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ON TWO DIFFERENT CLASSES OF WARPED PRODUCT SUBMANIFOLDS OF KENMOTSU MANIFOLDS

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Abstract. Warped product skew CR-submanifold of the form $M = M_1 \times_f M_\perp$ of a Kenmotsu manifold $\bar{M}$ (throughout the paper), where $M_1 = M_T \times M_\theta$ and M_T, $M_\perp$, M_θ represents invariant, anti-invariant and proper slant submanifold of $\bar{M}$, studied in [28] and another class of warped product skew CR-submanifold of the form $M = M_2 \times_f M_T$ of $\bar{M}$, where $M_2 = M_\perp \times M_\theta$ is studied in [19]. Also the warped product submanifold of the form $M = M_3 \times_f M_\theta$ of $\bar{M}$, where $M_3 = M_T \times M_\perp$ and M_T, $M_\perp$, M_θ represents invariant, anti-invariant and proper point wise slant submanifold of $\bar{M}$, were studied in [18]. As a generalization of the above mentioned three classes, we consider a class of warped product submanifold of the form $M = M_4 \times_f M_{\theta_3}$ of $\bar{M}$, where $M_4 = M_{\theta_1} \times M_{\theta_2}$ in which M_{θ_1} and M_{θ_2} are proper slant submanifolds of $\bar{M}$ and M_{θ_3} represents a proper pointwise slant submanifold of $\bar{M}$. A characterization is given on the existence of such warped product submanifolds which generalizes the characterization of warped product submanifolds of the form $M = M_1 \times_f M_\perp$, studied in [28], the characterization of warped product submanifolds of the form $M = M_2 \times_f M_T$, studied in [19], the characterization of warped product submanifolds of the form $M = M_3 \times_f M_\theta$, studied in [18] and also the characterization of warped product pointwise bi-slant submanifolds of $\bar{M}$, studied in [17]. Since warped product bi-slant submanifolds of $\bar{M}$ does not exist (Theorem 4.2 of [17]), the Riemannian product $M_4 = M_{\theta_1} \times M_{\theta_2}$ cannot be a warped product. So, for studying the bi-warped product submanifolds of $\bar{M}$ of the form $M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$, we have taken M_{θ_1}, M_{θ_2}, M_{θ_3} as pointwise slant submanifolds of $\bar{M}$ of distinct slant functions θ_1, θ_2, θ_3 respectively. The existence of such type of bi-warped product submanifolds of $\bar{M}$ is ensured by an example. Finally, a Chen-type inequality on the squared norm of the second fundamental form of such bi-warped product submanifolds of $\bar{M}$ is obtained which also generalizes the inequalities obtained in [33], [18] and [17], respectively.

Key words and phrases. Kenmotsu manifold, pointwise slant submanifolds, warped product, submanifolds, bi-warped product submanifolds.

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1. Introduction

The warped product [5] between two Riemannian manifolds (N_1, g_1) and (N_2, g_2) is the Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where

$$g = \pi_1^*(g_1) + (f \circ \pi_1)^2 \pi_2^*(g_2),$$

where π_1 and π_2 are canonical projections of $M_1 \times M_2$ onto M_1 and M_2, respectively and $\pi_i^*(g_i)$ is the pullback of g_i via π_i for $i = 1, 2$ and $f : N_1 \to \mathbb{R}^+$ is a smooth function.

A warped product manifold $N_1 \times_f N_2$ is said to be trivial if f is constant. For $M = N_1 \times_f N_2$, we have [5]

$$\nabla_U X = \nabla_X U = (X \ln f)U, \tag{1.1}$$

for any $X \in \Gamma(TN_1)$ and $U \in \Gamma(TN_2)$.

The study of warped product submanifold was initiated in [8–10]. Then many authors have studied warped product submanifolds of different ambient manifolds, see [15–17, 20]. In [31], Tanno classified almost contact metric manifolds in three different classes among which the third class was picked up by Kenmotsu in 1972 and he studied its differential geometric properties [21]. This class later named after him by Kenmotsu manifold which is very important class to study. Warped product submanifolds of Kenmotsu manifolds are also studied in ([1–3], [22], [23], [26], [27], [32]-[38]). Multiply warped products (see [11, 12, 38]) are generalizations of warped product and Riemannian product manifolds and bi-warped products are special classes of multiply warped products. Bi-warped product submanifolds of different ambient manifolds are studied in [33, 35]. For the study of slant immersion and slant submanifolds in contact metric manifolds we refer [6, 7, 24]. In [29] Park studied pointwise slant and pointwise semi slant submanifolds of almost contact Riemannian manifolds.

