t, x_2, y_2)|^2 \vee |b(t, x_1, y_1) - b(t, x_2, y_2)|^2 \leqslant K_m(|x_1 - x_2|^2 + \|y_1 - y_2\|^2).$$

Functions K_m are continuous, non-decreasing, concave, $K_m(0) = 0$ and $\int_{0+}^{+\infty} \frac{ds}{K_m(s)} = +\infty$. Also, for every $t \geqslant 0$, there exists a constant $K > 0$, such that

$$|a(t, 0, 0)|^2 \vee |b(t, 0, 0)|^2 \leqslant K.$$

LL. (Local Lipschitz condition) For any integer $m \geqslant 1$, there exists a positive constant $\overline{K}_m$, such that, for all $x_1, x_2 \in \mathbb{R}^n$, $y_1, y_2 \in BC(\mathbb{R}_-; \mathbb{R}^n)$ (or $y_1, y_2 \in C(J; \mathbb{R}^n)$), with $|x_1| \vee |x_2| \vee \|y_1\| \vee \|y_2\| \leqslant m$ and all $t \geqslant 0$,

$$|a(t, x_1, y_1) - a(t, x_2, y_2)|^2 \vee |b(t, x_1, y_1) - b(t, x_2, y_2)|^2 \leqslant \overline{K}_m(|x_1 - x_2|^2 + \|y_1 - y_2\|^2).$$

Theorem 2.1. *If one of the assumptions* **YW** *or* **LL** *hold, together with* **C1** *and* **C2** *and if there exists a function* $\overline{D} \in C(\mathbb{R}_+; \mathbb{R}_+)$, *such that (2.2) holds for every* $(t, y, z) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$, *then for any initial condition (1.2), there exists a unique global solution* $y(t)$, $t \in J \cup \mathbb{R}_+$, *of Eq. (1.1). Moreover, for every* $t \geqslant 0$

$$\sup_{s \in J \cup [0,t]} E\mathcal{V}(s, y(s)) \leqslant \frac{1}{2} c(t) d(t) e^{d(t)t},$$

where

$$c(t) = 1 + 2 \sup_{\theta \in J} E\mathcal{V}(\theta, \psi(\theta)) + 2 \sup_{s \in [0,t]} q(s)$$

$$(2.3) \qquad + 2 \sup_{s \in [0,t]} K_2(s)\left(r_1(p)E\|\psi\|^p + F^p r_2(p)\right) + 2r_3(p)E\widetilde{\mathcal{V}} \sup_{s \in [0,t]} \frac{K_2(s)}{K_1(s)},$$

$$d(t) = r_3(p) \sup_{s \in [0,t]} \frac{K_2(s)}{K_1(s)} \sup_{s \in [0,t]} \overline{D}(s) \vee 1,$$

and

$$(2.4) \qquad
\begin{aligned}
r_1(p) &= \begin{cases} \dfrac{\gamma^p}{1 - \gamma^p}, & p \in (0, 1], \\[2ex] \dfrac{\gamma}{1 - \gamma}, & p \in (1, +\infty), \end{cases}
&\qquad
r_2(p) &= \begin{cases} \dfrac{1}{1 - \gamma^p}, & p \in (0, 1], \\[2ex] \dfrac{(\sqrt{\gamma} - \gamma)^{1-p}}{1 - \gamma}, & p \in (1, +\infty), \end{cases} \\[6ex]
r_3(p) &= \begin{cases} \dfrac{1}{1 - \gamma^p}, & p \in (0, 1], \\[2ex] \dfrac{(1 - \sqrt{\gamma})^{1-p}}{1 - \gamma}, & p \in (1, +\infty). \end{cases}
\end{aligned}$$

3. Asymptotic Properties of the Solution

On the basis of the existing literature one can observe that there is no method for finding the explicit solution of Eq. (1.1). Because of that, studying its asymptotic properties is of great importance. In the following theorems, we establish some asymptotic bounds for the solution of NSDEwSDD in the p-th moment sense. In order to obtain these bounds, a condition stronger than (2.2) needs to be satisfied.

Remark 3.1. To simplify notation, we take $L_1(t, p, \varphi) := L_1(t)$, $R_1(t, p, \varphi) := R_1(t)$, $R_2(t, p, \varphi) := R_2(t)$, for $t \geqslant 0$, $\varphi \in (0, 1)$ and $p > 0$. Similarly, we also define the corresponding functions in the following theorems.

Theorem 3.1. *Let $y(t)$, $t \in J \cup \mathbb{R}_+$, be the solution of Eq. (1.1), with initial condition (1.2), and let the assumptions* **C1** *and* **C2** *hold. Suppose there exist functions $\lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}_+; \mathbb{R}_+)$, and $\varphi \in (0, 1)$, which satisfy conditions*

$$(3.1) \qquad \mathcal{L}\mathcal{V}(t, y, z) \leqslant -\lambda_1(t) + \lambda_2(t)\mathcal{V}(t, y) - \lambda_3(t)\mathcal{V}(t - \rho(t, y), z),$$

for every $(t, y, z) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$, and

$$(3.2) \qquad \max\{0, L_1(t)\} < \min\{R_1(t), R_2(t)\}, \quad t \geqslant 0,$$

(if $\max\{0, L_1(t)\} > 0$, $t \geqslant 0$, then we have $\leqslant$ instead of $<$), where

$$L_1(t) = \frac{K_2(t)\lambda_2(t)}{A_1(p, \varphi)} - K_1'(t),$$

$$(3.3) \qquad R_1(t) = \frac{\lambda_1(t) - \left(\dfrac{F^p}{\sqrt{A_2(p, \varphi)}} + \dfrac{A_1(p, \varphi)q(t)}{K_2(t)} \right) K_1'(t)}{\dfrac{F^p}{\sqrt{A_2(p, \varphi)}} + \dfrac{A_1(p, \varphi)q(t)}{K_2(t)} + \dfrac{E\widetilde{\mathcal{V}}}{K_1(0)}},$$

$$R_2(t) = \frac{A_2(p, \varphi)h(t)\lambda_3(t)}{\gamma^p A_1(p, \varphi)} - K_1'(t),$$

and

$$(3.4) \qquad h(t) = \inf_{u \in J \cup [0, t]} K_1(u),$$

(3.5)

$$A_1(p, \varphi) = \begin{cases} 1, & p \in (0, 1], \\ (1 - \varphi)^{p-1}, & p \in (1, +\infty), \end{cases} \qquad A_2(p, \varphi) = \begin{cases} 1, & p \in (0, 1], \\ \varphi^{2(p-1)}, & p \in (1, +\infty). \end{cases}$$

Then,

$$(3.6) \qquad \limsup_{t \to +\infty} E|y(t)|^p \leqslant r_1(p)E\|\psi\|^p + F^p r_2(p) + \frac{r_3(p)E\widetilde{\mathcal{V}}}{K_1(0)},$$

for $r_1(p)$, $r_2(p)$, $r_3(p)$ given by (2.4).

Proof. For $p > 1$, by applying the elementary inequality

$$(3.7) \qquad |a_1 + a_2|^p \leqslant \frac{|a_1|^p}{\xi^{p-1}} + \frac{|a_2|^p}{(1-\xi)^{p-1}}, \quad a_1 > 0, \, a_2 > 0, \quad \xi \in (0,1),$$

twice, as well as the assumption **C2**, we have

$$|y(t)|^p = |y(t) - D(t, y(t-\rho(t,y(t)))) + D(t, y(t-\rho(t,y(t))))|^p$$

$$(3.8) \qquad \leqslant \frac{|y(t) - D(t, y(t-\rho(t,y(t))))|^p}{(1-\xi)^{p-1}} + \frac{|D(t, y(t-\rho(t,y(t))))|^p}{\xi^{p-1}}$$

$$(3.9) \qquad \leqslant \frac{|y(t) - D(t, y(t-\rho(t,y(t))))|^p}{(1-\xi)^{p-1}}$$

$$+ \frac{1}{\xi^{p-1}} \left(\frac{\gamma^p |y(t-\rho(t,y(t)))|^p}{\xi^{p-1}} + \frac{F^p}{(1-\xi)^{p-1}} \right).$$

Since

$$\sup_{u \in J \cup [0,t]} E|y(u)|^p \leqslant E\|\psi\|^p + \sup_{u \in [0,t]} E|y(u)|^p,$$

we find that

$$\sup_{u \in [0,t]} E|y(u)|^p \leqslant \frac{1}{(1-\xi)^{p-1}} \sup_{u \in [0,t]} E|y(u) - D(u, y(u-\rho(u,y(u))))|^p$$

$$+ \frac{1}{\xi^{p-1}} \left(\frac{\gamma^p E\|\psi\|^p}{\xi^{p-1}} + \frac{\gamma^p}{\xi^{p-1}} \sup_{u \in [0,t]} E|y(u)|^p + \frac{F^p}{(1-\xi)^{p-1}} \right).$$

Thus,

$$\left(1 - \frac{\gamma^p}{\xi^{2(p-1)}}\right) \sup_{u \in [0,t]} E|y(u)|^p \leqslant \frac{1}{(1-\xi)^{p-1}} \sup_{u \in [0,t]} E|y(u) - D(u, y(u-\rho(u,y(u))))|^p$$

$$+ \frac{1}{\xi^{p-1}} \left(\frac{\gamma^p E\|\psi\|^p}{\xi^{p-1}} + \frac{F^p}{(1-\xi)^{p-1}} \right).$$

If we choose $\xi = \sqrt{\gamma}$, then $1 - \gamma^p \xi^{-2(p-1)} = 1 - \gamma > 0$, so the last inequality reduces to

$$\sup_{u \in [0,t]} E|y(u)|^p \leqslant \frac{\gamma}{1-\gamma} E\|\psi\|^p + \frac{F^p(\sqrt{\gamma} - \gamma)^{1-p}}{1-\gamma}$$

$$+ \frac{(1-\sqrt{\gamma})^{1-p}}{1-\gamma} \sup_{u \in [0,t]} E|y(u) - D(u, y(u-\rho(u,y(u))))|^p.$$

Using **C1** we get

$$(3.10) \qquad \sup_{u \in [0,t]} E|y(u)|^p \leqslant r_1(p) E\|\psi\|^p + F^p r_2(p)$$

$$+ r_3(p) \sup_{u \in [0,t]} \frac{E\mathcal{V}(u, y(u) - D(u, y(u-\rho(u,y(u)))))}{K_1(u)}.$$

As left side of inequality (3.2) is nonnegative, there exists an increasing function $\lambda \in C^1(\mathbb{R}_+; \mathbb{R}_+)$ with $\lambda(0) = 0$, such that $\lambda'(t) > -\frac{K_1'(t)}{K_1(t)}$ and

$$\frac{K_2(t)\lambda_2(t)}{A_1(p,\varphi)K_1(t)} \leqslant \lambda'(t) + \frac{K_1'(t)}{K_1(t)}$$

$$(3.11) \qquad \leqslant \min \left\{ \frac{A_2(p,\varphi)h(t)\lambda_3(t)}{\gamma^p A_1(p,\varphi)K_1(t)}, \frac{\dfrac{\lambda_1(t)}{K_1(t)} + \dfrac{E\widetilde{\mathcal{V}}K_1'(t)}{K_1(0)K_1(t)}}{\dfrac{F^p}{\sqrt{A_2(p,\varphi)}} + \dfrac{A_1(p,\varphi)q(t)}{K_2(t)} + \dfrac{E\widetilde{\mathcal{V}}}{K_1(0)}} \right\},$$

for $\varphi \in (0,1)$. The Itô formula gives us

$$\frac{e^{-\lambda(t)}\mathcal{V}(t, y(t) - D(t, y(t-\rho(t, y(t)))))}{K_1(t)} - \frac{\widetilde{\mathcal{V}}}{K_1(0)}$$

$$= \int_0^t \frac{e^{-\lambda(s)}}{K_1(s)} \left(-\frac{\lambda'(s)K_1(s) + K_1'(s)}{K_1(s)} \mathcal{V}(s, y(s) - D(s, y(s-\rho(s, y(s))))) \right.$$

$$\left. + \mathcal{L}\mathcal{V}(s, y(s), y(s-\rho(s, y(s)))) \right) ds + M_1(t),$$

where

$$M_1(t) = \int_0^t \frac{e^{-\lambda(s)}}{K_1(s)} \mathcal{V}_y(s, y(s) - D(s, y(s-\rho(s, y(s)))))b(s, y(s), y(s-\rho(s, y(s))))dW(s)$$

is a local martingale and $M_1(0) = 0$. Since $\lambda'(t)K_1(t) + K_1'(t) \geqslant 0$, for every $t \geqslant 0$, by applying (3.1) and **C1** respectively, the last inequality becomes

$$(3.12)$$
$$\frac{e^{-\lambda(t)}E\mathcal{V}(t, y(t) - D(t, y(t-\rho(t, y(t)))))}{K_1(t)} - \frac{E\widetilde{\mathcal{V}}}{K_1(0)}$$

$$\leqslant E \int_0^t \frac{e^{-\lambda(s)}}{K_1(s)} \left[-\frac{\lambda'(s)K_1(s) + K_1'(s)}{K_1(s)} \mathcal{V}(s, y(s) - D(s, y(s-\rho(s, y(s))))) - \lambda_1(s) \right.$$

$$\left. + \lambda_2(s)\mathcal{V}(s, y(s)) - \lambda_3(s)\mathcal{V}(s-\rho(s, y(s)), y(s-\rho(s, y(s)))) \right] ds$$

$$\leqslant E \int_0^t \frac{e^{-\lambda(s)}}{K_1(s)} \left[-(\lambda'(s)K_1(s) + K_1'(s))|y(s) - D(s, y(s-\rho(s, y(s))))|^p - \lambda_1(s) + \lambda_2(s)q(s) \right.$$

$$\left. + \lambda_2(s)K_2(s)|y(s)|^p - \lambda_3(s)K_1(s-\rho(s, y(s)))|y(s-\rho(s, y(s)))|^p \right] ds.$$

From (3.9), for $\varphi \in (0,1)$, we obtain

(3.13)
$$|y(t) - D(t, y(t-\rho(t, y(t))))|^p$$
$$\geqslant (1-\varphi)^{p-1}|y(t)|^p - \frac{\gamma^p(1-\varphi)^{p-1}|y(t-\rho(t, y(t)))|^p}{\varphi^{2(p-1)}} - \frac{F^p}{\varphi^{p-1}}.$$

By applying inequality (3.13) in (3.12) we get

(3.14)
$$\frac{e^{-\lambda(t)}E\mathcal{V}(t, y(t) - D(t, y(t-\rho(t, y(t)))))}{K_1(t)} - \frac{E\widetilde{\mathcal{V}}}{K_1(0)}$$
$$\leqslant \int_0^t \frac{e^{-\lambda(s)}}{K_1(s)}\left[\frac{F^p(\lambda'(s)K_1(s)+K_1'(s))}{\varphi^{p-1}} - \lambda_1(s) + \lambda_2(s)q(s)\right]ds$$
$$+ E\int_0^t \frac{e^{-\lambda(s)}}{K_1(s)}\left[\lambda_2(s)K_2(s) - (1-\varphi)^{p-1}(\lambda'(s)K_1(s)+K_1'(s))\right]|y(s)|^p\,ds$$
$$+ E\int_0^t e^{-\lambda(s)}\left[\frac{\gamma^p(1-\varphi)^{p-1}(\lambda'(s)K_1(s)+K_1'(s))}{\varphi^{2(p-1)}K_1(s)} - \frac{\lambda_3(s)h(s)}{K_1(s)}\right]|y(s-\rho(s, y(s)))|^p ds,$$

where $h(t)$ is given by (3.4). Coming out of (3.11), for every $t \geqslant 0$, we conclude that

$$\left(\frac{F^p}{\sqrt{A_2(p, \varphi)}} + \frac{A_1(p, \varphi)q(t)}{K_2(t)} + \frac{E\widetilde{\mathcal{V}}}{K_1(0)}\right)\lambda'(t)$$
$$\leqslant \frac{\lambda_1(t)}{K_1(t)} - \left(\frac{F^p}{\sqrt{A_2(p, \varphi)}} + \frac{A_1(p, \varphi)q(t)}{K_2(t)}\right)\frac{K_1'(t)}{K_1(t)},$$

which induces

(3.15)
$$\frac{F^p(\lambda'(t)K_1(t)+K_1'(t))}{\varphi^{p-1}K_1(t)} - \frac{\lambda_1(t)}{K_1(t)} + \frac{A_1(p, \varphi)q(t)}{K_1(t)K_2(t)}(\lambda'(t)K_1(t)+K_1'(t)) \leqslant -\frac{E\widetilde{\mathcal{V}}}{K_1(0)}\lambda'(t).$$

Also, from (3.11) we find that

$$\frac{K_2(t)\lambda_2(t)}{A_1(p, \varphi)} \leqslant \lambda'(t)K_1(t) + K_1'(t),$$

so (3.15) gives

$$\frac{1}{K_1(t)}\left[\frac{F^p(\lambda'(t)K_1(t) + K_1'(t))}{\varphi^{p-1}} - \lambda_1(t) + \lambda_2(t)q(t)\right] \leqslant -\frac{E\widetilde{\mathcal{V}}}{K_1(0)}\lambda'(t).$$

Consequently, by applying previous two inequalities and (3.11) in (3.14) we get

$$\frac{e^{-\lambda(t)}E\mathcal{V}(t, y(t) - D(t, y(t-\rho(t, y(t)))))}{K_1(t)} - \frac{E\widetilde{\mathcal{V}}}{K_1(0)} \leqslant -\frac{E\widetilde{\mathcal{V}}}{K_1(0)} \int_0^t e^{-\lambda(s)}\lambda'(s)ds$$
$$= -\frac{E\widetilde{\mathcal{V}}}{K_1(0)}(1 - e^{-\lambda(t)}),$$

such that

$$(3.16) \qquad \frac{E\mathcal{V}(t, y(t) - D(t, y(t-\rho(t, y(t)))))}{K_1(t)} \leqslant \frac{E\widetilde{\mathcal{V}}}{K_1(0)}$$

and (3.10) admits

$$\sup_{u\in[0,t]} E|y(u)|^p \leqslant r_1(p)E\|\psi\|^p + F^p r_2(p) + \frac{r_3(p)E\widetilde{\mathcal{V}}}{K_1(0)},$$

so (3.6) holds.

For $p \in (0, 1]$, from (3.8) and the inequality

$$(3.17) \qquad\qquad |a_1 + a_2|^p \leqslant |a_1|^p + |a_2|^p, \quad a_1 > 0, \, a_2 > 0,$$

we have

$$\sup_{u\in[0,t]} E|y(u)|^p \leqslant \sup_{u\in[0,t]} E|y(u) - D(u, y(u-\rho(u, y(u))))|^p$$
$$+ \gamma^p \Big(E\|\psi\|^p + \sup_{u\in[0,t]} E|y(u)|^p \Big) + F^p,$$

such that

$$\sup_{u\in[0,t]} E|y(u)|^p \leqslant \frac{1}{1-\gamma^p} \Big(\gamma^p E\|\psi\|^p + F^p + \sup_{u\in[0,t]} E|y(u) - D(u, y(u-\rho(u, y(u))))|^p \Big).$$

Consequently, taking into account (2.4), we derive (3.10). By applying (3.17) we attain

$$(3.18) \qquad |y(t) - D(t, y(t-\rho(t, y(t))))|^p \geqslant |y(t)|^p - \gamma^p|y(t-\rho(t, y(t)))|^p - F^p.$$

Substituting (3.18) into (3.12), we obtain

$$\frac{e^{-\lambda(t)}E\mathcal{V}(t, y(t) - D(t, y(t-\rho(t, y(t)))))}{K_1(t)} - \frac{E\widetilde{\mathcal{V}}}{K_1(0)}$$
$$\leqslant \int_0^t \frac{e^{-\lambda(s)}}{K_1(s)} \Big[F^p(\lambda'(s)K_1(s) + K_1'(s)) - \lambda_1(s) + \lambda_2(s)q(s) \Big] ds$$
$$+ E \int_0^t \frac{e^{-\lambda(s)}}{K_1(s)} \Big[\lambda_2(s)K_2(s) - (\lambda'(s)K_1(s) + K_1'(s)) \Big] |y(s)|^p \, ds$$
$$+ E \int_0^t e^{-\lambda(s)} \left[\frac{\gamma^p(\lambda'(s)K_1(s) + K_1'(s))}{K_1(s)} - \frac{\lambda_3(s)h(s)}{K_1(s)} \right] |y(s-\rho(s, y(s)))|^p ds.$$

Utilizing (3.2), similarly as for $p > 1$, we get (3.6).   □

Especially, if $p > 1$, when $F = 0$ in assumption **C1**, we can prove a result similar to (3.6) with the assumption slightly different from (3.2). In this case, for $p \in (0, 1]$ no new bound is achieved, other than one in (3.6), taking that $F = 0$.