Recently, Roy et al. studied the characterization theorem on warped product submanifold of Sasakian manifolds in [30]. Motivated by the above studies, in this present paper we have studied warped product submanifolds of $\bar{M}$ of the form $M = M_4 \times_f M_{\theta_3}$ of $\bar{M}$ such that $\xi \in \Gamma(TM_4)$, where $M_4 = M_{\theta_1} \times M_{\theta_2}$, M_{θ_1}, M_{θ_2} are proper slant submanifolds of $\bar{M}$ and here M_{θ_3} represents a proper pointwise slant submanifold of $\bar{M}$. Next we have studied bi-warped product submanifolds of $\bar{M}$ of the form $M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$, where M_{θ_1}, M_{θ_2}, M_{θ_3} are pointwise slant submanifolds of $\bar{M}$ of distinct slant functions θ_1, θ_2 and θ_3, respectively.

The paper is organized as follows. Section 2 deals with some preliminary useful results for construction of the paper, Section 3 is concerned with the study of a class of submanifold M of $\bar{M}$ such that $TM = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \mathcal{D}^{\theta_3} \oplus \langle\xi\rangle$, where $\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_2}$ are slant distributions and $\mathcal{D}^{\theta_3}$ is pointwise slant distribution. In Section 4, we have studied warped product submanifolds of the form $M = M_4 \times_f M_{\theta_3}$ of $\bar{M}$ where $M_4 = M_{\theta_1} \times M_{\theta_2}$ such that ξ is orthogonal to M_{θ_3} with an supporting example. In Section 5, a characterization theorem of the mentioned class has been obtained, Section 6 deals with bi-warped product submanifolds $M = M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$ of $\bar{M}$, where M_{θ_1}, , M_{θ_2}, M_{θ_3} are pointwise slant submanifolds of $\bar{M}$ and constructed an example. In Section 7, we have obtained a generalized inequality for such class of bi-warped product submanifolds of $\bar{M}$. The last section is the conclusion part of the paper where we have shown how the results of this paper generalizes several results of different works.

2. Preliminaries

An odd dimensional smooth manifold $\bar{M}^{2m+1}$ is said to be an almost contact metric manifold [4] if it admits a $(1,1)$ tensor field ϕ, a vector field ξ, an 1-form η and a Riemannian metric g which satisfy

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$g(\phi X, Y) = -\, g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for all vector fields X, Y on $\bar{M}^{2m+1}$.

An almost contact metric manifold $\bar{M}^{2m+1}(\phi, \xi, \eta, g)$ is said to be Kenmotsu manifold if the following conditions hold [21]:

$$\bar{\nabla}_X \xi = X - \eta(X)\xi, \tag{2.4}$$

$$(\bar{\nabla}_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.5}$$

where $\bar{\nabla}$ denotes the Riemannian connection of g.

Let M be an n-dimensional submanifold of a Kenmotsu manifold $\bar{M}$. Throughout the paper we assume that the submanifold M of $\bar{M}$ is tangent to the structure vector field ξ.

Let ∇ and $\nabla^\perp$ be the induced connections on the tangent bundle TM and the normal bundle $T^\perp M$ of M respectively. Then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.6}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^\perp_X V, \tag{2.7}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, where h and A_V are second fundamental form and the shape operator (corresponding to the normal vector field V) respectively for the immersion of M into $\bar{M}$. The second fundamental form h and the shape operator A_V are related by $g(h(X, Y), V) = g(A_V X, Y)$ for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, where g is the Riemannian metric on $\bar{M}$ as well as on M.

The mean curvature H of M is given by $H = \frac{1}{n}\mathrm{trace}\, h$. A submanifold of a Kenmotsu manifold $\bar{M}$ is said to be totally umbilical if $h(X, Y) = g(X, Y)H$ for any $X, Y \in \Gamma(TM)$. If $h(X, Y) = 0$ for all $X, Y \in \Gamma(TM)$, then M is totally geodesic and if $H = 0$, then M is minimal in $\bar{M}$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent bundle TM and $\{e_{n+1}, \ldots, e_{2m+1}\}$ an orthonormal basis of the normal bundle $T^{\perp}M$. We put

$$h_{ij}^r = g(h(e_i, e_j), e_r) \quad \text{and} \quad \|h\|^2 = g(h(e_i, e_j), h(e_i, e_j)),$$

for $r \in \{n+1, \ldots, 2m+1\}$, $i, j = 1, 2, \ldots, n$.

For a differentiable function f on M, the gradient $\boldsymbol{\nabla} f$ is defined by

$$g(\boldsymbol{\nabla} f, X) = Xf,$$

for any $X \in \Gamma(TM)$. As a consequence, we get

$$\|\boldsymbol{\nabla} f\|^2 = \sum_{i=1}^{n} (e_i(f))^2. \tag{2.8}$$

For any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we can write

(a) $\phi X = PX + QX$;

(b) $\phi V = bV + cV$,

where PX, bV are the tangential components and QX, cV are the normal components.