Proposition 3.1. *Let* $y(t)$, $t \in J \cup \mathbb{R}_+$, *be the solution of Eq.* (1.1), *with initial condition* (1.2). *Suppose that* $p > 1$ *and that the assumptions* **C1** *and* **C2** *hold with* $F = 0$. *Also, suppose that there exist functions* $\lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}_+; \mathbb{R}_+)$, *and* $\varphi \in (0, 1)$, *such that* (3.1) *holds, for every* $(t, y, z) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$, *as well as*

$$(3.19) \qquad \max\{0, L_1(t)\} < \min\{R_1(t), R_3(t)\}, \quad t \geqslant 0,$$

(if $\max\{0, L_1(t)\} > 0$, $t \geqslant 0$, *then we have* $\leqslant$ *instead of* $<$), *with* $L_1(t)$, $R_1(t)$ *given by* (3.3), *while*

$$R_3(t) = \frac{\sqrt{A_2(p, \varphi)}h(t)\lambda_3(t)}{\gamma^p A_1(p, \varphi)} - K_1'(t),$$

where $h(t)$ *is given by* (3.4) *and* $A_1(p, \varphi)$, $A_2(p, \varphi)$ *are given by* (3.5). *Then,*

$$(3.20) \qquad \limsup_{t \to +\infty} E|y(t)|^p \leqslant \frac{\gamma}{1-\gamma} E\|\psi\|^p + \frac{E\widetilde{\mathcal{V}}}{(1-\gamma)^p K_1(0)}.$$

Proof. For $p > 1$, instead of (3.9) we get

$$(3.21) \qquad |y(t)|^p \leqslant \frac{|y(t) - D(t, y(t-\rho(t, y(t))))|^p}{(1-\xi)^{p-1}} + \frac{\gamma^p |y(t-\rho(t, y(t))))|^p}{\xi^{p-1}},$$

so for $\xi = \gamma$, in the same manner as for (3.10), we derive

$$(3.22)$$

$$\sup_{u \in [0,t]} E|y(u)|^p \leqslant (1-\gamma)^{-p} \sup_{u \in [0,t]} \frac{E\mathcal{V}\Big(u, y(u) - D(u, y(u-\rho(u, y(u))))\Big)}{K_1(u)} + \frac{\gamma}{1-\gamma} E\|\psi\|^p.$$

From (3.21), for $\varphi \in (0, 1)$, we have

$$(3.23) \qquad |y(t) - D(t, y(t-\rho(t, y(t))))|^p$$

$$\geqslant (1-\varphi)^{p-1}|y(t)|^p - \frac{\gamma^p(1-\varphi)^{p-1}|y(t-\rho(t, y(t)))|^p}{\varphi^{p-1}}$$

and then (3.12) has the form

$$\frac{e^{-\lambda(t)}E\mathcal{V}(t, y(t) - D(t, y(t-\rho(t, y(t)))))}{K_1(t)} - \frac{E\widetilde{\mathcal{V}}}{K_1(0)}$$

$$\leqslant \int_0^t \frac{e^{-\lambda(s)}}{K_1(s)}[-\lambda_1(s) + \lambda_2(s)q(s)]\, ds$$

$$+ E\int_0^t \frac{e^{-\lambda(s)}}{K_1(s)}\Big[\lambda_2(s)K_2(s) - (1-\varphi)^{p-1}(\lambda'(s)K_1(s) + K_1'(s))\Big]|y(s)|^p\, ds$$

$$+ E\int_0^t e^{-\lambda(s)}\left[\frac{\gamma^p(1-\varphi)^{p-1}(\lambda'(s)K_1(s) + K_1'(s))}{\varphi^{p-1}K_1(s)} - \frac{\lambda_3(s)h(s)}{K_1(s)}\right]|y(s-\rho(s, y(s)))|^p ds,$$

which, by using (3.19), leads to (3.16). Lastly, applying (3.16) in (3.22) gives (3.20).
□

Remark 3.2. If $K_1(t) \equiv K_1$, $t \in J \cup \mathbb{R}_+$, in the assumption **C1**, where K_1 is a positive constant, the conditions of Theorem 3.1 and Proposition 3.1 are simplified with the fact that $K_1'(t) = 0$, $t \in J \cup \mathbb{R}_+$.

In the next theorem, condition (3.1) is weakened, while the condition for functions λ_i, $i = 1, 2, 3$, is incomparable to (3.2). This expands the family of coefficients of Eq. (1.1), for which p-moment asymptotic bound of the solution can be determined. However, this bound is greater than one previously obtained in Theorem 3.1.

Theorem 3.2. *Let $y(t)$, $t \in J \cup \mathbb{R}_+$, be the solution of Eq. (1.1), with initial condition (1.2), and let the assumptions **C1** and **C2** hold, where K_1 is an increasing function, such that $\overline{h} = \inf_{\theta \in J} K_1(\theta)$. Also, assume there exist functions $\lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}_+; \mathbb{R}_+)$, and $\varphi \in (0, 1)$, which satisfy conditions*

$$(3.24) \qquad \mathcal{L}\mathcal{V}(t, y, z) \leqslant \lambda_1(t) + \lambda_2(t)\mathcal{V}(t, y) - \lambda_3(t)\mathcal{V}(t - \rho(t, y), z),$$

for every $(t, y, z) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$, and

$$(3.25) \qquad \max\{\overline{L}_1(t), \overline{L}_2(t)\} \leqslant \overline{R}_1(t), \quad t \geqslant 0,$$

where

$$\overline{R}_1(t) = K_1'(t) - \frac{K_2(t)\lambda_2(t)}{A_1(p, \varphi)}, \quad \overline{L}_1(t) = K_1'(t) - \frac{\overline{h}A_2(p, \varphi)\lambda_3(t)}{\gamma^p A_1(p, \varphi)},$$

$$(3.26) \qquad \overline{L}_2(t) = \frac{\lambda_1(t) + \left(\dfrac{F^p}{\sqrt{A_2(p, \varphi)}} + \dfrac{A_1(p, \varphi)q(t)}{K_2(t)} \right) K_1'(t)}{\dfrac{F^p}{\sqrt{A_2(p, \varphi)}} + \dfrac{A_1(p, \varphi)q(t)}{K_2(t)} + v},$$

for positive constant v, with $A_1(p, \varphi)$ and $A_2(p, \varphi)$ given by (3.5). Then,

$$(3.27) \qquad \limsup_{t \to +\infty} E|y(t)|^p \leqslant r_1(p)E\|\psi\|^p + F^p r_2(p) + r_3(p)\left(\frac{E\widetilde{\mathcal{V}}}{K_1(0)} + v \right),$$

where $r_1(p)$, $r_2(p)$ and $r_3(p)$ are given by (2.4).

Proof. Let $\lambda \in C^1(\mathbb{R}_+; \mathbb{R}_+)$ be an increasing function with $\lambda(0) = 0$. Using the Itô formula yields

$$(3.28) \qquad \frac{e^{\lambda(t)}\mathcal{V}(t, y(t) - D(t, y(t - \rho(t, y(t)))))}{K_1(t)} - \frac{\widetilde{\mathcal{V}}}{K_1(0)}$$

$$= \int_0^t \frac{e^{\lambda(s)}}{K_1(s)} \Bigg(\frac{\lambda'(s)K_1(s) - K_1'(s)}{K_1(s)} \mathcal{V}(s, y(s) - D(s, y(s - \rho(s, y(s)))))$$

$$+ \mathcal{L}\mathcal{V}(s, y(s), y(s - \rho(s, y(s)))) \Bigg) ds + M_2(t),$$

where

$$M_2(t) = \int_0^t e^{\lambda(s)} \mathcal{V}_y(s, y(s) - D(s, y(s-\rho(s, y(s)))))) b(s, y(s), y(s-\rho(s, y(s)))) dW(s)$$

is a local martingale and $M_2(0) = 0$. As (3.25) holds, we choose $\lambda'(t)$, such that, for every $t \geqslant 0$,

$$\max \left\{ \left(\frac{K_1'(t)}{K_1(t)} - \frac{\bar{h} A_2(p, \varphi) \lambda_3(t)}{\gamma^p A_1(p, \varphi) K_1(t)} \right), \frac{\lambda_1(t) + \left(\frac{F^p}{\sqrt{A_2(p, \varphi)}} + \frac{A_1(p, \varphi) q(t)}{K_2(t)} \right) K_1'(t)}{\left(\frac{F^p}{\sqrt{A_2(p, \varphi)}} + \frac{A_1(p, \varphi) q(t)}{K_2(t)} + v \right) K_1(t)} \right\}$$

(3.29)

$$\leqslant \lambda'(t) \leqslant \left(\frac{K_1'(t)}{K_1(t)} - \frac{K_2(t) \lambda_2(t)}{A_1(p, \varphi) K_1(t)} \right),$$

where $A_1(p, \varphi)$ and $A_2(p, \varphi)$ are given by (3.5). From (3.29) we see that, for every $t \geqslant 0$, $\lambda'(t) K_1(t) - K_1'(t) \leqslant 0$, so by using (3.24) and **C1** respectively, from (3.28) it follows that

(3.30)

$$\frac{e^{\lambda(t)} E \mathcal{V}(t, y(t) - D(t, y(t-\rho(t, y(t)))))}{K_1(t)} - \frac{E \widetilde{\mathcal{V}}}{K_1(0)}$$

$$\leqslant E \int_0^t \frac{e^{\lambda(s)}}{K_1(s)} \Big[\big(\lambda'(s) K_1(s) - K_1'(s) \big) |y(s) - D(s, y(s-\rho(s, y(s))))|^p + \lambda_1(s) + \lambda_2(s) q(s)$$

$$+ \lambda_2(s) K_2(s) |y(s)|^p - K_1(s-\rho(s, y(s))) \lambda_3(s) |y(s-\rho(s, y(s)))|^p \Big] ds,$$

so, for $p > 1$, the inequality (3.13) and the fact that K_1 is an increasing function imply

$$\frac{e^{\lambda(t)} E \mathcal{V}(t, y(t) - D(t, y(t-\rho(t, y(t)))))}{K_1(t)} - \frac{E \widetilde{\mathcal{V}}}{K_1(0)}$$

$$\leqslant \int_0^t e^{\lambda(s)} \left[\frac{F^p}{\sqrt{A_2(p, \varphi)}} \left(\frac{K_1'(s)}{K_1(s)} - \lambda'(s) \right) + \frac{\lambda_1(s) + \lambda_2(s) q(s)}{K_1(s)} \right] ds$$

$$+ \int_0^t e^{\lambda(s)} \left[\frac{\lambda_2(s) K_2(s)}{K_1(s)} - A_1(p, \varphi) \left(\frac{K_1'(s)}{K_1(s)} - \lambda'(s) \right) \right] E|y(s)|^p ds$$

$$+ \int_0^t e^{\lambda(s)} \left[\frac{\gamma^p A_1(p, \varphi)}{A_2(p, \varphi)} \left(\frac{K_1'(s)}{K_1(s)} - \lambda'(s) \right) - \frac{\bar{h} \lambda_3(s)}{K_1(s)} \right] E|y(s-\rho(s, y(s)))|^p ds.$$

From (3.29) we get

$$(3.31) \quad \frac{e^{\lambda(t)} E\mathcal{V}(t, y(t) - D(t, y(t-\rho(t, y(t)))))}{K_1(t)} - \frac{E\widetilde{\mathcal{V}}}{K_1(0)}$$

$$\leqslant \int_0^t e^{\lambda(s)} \left[\frac{F^p}{\sqrt{A_2(p, \varphi)}} \left(\frac{K_1'(s)}{K_1(s)} - \lambda'(s) \right) + \frac{\lambda_1(s) + \lambda_2(s)q(s)}{K_1(s)} \right] ds.$$

Then, from (3.29) we conclude that

$$\lambda_2(t) \leqslant \frac{A_1(p, \varphi)K_1(t)}{K_2(t)} \left(\frac{K_1'(t)}{K_1(t)} - \lambda'(t) \right),$$

so (3.31) generates

$$(3.32) \quad \frac{e^{\lambda(t)} E\mathcal{V}(t, y(t) - D(t, y(t-\rho(t, y(t)))))}{K_1(t)} - \frac{E\widetilde{\mathcal{V}}}{K_1(0)}$$

$$\leqslant \int_0^t e^{\lambda(s)} \left[\left(\frac{K_1'(s)}{K_1(s)} - \lambda'(s) \right) \left(\frac{F^p}{\sqrt{A_2(p, \varphi)}} + \frac{A_1(p, \varphi)q(s)}{K_2(s)} \right) + \frac{\lambda_1(s)}{K_1(s)} \right] ds.$$

From (3.29) we have

$$\left(\frac{K_1'(s)}{K_1(s)} - \lambda'(s) \right) \left(\frac{F^p}{\sqrt{A_2(p, \varphi)}} + \frac{A_1(p, \varphi)q(s)}{K_2(s)} \right) + \frac{\lambda_1(s)}{K_1(s)} \leqslant v\lambda'(t)$$

and applying last inequality in (3.32) gives us

$$(3.33) \quad \sup_{u \in [0,t]} \frac{E\mathcal{V}(u, y(u) - D(t, y(u-\rho(u, y(u)))))}{K_1(u)} \leqslant \frac{E\widetilde{\mathcal{V}}}{K_1(0)} + v.$$

Since **C1** and **C2** hold, we can derive (3.10), and, by (3.33), (3.27) holds.

In the same manner, for $p \in (0, 1]$, by applying the inequality (3.18) in (3.30) and using the assumption (3.25), we get (3.32), so (3.27) holds. $\qquad \square$

Remark 3.3. Apparently, $\overline{L}_1(t) = -R_2(t)$, as well as $\overline{R}_1(t) = -L_1(t)$. New functions are introduced in order to simplify following of the conditions.

Proposition 3.2. *Let* $y(t)$, $t \in J \cup \mathbb{R}_+$, *be the solution of Eq.* (1.1), *with initial condition* (1.2). *Suppose that* $p > 1$ *and that the assumptions* **C1** *and* **C2** *hold with* $F = 0$, *where* K_1 *is an increasing function, such that* $\overline{h} = \inf_{\theta \in J} K_1(\theta)$. *Also, assume that there exist functions* $\lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}_+; \mathbb{R}_+)$, *and* $\varphi \in (0, 1)$, *which satisfy conditions* (3.24), *for every* $(t, y, z) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$, *and*

$$(3.34) \quad \max\{\overline{L}_1(t), \overline{L}_3(t)\} \leqslant \overline{R}_1(t), \quad t \geqslant 0,$$

where $\overline{L}_1(t)$ *and* $\overline{R}_1(t)$ *are determined in* (3.26) *and*

$$\overline{L}_3(t) = K_1'(t) - \frac{\overline{h}\sqrt{A_2(p, \varphi)}\lambda_3(t)}{\gamma^p A_1(p, \varphi)},$$

for positive constant v, with $A_1(p, \varphi)$ and $A_2(p, \varphi)$ given by (3.5). Then,

$$(3.35) \qquad \limsup_{t \to +\infty} E|y(t)|^p \leqslant \frac{\gamma}{1 - \gamma} E\|\psi\|^p + (1 - \gamma)^{-p} \left(\frac{E\widetilde{\mathcal{V}}}{K_1(0)} + v \right).$$

Proof. Similarly as in proof of Theorem 3.2, using (3.23) and (3.34) in (3.30), give us (3.33) and therefore (3.35) holds. $\qquad \square$

Theorem 3.3. *Let $y(t)$, $t \in J \cup \mathbb{R}_+$, be the solution of Eq. (1.1), with initial condition (1.2), and let the assumptions **C1** and **C2** hold. Also, assume that there exist functions $\lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}_+; \mathbb{R}_+)$, such that, for every $(t, y, z) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$, (3.24) holds. If there exist $\varphi \in (0, 1)$, a function $\nu \in C(\mathbb{R}_+; \mathbb{R}_+)$, with $\int_0^{+\infty} \nu(t)dt < +\infty$, and positive constant $\overline{v}$, for which one of the conditions hold.*
  (i) For every $t \geqslant 0$,

$$(3.36) \qquad \left(\frac{1}{\sqrt{A_2(p, \varphi)}} - 1 \right) q(t) + \overline{v} K_1(t) < \frac{F^p K_2(t)}{A_1^2(p, \varphi)},$$

such that,

$$(3.37) \qquad \max \left\{ 0, \frac{K_1'(t)}{K_1(t)} \right\} < \min\{\widetilde{R}_1(t), \widetilde{R}_2(t), \widetilde{R}_3(t)\}$$

(if $\max \left\{ 0, \frac{K_1'(t)}{K_1(t)} \right\} > 0$, $t \geqslant 0$, then we have $\leqslant$ instead of $<$), where

$$\widetilde{R}_1(t) = \frac{\lambda_1(t) + \dfrac{q(t)K_1(t)\nu(t)}{K_2(t)} + \dfrac{K_1'(t)}{K_1(t)} \left[\left(\dfrac{1}{\sqrt{A_2(p, \varphi)}} - 1 \right) q(t) - \dfrac{F^p K_2(t)}{A_1^2(p, \varphi)} \right]}{\overline{v} K_1(t) + \left(\dfrac{1}{\sqrt{A_2(p, \varphi)}} - 1 \right) q(t) - \dfrac{F^p K_2(t)}{A_1^2(p, \varphi)}},$$

$$(3.38) \quad \widetilde{R}_2(t) = \frac{K_1'(t)}{K_1(t)} + \sqrt{A_2(p, \varphi)} \left(\frac{K_1(t)\nu(t)}{K_2(t)} - \lambda_2(t) \right),$$

$$\widetilde{R}_3(t) = \frac{K_1'(t)}{K_1(t)} + \frac{A_1(p, \varphi)\sqrt{A_2(p, \varphi)}h(t)\lambda_3(t)}{\gamma^p K_2(t)}.$$

  (ii) For every $t \geqslant 0$,

$$(3.39) \qquad \left(\frac{1}{\sqrt{A_2(p, \varphi)}} - 1 \right) q(t) + \overline{v} K_1(t) > \frac{F^p K_2(t)}{A_1^2(p, \varphi)},$$

such that

$$(3.40) \qquad \max \left\{ 0, \frac{K_1'(t)}{K_1(t)}, \widetilde{R}_1(t) \right\} < \min\{\widetilde{R}_2(t), \widetilde{R}_3(t)\}$$

(if $\max\left\{0, \frac{K_1'(t)}{K_1(t)}, \widetilde{R}_1(t)\right\} > 0$, $t \geqslant 0$, *then we have* $\leqslant$ *instead of* $<$*), where* $\widetilde{R}_1(t)$, $\widetilde{R}_2(t)$ *and* $\widetilde{R}_3(t)$ *are given by* (3.38).

Here, $A_1(p, \varphi)$ *and* $A_2(p, \varphi)$ *are given by* (3.5) *and* $h(t)$ *given by* (3.4). *Then,*

$$(3.41) \quad \limsup_{t \to +\infty} E|y(t)|^p \leqslant \left[r_1(p)E\|\psi\|^p + F^p r_2(p) + r_3(p)\left(\frac{E\widetilde{\mathcal{V}}}{K_1(0)} + \overline{v}\right)\right] e^{r_3(p)\int_0^{+\infty} \nu(t)dt},$$

where $r_1(p)$, $r_2(p)$, $r_3(p)$ *are given by* (2.4).

Proof. (i) Let $\lambda \in C^1(\mathbb{R}_+; \mathbb{R}_+)$ be an increasing function with $\lambda(0) = 0$. From (3.36) and (3.37), there exists $\lambda'(t)$, such that, for every $t \geqslant 0$,

$$\max\left\{-\frac{K_1'(t)}{K_1(t)}, 0\right\} \leqslant \lambda'(t) - \frac{K_1'(t)}{K_1(t)}$$

$$\leqslant \min\left\{\frac{\lambda_1(t) + \dfrac{q(t)K_1(t)\nu(t)}{K_2(t)} - \overline{v}K_1'(t)}{\overline{v}K_1(t) + \left(\dfrac{1}{\sqrt{A_2(p,\varphi)}} - 1\right)q(t) - \dfrac{F^p K_2(t)}{A_1^2(p,\varphi)}},\right.$$

$$(3.42) \qquad \left. \sqrt{A_2(p,\varphi)}\left(\frac{K_1(t)\nu(t)}{K_2(t)} - \lambda_2(t)\right), \frac{A_1(p,\varphi)\sqrt{A_2(p,\varphi)}h(t)\lambda_3(t)}{\gamma^p K_2(t)}\right\},$$

with $A_1(p, \varphi)$ and $A_2(p, \varphi)$ given by (3.5), which provides that $K_1(t)\lambda'(t) - K_1'(t) \geqslant 0$. By using the Itô formula we can derive (3.28) and then, by using (3.24) and **C1** respectively, it follows that

$$(3.43) \quad \frac{e^{\lambda(t)}E\mathcal{V}(t, y(t) - D(t, y(t-\rho(t, y(t)))))}{K_1(t)} - \frac{E\widetilde{\mathcal{V}}}{K_1(0)}$$

$$\leqslant E\int_0^t \frac{e^{\lambda(s)}}{K_1(s)}\left[\frac{\lambda'(s)K_1(s) - K_1'(s)}{K_1(s)}\Big(q(s) + K_2(s)|y(s) - D(s, y(s-\rho(s, y(s))))|^p\Big)\right.$$

$$+ \lambda_1(s) + \lambda_2(s)\Big(q(s) + K_2(s)|y(s)|^p\Big)$$

$$\left. - K_1(s-\rho(s, y(s)))\lambda_3(s)|y(s-\rho(s, y(s)))|^p\right]ds.$$

For $\varphi \in (0, 1)$ and $p > 1$, inequality (3.7) and **C2** give us

$$|y(t) - D(t, y(t-\rho(t, y(t))))|^p \leqslant \frac{|y(t)|^p}{\varphi^{p-1}} + \frac{|D(t, y(t-\rho(t, y(t))))|^p}{(1-\varphi)^{p-1}}$$

$$\leqslant \frac{|y(t)|^p}{\varphi^{p-1}} + \frac{\gamma^p|y(t-\rho(t, y(t)))|^p}{\varphi^{p-1}(1-\varphi)^{p-1}} + \frac{F^p}{(1-\varphi)^{2(p-1)}}.$$

Applying the last inequality in (3.43) we get

(3.44)

$$
\frac{e^{\lambda(t)}E\mathcal{V}(t, y(t) - D(t, y(t-\rho(t,y(t)))))}{K_1(t)} - \frac{E\widetilde{\mathcal{V}}}{K_1(0)}
$$

$$
\leqslant \int_0^t \frac{e^{\lambda(s)}}{K_1(s)}\left[\frac{K_1(s)\lambda'(s) - K_1'(s)}{K_1(s)}\left(q(s) + \frac{F^p K_2(s)}{A_1^2(p,\varphi)}\right) + \lambda_1(s) + \lambda_2(s)q(s)\right]ds
$$

$$
+ \int_0^t \frac{e^{\lambda(s)}}{K_1(s)}\left[\frac{(K_1(s)\lambda'(s) - K_1'(s))K_2(s)}{\sqrt{A_2(p,\varphi)}K_1(s)} + \lambda_2(s)K_2(s)\right]E|y(s)|^p ds
$$

$$
+ \int_0^t \frac{e^{\lambda(s)}}{K_1(s)}\left[\frac{\gamma^p\left(K_1(s)\lambda'(s) - K_1'(s)\right)K_2(s)}{A_1(p,\varphi)\sqrt{A_2(p,\varphi)}K_1(s)} - h(s)\lambda_3(s)\right]E|y(s-\rho(s,y(s)))|^p ds,
$$

where $h(t)$ is given by (3.4). Inequalities in (3.42) provide that, for every $t \geqslant 0$,

$$
\frac{\gamma^p\left(K_1(t)\lambda'(t) - K_1'(t)\right)K_2(t)}{A_1(p,\varphi)\sqrt{A_2(p,\varphi)}K_1(t)} - h(t)\lambda_3(t) \leqslant 0,
$$

$$
\frac{(K_1(t)\lambda'(t) - K_1'(t))K_2(t)}{\sqrt{A_2(p,\varphi)}K_1(t)} + \lambda_2(t)K_2(t) \leqslant K_1(t)\nu(t),
$$

such that (3.44) becomes

(3.45)
$$
\frac{e^{\lambda(t)}E\mathcal{V}(t, y(t) - D(t, y(t-\rho(t,y(t)))))}{K_1(t)} - \frac{E\widetilde{\mathcal{V}}}{K_1(0)}
$$

$$
\leqslant \int_0^t \frac{e^{\lambda(s)}}{K_1(s)}\left[\frac{K_1(s)\lambda'(s) - K_1'(s)}{K_1(s)}\left(\left(1 - \frac{1}{\sqrt{A_2(p,\varphi)}}\right)q(s) + \frac{F^p K_2(s)}{A_1^2(p,\varphi)}\right)\right.
$$

$$
\left. + \lambda_1(s) + \frac{q(s)K_1(s)\nu(s)}{K_2(s)}\right]ds + \int_0^t e^{\lambda(s)}\nu(s)E|y(s)|^p ds.
$$

Also, from (3.42) we conclude that

$$
\frac{K_1(t)\lambda'(t) - K_1'(t)}{K_1(t)}\left(\left(1 - \frac{1}{\sqrt{A_2(p,\varphi)}}\right)q(t) + \frac{F^p K_2(t)}{A_1^2(p,\varphi)}\right) + \lambda_1(t) + \frac{q(t)K_1(t)\nu(t)}{K_2(t)}
$$

$$
\leqslant \overline{v}K_1(t)\lambda'(t),
$$

and (3.45) has the form

$$
\frac{e^{\lambda(t)}E\mathcal{V}(t, y(t) - D(t, y(t-\rho(t,y(t)))))}{K_1(t)} - \frac{E\widetilde{\mathcal{V}}}{K_1(0)} \leqslant \overline{v}\int_0^t e^{\lambda(s)}\lambda'(s)ds
$$

$$
+ \int_0^t e^{\lambda(s)}\nu(s)E|y(s)|^p ds.
$$

As $e^{\lambda(s)-\lambda(t)} \leqslant 1$, for $0 \leqslant s \leqslant t$, considering the last inequality we have

$$(3.46) \qquad \frac{E\mathcal{V}(t, y(t) - D(t, y(t-\rho(t, y(t)))))}{K_1(t)} \leqslant \frac{E\widetilde{\mathcal{V}}e^{-\lambda(t)}}{K_1(0)} + \overline{v}(1 - e^{-\lambda(t)})$$
$$+ \int_0^t \nu(s)E|y(s)|^p ds.$$

According to (3.46), (3.10) provides

$$\sup_{u \in [0,t]} E|y(u)|^p \leqslant r_1(p)E\|\psi\|^p + F^p r_2(p) + r_3(p)\left(\frac{E\widetilde{\mathcal{V}}}{K_1(0)} + \overline{v} + \int_0^t \nu(s)\sup_{r \in [0,s]} E|y(r)|^p ds\right),$$

so applying Theorem 1.1 in the last inequality yields (3.41).

Furthermore, for $p \in (0, 1]$ inequalities (3.17) and **C2** generate

$$|y(t) - D(t, y(t-\rho(t, y(t))))|^p \leqslant |y(t)|^p + \gamma^p|y(t-\rho(t, y(t)))|^p + F^p.$$

The last inequality used in (3.43) gives (3.44), so (3.41) holds.

(*ii*) If (3.39) holds, there exists an increasing function $\overline{\lambda} \in C(\mathbb{R}_+; \mathbb{R}_+)$ with $\overline{\lambda}(0) = 0$, such that

$$\max\left\{-\frac{K_1'(t)}{K_1(t)}, 0, \widetilde{R}_1(t) - \frac{K_1'(t)}{K_1(t)}\right\} \leqslant \overline{\lambda}'(t) - \frac{K_1'(t)}{K_1(t)} \leqslant \min\left\{\widetilde{R}_2(t) - \frac{K_1'(t)}{K_1(t)}, \widetilde{R}_3(t) - \frac{K_1'(t)}{K_1(t)}\right\},$$

where $\widetilde{R}_1(t)$, $\widetilde{R}_2(t)$ and $\widetilde{R}_3(t)$ are given by (3.38). By repeating completely the procedure as in the first part of the proof, we get (3.41). $\qquad \square$

Proposition 3.3. *Let $y(t)$, $t \in J \cup \mathbb{R}_+$, be the solution of Eq. (1.1), with initial condition (1.2). Suppose that $p > 1$ and that the assumptions **C1** and **C2** hold with $F = 0$. Also, assume that there exist functions $\lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}_+; \mathbb{R}_+)$, such that, for every $(t, y, z) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$, (3.24) holds. If there exist $\varphi \in (0, 1)$, $\nu \in C(\mathbb{R}_+; \mathbb{R}_+)$, with $\int_0^{+\infty} \nu(t)dt < +\infty$, and positive constant $\overline{v}$ satisfying one of the following conditions.*

(i) For every $t \geqslant 0$,

$$\max\left\{0, \frac{K_1'(t)}{K_1(t)}\right\} < \min\{\widetilde{R}_1(t), \widetilde{R}_2(t), \widetilde{R}_4(t)\}$$

(if $\max\left\{0, \frac{K_1'(t)}{K_1(t)}\right\} > 0$, $t \geqslant 0$, then we have $\leqslant$ instead of $<$) and (3.36), where $\widetilde{R}_1(t)$ and $\widetilde{R}_2(t)$ are determined in (3.38), while

$$\widetilde{R}_4(t) = \frac{A_1(p, \varphi)h(t)\lambda_3(t)}{\gamma^p K_2(t)} + \frac{K_1'(t)}{K_1(t)}.$$

(ii) For every $t \geqslant 0$,

$$\max\left\{0, \frac{K_1'(t)}{K_1(t)}, \widetilde{R}_1(t)\right\} < \min\{\widetilde{R}_2(t), \widetilde{R}_4(t)\}$$

(if $\max\left\{0, \frac{K_1'(t)}{K_1(t)}, \widetilde{R}_1(t)\right\} > 0$, $t \geqslant 0$, *then we have* $\leqslant$ *instead of* $<$) *and* (3.39).

Here $A_1(p, \varphi)$ and $A_2(p, \varphi)$ are defined in (3.5) and $h(t)$ is given by (3.4). Then,

$$(3.47) \qquad \limsup_{t \to +\infty} E|y(t)|^p \leqslant \left[\frac{\gamma}{1-\gamma} E\|\psi\|^p + r_3(p)\left(\frac{E\widetilde{\mathcal{V}}}{K_1(0)} + \overline{v}\right)\right] e^{r_3(p)\int_0^{+\infty} \nu(t)dt},$$

where $r_1(p)$, $r_2(p)$, $r_3(p)$ *are given by* (2.4).

Proof. For $\varphi \in (0, 1)$, inequality (3.7) and **C2** give us

$$|y(t) - D(t, y(t - \rho(t, y(t))))|^p \leqslant \frac{|y(t)|^p}{\varphi^{p-1}} + \frac{\gamma^p|y(t-\rho(t, y(t)))|^p}{(1 - \varphi)^{p-1}},$$

so, as in the proof of Theorem 3.3, applying the last inequality together with (3.37) provides (3.46). Using (3.46) in (3.22) leads to (3.47).

In the case where (3.39) holds instead of (3.36), the proof is similar. $\square$

4. EXAMPLE

Example 4.1. We inspect neutral stochastic differential equation of the form (1.1), with local Lipschitz drift and diffusion coefficients

$$a : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad a(t, y, z) = -l'(t)\left(y - \frac{z}{4}\right)\left[\frac{9}{8} + e^{\frac{3l(t)}{8}}\left(e^{-\frac{l(t)}{4}} + l(t)\left(y - \frac{z}{4}\right)^2\right)\right],$$

$$b : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad b(t, y, z) = \frac{1}{4}\sqrt{\frac{l'(t)}{l(t)}} e^{\frac{5l(t)}{16}}\left[e^{-\frac{l(t)}{4}} + l(t)\left(y - \frac{z}{4}\right)^2\right]^{\frac{3}{2}},$$

for the increasing function

$$l : \left[-\frac{4}{5}, +\infty\right) \to \left[\frac{1}{6}, 1\right), \quad l(t) = \frac{1+t}{2+t}.$$

The neutral term is of the form

$$D : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, \quad D(t, z) = \frac{z}{4},$$

where $\psi(\theta) = \theta + 1$, $\theta \in \left[-\frac{4}{5}, 0\right]$, is the initial condition and state-dependent delay function is given by

$$\rho : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+, \quad \rho(t, y) = t + \frac{4(y^2 + 1)}{5y^2 + 6}.$$

As

$$|D(t, z_1) - D(t, z_2)| = \frac{1}{4}|z_1 - z_2|, \quad (t, z_1), (t, z_2) \in \mathbb{R}_+ \times \mathbb{R},$$

assumption **C2** is satisfied, with $F = 0$. Taking

$$\mathcal{V} : \left[-\frac{4}{5}, +\infty\right) \times \mathbb{R} \to \mathbb{R}_+, \quad \mathcal{V}(t, y) = \sqrt{e^{-\frac{l(t)}{4}} + l(t)y^2} + e^{-l(t)} - e^{-1},$$

we can estimate

$$(4.1) \qquad \sqrt{l(t)}|y| \leqslant \mathcal{V}(t,y) \leqslant e^{-\frac{l(t)}{8}} + e^{-l(t)} - e^{-1} + \sqrt{l(t)}|y|, \quad (t,y) \in \mathbb{R}_+ \times \mathbb{R},$$

$$\mathcal{V}_t(t,y) = \frac{\left(y^2 - \frac{1}{4}e^{-\frac{l(t)}{4}}\right)l'(t)}{2\sqrt{e^{-\frac{l(t)}{4}} + l(t)y^2}} - l'(t)e^{-l(t)}, \quad \mathcal{V}_y(t,y) = \frac{l(t)y}{\sqrt{e^{-\frac{l(t)}{4}} + l(t)y^2}},$$

$$\mathcal{V}_{yy}(t,y) = \frac{l(t)e^{-\frac{l(t)}{4}}}{\sqrt{\left(e^{-\frac{l(t)}{4}} + l(t)y^2\right)^3}}.$$

Comparing (4.1) with (2.1), we conclude that

$$p = 1, \quad K_1(t) = K_2(t) = \sqrt{l(t)}, \quad q(t) = e^{-\frac{l(t)}{8}} + e^{-l(t)} - e^{-1}, \quad t \geqslant 0,$$

$$E\widetilde{\mathcal{V}} = E\mathcal{V}(0, \psi(0)) - D(0, \psi(-\rho(0, \psi(0)))) = \sqrt{e^{-\frac{1}{8}} + \frac{1}{2} \cdot \left(\frac{3}{44}\right)^2} + e^{-\frac{1}{2}} - e^{-1}.$$

By applying some elementary inequalities, for every $t \geqslant 0$ and $y, z \in \mathbb{R}$, we estimate the operator in (1.3) as follows

$$\mathcal{L}\mathcal{V}(t,y,z) = -\frac{l'(t)}{2}\frac{\frac{1}{4}e^{-\frac{l(t)}{4}} - (y - \frac{z}{4})^2}{\sqrt{e^{\frac{l(t)}{4}} + l(t)(y - \frac{z}{4})^2}} - l'(t)e^{-l(t)}$$

$$- \frac{l(t)l'(t)(y - \frac{z}{4})^2}{\sqrt{e^{-\frac{l(t)}{4}} + l(t)(y - \frac{z}{4})^2}}\left[\frac{9}{8} + e^{\frac{l(t)}{8}} + e^{\frac{3l(t)}{8}}l(t)\left(y - \frac{z}{4}\right)^2\right]$$

$$+ \frac{l'(t)e^{\frac{3l(t)}{8}}}{32\sqrt{\left(e^{-\frac{l(t)}{4}} + l(t)(y - \frac{z}{4})^2\right)^3}}\left[e^{-\frac{l(t)}{4}} + l(t)\left(y - \frac{z}{4}\right)^2\right]^3$$

$$\leqslant -\frac{l'(t)}{8}\sqrt{e^{-\frac{l(t)}{4}} + l(t)\left(y - \frac{z}{4}\right)^2} + \frac{l'(t)}{2}\frac{\left(\frac{l(t)}{4} + 1\right)\left(y - \frac{z}{4}\right)^2}{\sqrt{e^{-\frac{l(t)}{4}} + l(t)(y - \frac{z}{4})^2}} - l'(t)e^{-l(t)}$$

$$- \frac{l(t)l'(t)\left(\frac{9}{8} + e^{\frac{l(t)}{8}}\right)\left(y - \frac{z}{4}\right)^2}{\sqrt{e^{-\frac{l(t)}{4}} + l(t)(y - \frac{z}{4})^2}} - \frac{l^2(t)l'(t)e^{\frac{3l(t)}{8}}(y - \frac{z}{4})^4}{\sqrt{e^{-\frac{l(t)}{4}} + l(t)(y - \frac{z}{4})^2}}$$

$$+ \frac{l'(t)e^{\frac{3l(t)}{8}}}{16\sqrt{e^{-\frac{l(t)}{4}} + l(t)(y - \frac{z}{4})^2}}\left[e^{-\frac{l(t)}{2}} + l^2(t)\left(y - \frac{z}{4}\right)^4\right].$$

As, for $(t, y) \in \mathbb{R}_+ \times \mathbb{R}^n$, $-\frac{4}{5} < t - \rho(t, y) \leqslant -\frac{2}{3}$, we have that $\frac{1}{6} < l(t - \rho(t, y)) \leqslant \frac{1}{4}$, so using reverse triangle inequality gives

$$
-\sqrt{e^{-\frac{l(t)}{4}} + l(t)\left(y - \frac{z}{4}\right)^2}
$$

$$
\leqslant -\frac{1}{\sqrt{2}}\left(\sqrt{e^{-\frac{l(t)}{4}}} + \sqrt{l(t)}\left(-|y| + \frac{|z|}{4}\right)\right)
$$

$$
\leqslant -\sqrt{2}e^{-\frac{l(t)}{8}} + \mathcal{V}(t, y) + e^{-1} - e^{-l(t)}
$$

$$
-\frac{1}{4}\sqrt{\frac{l(t)}{2l(t - \rho(t, y))}}\left(\mathcal{V}(t - \rho(t, y), z) - e^{-\frac{l(t-\rho(t,y))}{8}} - e^{-l(t-\rho(t,y))} + e^{-1}\right)
$$

$$
\leqslant -\sqrt{2}e^{-\frac{l(t)}{8}} + e^{-1} - e^{-l(t)} + \frac{\sqrt{3l(t)}}{4}\left(e^{-\frac{1}{48}} + e^{-\frac{1}{6}} - e^{-1}\right) + \mathcal{V}(t, y) - \frac{\sqrt{2l(t)}}{4}\mathcal{V}(t - \rho(t, y), z),
$$

so

$$
\mathcal{L}\mathcal{V}(t, y, z)
$$

$$
\leqslant \frac{l'(t)}{8}\left[-\sqrt{2}e^{-\frac{l(t)}{8}} + e^{-1} - e^{-l(t)} + \frac{\sqrt{3l(t)}}{4}\left(e^{-\frac{1}{48}} + e^{-\frac{1}{6}} - e^{-1}\right)\right] + \frac{l'(t)}{8}\mathcal{V}(t, y)
$$

$$
-\frac{l'(t)\sqrt{2l(t)}}{32}\mathcal{V}(t - \rho(t, y), z) - l'(t)e^{-l(t)} + \frac{l'(t)}{16} - \frac{15l^2(t)l'(t)e^{\frac{3l(t)}{8}}\left(y - \frac{z}{4}\right)^4}{16\sqrt{e^{-\frac{l(t)}{4}} + l(t)(y - \frac{z}{4})^2}}
$$

$$
+\frac{l'(t)\left(y - \frac{z}{4}\right)^2}{2\sqrt{e^{-\frac{l(t)}{4}} + l(t)(y - \frac{z}{4})^2}}\left[1 - 2l(t) - 2l(t)e^{\frac{l(t)}{8}}\right]
$$

$$
\leqslant \frac{l'(t)}{8}\left[-\sqrt{2}e^{-\frac{l(t)}{8}} + e^{-1} - 9e^{-l(t)} + \frac{\sqrt{3l(t)}}{4}\left(e^{-\frac{1}{48}} + e^{-\frac{1}{6}} - e^{-1}\right) + \frac{1}{2}\right] + \frac{l'(t)}{8}\mathcal{V}(t, y)
$$

$$
-\frac{l'(t)\sqrt{2l(t)}}{32}\mathcal{V}(t - \rho(t, y), z).
$$

Since $-\sqrt{2}e^{-\frac{l(t)}{8}} + e^{-1} - 9e^{-l(t)} + \frac{\sqrt{3l(t)}}{4}\left(e^{-\frac{1}{48}} + e^{-\frac{1}{6}} - e^{-1}\right) + \frac{1}{2} < 0$, for $t \geqslant 0$, it is clear that (3.24) holds with

$$
\lambda_1(t) = 0, \quad \lambda_2(t) = \frac{l'(t)}{8}, \quad \lambda_3(t) = \frac{l'(t)\sqrt{2l(t)}}{32}.
$$

Taking $\overline{v} = 0.5$, (3.39) obviously holds. Also, for $\nu(t) = \frac{1}{4(2+t)^2}$, we have that

$$\widetilde{R}_1(t) = \frac{e^{-\frac{l(t)}{8}} + e^{-l(t)} - e^{-1}}{4(2+t)^2\sqrt{l(t)}}, \quad \widetilde{R}_2(t) = \frac{1}{4(2+t)^2} + \frac{l'(t)}{2}\left(\frac{1}{l(t)} - \frac{1}{4}\right),$$

$$\widetilde{R}_3(t) = \frac{l'(t)}{2}\left(\frac{1}{4\sqrt{3}} + \frac{1}{l(t)}\right)$$

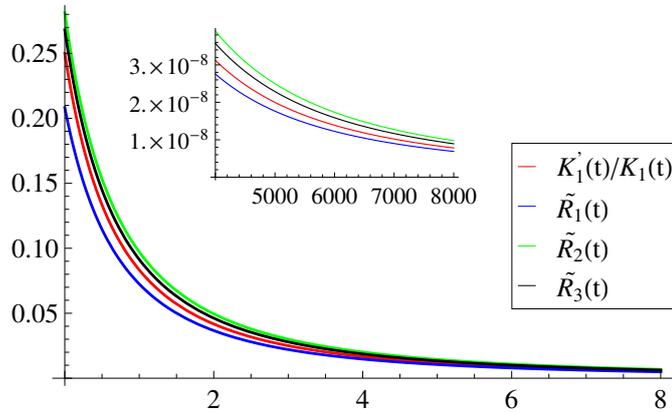and inequality (3.40) holds, which can be noticed in Figure 1.

FIGURE 1. Functions in (3.40)

According to Theorem 3.3 and (3.41), it follows that

$$\limsup_{t \to +\infty} E|y(t)| \leqslant \left[\frac{1}{3} + \frac{4}{3}\left(\sqrt{2\left(e^{-\frac{1}{8}} + \frac{1}{2}\cdot\left(\frac{3}{44}\right)^2\right)} + \sqrt{2}(e^{-\frac{1}{2}} - e^{-1}) + 0.5\right)\right]e^{\frac{1}{6}} \approx 3.81.$$

In this example, it should be pointed out that Theorem 3.1 and Theorem 3.2 do not hold, because (3.2) and (3.25) are not satisfied. Also, having in mind that $K_1(t) \geqslant \frac{1}{\sqrt{2}}$, we can take $K_1(t) \equiv \frac{1}{\sqrt{2}}$, $t \in \mathbb{R}_+$, while $\overline{h} = \frac{1}{\sqrt{6}}$. However, in that case, the conditions of Theorem 3.1, 3.2 or 3.3 are not satisfied.

5. CONCLUSION

The main contribution of the paper are the asymptotic properties of the solution to Eq. (1.1). Techniques used in proofs of assertions are influenced by the properties of delay function, which depend on the state of the system, and by the fact that it appears in some integrals. Moreover, it is important to mention that we provide more general Khasminskii-type conditions for the existence and uniqueness of the solution to Eq. (1.1), compared to the conditions in [15], which expands the number of equations for which the existence of unique solution is assured. It should be noted that studying different types of stability of the trivial solution of Eq. (1.1) is not part of this paper, but is the subject of our current research. Since the existence of a stable

trivial solution would require stronger assumptions than those given in Section 3, it is questionable whether the study of stability under conditions in this paper would be useful in the sense that not many equations of type (1.1) would satisfy them. With the intention of achieving the stability of different types under optimal conditions, we continue our research by trying to implement different techniques in our proofs. Also, future plans for our research include numerical methods for solving Eq. (1.1).

APPENDIX

As the existence and uniqueness of solution to Eq. (1.1) under the assumption **C1** are not the main focus of this paper, but still they are important to provide to discuss the asymptotic properties of Eq. (1.1), we give the proof of Theorem 2.1 here.

Proof of Theorem 2.1. As $\psi \in C^b_{\mathcal{F}_0}(J; \mathbb{R}^n)$, there exists sufficiently large $m_0 > 0$, such that $\|\psi\| \leqslant m_0$. For each integer $m \geqslant m_0$, define the increasing sequence of the stopping times $\{\nu_m\}_{m \geqslant m_0}$, with

$$\nu_m = \inf\{t \in [0, \nu_e) : |y(t)| \geqslant m\},$$

where ν_e is the explosion time, taking $\inf \emptyset = +\infty$, for an empty set $\emptyset$ and $\nu_\infty = \lim_{m \to +\infty} \nu_m$. Obviously, $\nu_\infty \leqslant \nu_e$ a.s. For arbitrary $\eta > 0$ and $p > 1$, by using inequality

$$|a_1 + a_2|^p \leqslant \left(1 + \eta^{\frac{1}{p-1}}\right)^{p-1} \left(|a_1|^p + \frac{|a_2|^p}{\eta}\right), \quad a_1, a_2 \in \mathbb{R},$$

(see [8], Lemma 4.1) and **C2**, we get

$$
\begin{aligned}
(5.1) \quad &|y(\nu_m \wedge t)|^p \\
&\leqslant \left(1 + \eta^{\frac{1}{p-1}}\right)^{p-1} |y(\nu_m \wedge t) - D(\nu_m \wedge s, y(\nu_m \wedge s - \rho(\nu_m \wedge s, y(\nu_m \wedge s))))|^p \\
&\quad + \eta^{-1} \left(1 + \eta^{\frac{1}{p-1}}\right)^{p-1} |D(\nu_m \wedge s, y(\nu_m \wedge s - \rho(\nu_m \wedge s, y(\nu_m \wedge s))))|^p \\
&\leqslant \left(1 + \eta^{\frac{1}{p-1}}\right)^{p-1} |y(\nu_m \wedge t) - D(\nu_m \wedge s, y(\nu_m \wedge s - \rho(\nu_m \wedge s, y(\nu_m \wedge s))))|^p \\
&\quad + \eta^{-1} \left(1 + \eta^{\frac{1}{p-1}}\right)^{2(p-1)} \left(\eta^{-1} \gamma^p |y(\nu_m \wedge s - \rho(\nu_m \wedge s, y(\nu_m \wedge s)))|^p + \sup_{s \in [0,t]} |D(s, 0)|^p\right).
\end{aligned}
$$

As

$$
\begin{aligned}
&\sup_{s \in [0,t]} K_2(s) E |y(\nu_m \wedge s - \rho(\nu_m \wedge s, y(\nu_m \wedge s)))|^p \\
&\leqslant E \|\psi\|^p \sup_{s \in J} K_2(s) + \sup_{s \in [0,t]} K_2(s) E |y(\nu_m \wedge s)|^p,
\end{aligned}
$$

from (5.1), we carry out

$$\sup_{s\in[0,t]} K_2(s)E|y(\nu_m\wedge s)|^p$$

$$\leqslant\left(1+\eta^{\frac{1}{p-1}}\right)^{p-1}\sup_{s\in[0,t]} K_2(s)E|y(\nu_m\wedge s)-D(\nu_m\wedge s,y(\nu_m\wedge s-\rho(\nu_m\wedge s,y(\nu_m\wedge s))))|^p$$

$$+\eta^{-2}\gamma^p\left(1+\eta^{\frac{1}{p-1}}\right)^{2(p-1)}\left(E\|\psi\|^p\sup_{s\in J} K_2(s)+\sup_{s\in[0,t]} K_2(s)E|y(\nu_m\wedge s)|^p\right)$$

$$+\eta^{-1}\left(1+\eta^{\frac{1}{p-1}}\right)^{2(p-1)}\sup_{s\in[0,t]} K_2(s)\sup_{s\in[0,t]}|D(s,0)|^p,$$

so

$$\left(1-\eta^{-2}\gamma^p\left(1+\eta^{\frac{1}{p-1}}\right)^{2(p-1)}\right)\sup_{s\in[0,t]} K_2(s)E|y(\nu_m\wedge s)|^p$$

$$\leqslant\left(1+\eta^{\frac{1}{p-1}}\right)^{p-1}\sup_{s\in[0,t]} K_2(s)E|y(\nu_m\wedge s)-D(\nu_m\wedge s,y(\nu_m\wedge s-\rho(\nu_m\wedge s,y(\nu_m\wedge s))))|^p$$

$$+\eta^{-1}\left(1+\eta^{\frac{1}{p-1}}\right)^{2(p-1)}\left(\eta^{-1}\gamma^p E\|\psi\|^p\sup_{s\in[0,t]} K_2(s)+F^p\sup_{s\in[0,t]} K_2(s)\right).$$

Choosing $\eta=\left(\frac{\sqrt{\gamma}}{1-\sqrt{\gamma}}\right)^{p-1}$, we have $1-\eta^{-2}\gamma^p\left(1+\eta^{\frac{1}{p-1}}\right)^{2(p-1)}=1-\gamma>0$, so the last inequality becomes

(5.2)

$$\sup_{s\in[0,t]} K_2(s)E|y(\nu_m\wedge s)|^p$$

$$\leqslant\frac{(1-\sqrt{\gamma})^{1-p}}{1-\gamma}\sup_{s\in[0,t]} K_2(s)E|y(\nu_m\wedge s)-D(\nu_m\wedge s,y(\nu_m\wedge s-\rho(\nu_m\wedge s,y(\nu_m\wedge s))))|^p$$

$$+\frac{\gamma}{1-\gamma}E\|\psi\|^p\sup_{s\in[0,t]} K_2(s)+\frac{F^p(\sqrt{\gamma}-\gamma)^{1-p}}{1-\gamma}\sup_{s\in[0,t]} K_2(s).$$

For $p\in(0,1]$, implementing inequality (3.17), instead of (5.1) we have

$$\sup_{s\in[0,t]} K_2(s)E|y(\nu_m\wedge s)|^p$$

$$\leqslant\sup_{s\in[0,t]} K_2(s)E|y(\nu_m\wedge s)-D(\nu_m\wedge s,y(\nu_m\wedge s-\rho(\nu_m\wedge s,y(\nu_m\wedge s))))|^p$$

$$+\gamma^p\left(E\|\psi\|^p\sup_{s\in[0,t]} K_2(s)+\sup_{s\in[0,t]} K_2(s)E|y(\nu_m\wedge s)|^p\right)+F^p\sup_{s\in[0,t]} K_2(s),$$

which leads to

(5.3)

$$\sup_{s\in[0,t]} K_2(s)E|y(\nu_m \wedge s)|^p$$

$$\leqslant \frac{1}{1-\gamma^p}\Big(\sup_{s\in[0,t]} K_2(s)E|y(\nu_m \wedge s) - D(\nu_m \wedge s, y(\nu_m \wedge s-\rho(\nu_m \wedge s, y(\nu_m \wedge s))))|^p$$

$$+ \gamma^p E\|\psi\|^p \sup_{s\in[0,t]} K_2(s) + F^p \sup_{s\in[0,t]} K_2(s)\Big).$$

The generalized Itô formula, (1.3) and (2.2) give us

$$d\mathcal{V}\Big(t, y(t) - D(t,\ y(t-\rho(t, y(t))))\Big)$$

$$=\mathcal{L}\mathcal{V}\Big(t, y(t), y(t-\rho(t, y(t)))\Big)dt + dM(t)$$

(5.4) $$\leqslant \overline{D}(t)\Big(1 + \mathcal{V}(t, y(t)) + \mathcal{V}(t-\rho(t, y(t)), y(t-\rho(t, y(t))))\Big)\,dt + dM(t),$$

where

$$M(t) = \int_0^t V_y(s, y(s) - D(s, y(s-\rho(s, y(s)))))\, b(s, y(s), y(s-\rho(s, y(s))))\,dW(s)$$

is a local martingale and $M(0) = 0$. For every $m \geqslant m_0$, integrating and taking the expectation of both sides of (5.4), for every $t \geqslant 0$ we have

(5.5) $$E\mathcal{V}\Big(\nu_m \wedge t, y(\nu_m \wedge t) - D(\nu_m \wedge t, y(\nu_m \wedge t - \rho(\nu_m \wedge t, y(\nu_m \wedge t))))\Big) - \widetilde{\mathcal{V}}$$

$$\leqslant \sup_{s\in[0,t]} \overline{D}(s) \int_0^t \Big(1+2\sup_{s\in J\cup[0,s]} E\mathcal{V}(\nu_m \wedge s, y(\nu_m \wedge s))\Big)\,ds.$$

Using **C1**, for $p > 0$, we get

$$\sup_{s\in J\cup[0,t]} E\mathcal{V}(\nu_m \wedge s, y(\nu_m \wedge s)) \leqslant \sup_{\zeta\in J} E\mathcal{V}(\zeta, \psi(\zeta)) + \sup_{s\in[0,t]} q(s) + \sup_{s\in[0,t]} K_2(s)E|y(\nu_m \wedge s))|^p,$$

so, from (2.1), (5.2), (5.3) and (5.5), we derive

$$\sup_{s\in J\cup[0,t]} E\mathcal{V}(\nu_m \wedge s, y(\nu_m \wedge s))$$

$$\leqslant \sup_{\zeta\in J} E\mathcal{V}(\zeta, \psi(\zeta)) + \sup_{s\in[0,t]} q(s) + \sup_{s\in[0,t]} K_2(s)\Big(r_1(p)E\|\psi\|^p + r_2(p)F^p\Big)$$

$$+ r_3(p) \sup_{s\in[0,t]} K_2(s)E|y(\nu_m \wedge s) - D(\nu_m \wedge s, y(\nu_m \wedge s-\rho(\nu_m \wedge s, y(\nu_m \wedge s))))|^p$$

$$\leqslant \sup_{\zeta\in J} E\mathcal{V}(\zeta, \psi(\zeta)) + \sup_{s\in[0,t]} q(s) + \sup_{s\in[0,t]} K_2(s)\Big(r_1(p)E\|\psi\|^p + r_2(p)F^p\Big)$$

$$+ r_3(p) \sup_{s\in[0,t]} \frac{K_2(s)}{K_1(s)}\Big(\widetilde{\mathcal{V}} + \sup_{s\in[0,t]} \overline{D}(s) \int_0^t \Big(1 + 2\sup_{s\in J\cup[0,s]} E\mathcal{V}(\nu_m \wedge s, y(\nu_m \wedge s))\Big)\,ds\Big),$$

where $r_1(p)$, $r_2(p)$ and $r_3(p)$ are defined in (2.4). Then, we have that

$$(5.6) \qquad 1 + 2 \sup_{s \in J \cup [0,t]} E\mathcal{V}(\nu_m \wedge s, y(\nu_m \wedge s))$$

$$\leqslant c(t) + d(t) \int_0^t \left(1 + 2 \sup_{s \in J \cup [0,s]} E\mathcal{V}(\nu_m \wedge s, y(\nu_m \wedge s)) \right) ds,$$

where $c(t)$ and $d(t)$ are given in (2.3). Since (5.6) has the form (1.4), by applying Lemma 1.2, we conclude that

$$(5.7) \qquad \sup_{s \in J \cup [0,t]} E\mathcal{V}(\nu_m \wedge s, y(\nu_m \wedge s)) \leqslant \frac{1}{2} c(t) d(t) e^{d(t)t}, \quad t > 0.$$

For $t = 0$, from (5.6) we conclude that

$$\sup_{\zeta \in J} E\mathcal{V}(\zeta, \psi(\zeta)) \leqslant \frac{c(0) - 1}{2} \leqslant \frac{c(0)d(0)}{2}.$$

From (5.7) and (2.1), for any $t \geqslant 0$, we obtain

$$\frac{1}{2} c(t) d(t) e^{d(t)t} \geqslant \sup_{s \in J \cup [0,t]} E\mathcal{V}(\nu_m \wedge s, y(\nu_m \wedge s)) \geqslant E\mathcal{V}(\nu_m \wedge t, y(\nu_m \wedge t))$$

$$\geqslant EK_1(\nu_m)|y(\nu_m)|^p I_{\{\nu_m \leqslant t\}} \geqslant m^p \inf_{s \in [0,t]} K_1(s) E I_{\{\nu_m \leqslant t\}}$$

$$\geqslant m^p \inf_{s \in [0,t]} K_1(s) P\{\nu_m \leqslant t\}.$$

Then,

$$P\{\nu_m \leqslant t\} \leqslant \frac{c(t) d(t) e^{d(t)t}}{2m^p \inf_{s \in [0,t]} K_1(s)}.$$

As $t \geqslant 0$ is arbitrary, letting $m \to +\infty$ in the previous inequality gives $P\{\nu_\infty = +\infty\} = 1$, so there exists unique global solution of Eq. (1.1). $\qquad \square$

References

[1] B. Akhtari, *Numerical solution of stochastic state-dependent delay differential equations: convergence and stability*, Adv. Differ. Equ. **396** (2019). https://doi.org/10.1186/s13662-019-2323-x

[2] H. Bao and J. Cao, *Existence and uniqueness of solutions to neutral stochastic functional differential equations with infinite delay*, Appl. Math. Comput. **215** (2009) 1732–1743. https://doi.org/10.1016/j.amc.2009.07.025

[3] H. B. Chen, *The existence and uniqueness for the solution of neutral stochastic functional differential equations with infinite delay*, J. Math. Res. Exposit. **30**(4) (2010) 589–598. `https://doi.org/10.3770/j.issn:1000-341X.2010.04.003`

[4] S. S. Dragomir, *Some Gronwall Type Inequalities and Applications*, Nova Science Publishers, 2003

[5] W. Fei, L. Hu, X. Mao and M. Shen, *Generalized criteria on delay-dependent stability of highly nonlinear hybrid stochastic systems*, Int. J. Robust Nonlinear Control **29** (2019), 1201–1215. `https://doi.org/10.1002/rnc.4402doi:10.1002/rnc.4402`

[6] Y. I. Kazmerchuk and J. H. Wu, *Stochastic state-dependent delay differential equations with applications in finance*, Funct. Differ. Equ. **11** (1–2) (2004) 77–86.

[7] Y.-H. Kim, *An existence of the solution to neutral stochastic functional differential equations under the Hölder condition*, Aust. J. Math. Anal. Appl. **16**(2) (2019) 1–10.

[8] X. Mao, *Stochastic Differential Equations and Applications*, Woodhead Publishing, 2007.

[9] M. Milošević, *Highly nonlinear neutral stochastic differential equations with time-dependent delay and the Euler-Maruyama method*, Math. Comput. Modelling **54** (2011), 2235–2251. `https://doi.org/10.1016/j.mcm.2011.05.033`

[10] M. Milošević, *Almost sure exponential stability of solutions to highly nonlinear neutral stochastic differential equations with time-dependent delay and the Euler-Maruyama approximation*, Math. Comput. Modelling **57** (2013), 887–899. `https://doi.org/10.1016/j.mcm.2012.09.016`

[11] M. Obradović and M. Milošević, *Stability of a class of neutral stochastic differential equations with unbounded delay and Markovian switching and the Euler-Maruyama method*, J. Comput. Appl. Math, **309**, (2017), 244–266. `https://doi.org/10.1016/j.cam.2016.06.038`

[12] A. Petrović and M. Milošević, *The truncated Euler-Maruyama method for highly nonlinear neutral stochastic differential equations with time-dependent delay*, Filomat **35**(7) (2021), 2457–2484. `https://doi.org/10.2298/FIL2107457P`

[13] Y. Qin, *Integral and Discrete Inequalities and Their Applications Volume I: Linear Inequalities*, Birkhauser, Basel, 2016.

[14] L. Shaikhet, *Stability of stochastic differential equations with distributed and state-dependent delays*, J. App. Math. Comp. **4**(4) (2020), 181–188. `http://dx.doi.org/10.26855/jamc.2020.12.009`

[15] T. Trifunović, M. Jovanović, M. Milošević, *The Generalized Khasminskii-type Conditions in Establishing Existence, Uniqueness and Moment Estimates of Solution to Neutral Stochastic Functional Differential Equations*, Filomat **37**(24) (2023), 8157–8174. `https://doi.org/10.2298/FIL2324157T`

[1]FACULTY OF SCIENCE,
UNIVERSITY OF KRAGUJEVAC,
RADOJA DOMANOVIĆA 12, KRAGUJEVAC, SERBIA
Email address: `teodora.trifunovic@pmf.kg.ac.rs`
ORCID iD: `https://orcid.org/0000-0003-2642-9828`

[2]FACULTY OF SCIENCES AND MATHEMATICS,
UNIVERSITY OF NIŠ,
VIŠEGRADSKA 33, NIŠ, SERBIA
Email address: `mima@pmf.ni.ac.rs`
ORCID iD: `https://orcid.org/0000-0002-8535-7513`
Email address: `27marija.milosevic@gmail.com`
ORCID iD: `https://orcid.org/0000-0002-5399-3320`