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be slant if for each non-zero vector $X \in T_pM$, the angle θ between ϕX and T_pM is constant, i.e., it does not depend on the choice of $p \in M$.

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be pointwise slant [13] if for any non-zero vector $X \in T_pM$ at $p \in M$, such that X is not proportional to ξ_p, the angle $\theta(X)$ between ϕX and $T_p^*M = T_pM - \{0\}$ is independent of the choice of non-zero $X \in T_p^*M$.

For pointwise slant submanifold, θ is a function on M, which is known as slant function of M. Invariant and anti-invariant submanifolds are particular cases of pointwise slant submanifolds with slant function $\theta = 0$ and $\frac{\pi}{2}$ respectively. Also a pointwise slant submanifold M will be slant if θ is constant on M. Thus a pointwise slant submanifold is proper if neither $\theta = 0, \frac{\pi}{2}$ nor constant. It may be noted that [25] M is a pointwise slant submanifold of $\bar{M}$ if and only if exists a constant $\lambda \in [0, 1]$ such that

$$P^2 = \lambda(-I + \eta \otimes \xi). \tag{2.9}$$

Furthermore, $\lambda = \cos^2\theta$ for slant function θ. If M be a pointwise slant submanifold of $\bar{M}$, then we have [34]:

$$bQX = \sin^2\theta\{-X + \eta(X)\xi\}, \quad cQX = -QPX. \tag{2.10}$$

Let M_1, M_2, M_3 be Riemannian manifolds and let $M = M_1 \times_{f_1} M_2 \times_{f_2} M_3$ be the product manifold of M_1, M_2, M_3 such that $f_1, f_2 : M_1 \to \mathbb{R}^+$ are real valued smooth functions. For each i, denote by $\pi_i : M \to M_i$ the canonical projection of M onto M_i, $i = 1, 2, 3$. Then the metric on M, called a bi-warped metric is given by

$$g(X, Y) = g(\pi_{1*}X, \pi_{2*}Y) + (f_1 \circ \pi_1)^2 g(\pi_{2*}X, \pi_{2*}Y) + (f_2 \circ \pi_1)^2 g(\pi_{3*}X, \pi_{3*}Y),$$

for any $X, Y \in \Gamma(TM)$ and $*$ denotes the symbol for tangent maps. The manifold M endowed with this product metric is called a bi-warped product manifold. Here f_1, f_2 are non-constant functions, called warping functions on M. Clearly, if both f_1, f_2 are constant on M, then M is simply a Riemannian product manifold and if anyone of the functions is constant, then M is a single warped product manifold. If neither f_1 nor f_2 is constant, then M is a proper bi-warped product manifold.

Let $M = M_1 \times_{f_1} M_2 \times_{f_2} M_3$ be a warped product submanifold of $\bar{M}$. Then we have [35]

$$\nabla_X Z = \sum_{i=1}^{2} (X(\ln f_i)) Z^i,$$

for any $X \in \mathcal{D}^1$, the tangent space of M_1 and $Z \in \Gamma(TN)$, where $N =_{f_1} M_2 \times_{f_2} M_3$ and Z^i is M_i components of Z for each $i = 2, 3$ and ∇ is the Levi-Civita connection on M.

3. Submanifolds of $\bar{M}$

In this section we consider submanifold M of $\bar{M}$ such that

$$TM = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \mathcal{D}^{\theta_3} \oplus \langle \xi \rangle,$$
$$T^\perp M = Q\mathcal{D}^{\theta_1} \oplus Q\mathcal{D}^{\theta_2} \oplus Q\mathcal{D}^{\theta_3} \oplus \nu,$$

where ν is a ϕ-invariant normal subbundle of $T^\perp M$.

If M is such submanifold of $\bar{M}$, then for any $X \in \Gamma(TM)$ we have

$$X = T_1 X + T_2 X + T_3 X, \tag{3.1}$$

where T_1, T_2 and T_3 are the projections from TM onto $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_3}$, respectively.

If we put $P_1 = T_1 \circ P$, $P_2 = T_2 \circ P$ and $P_3 = T_3 \circ P$ then from (3.1), we get

$$\phi X = P_1 X + P_2 X + P_3 X + QX, \tag{3.2}$$

for $X \in \Gamma(TM)$.

From (2.9) and (3.2), we get

$$P_i^2 = \cos^2 \theta_i (-I + \eta \otimes \xi), \quad \text{for } i = 1, 2, 3. \tag{3.3}$$

Now for the sake of further study we obtain the following useful results.

Lemma 3.1. *Let M be a submanifold of $\bar{M}$ such that $TM = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \mathcal{D}^{\theta_3}$ and $\xi \in \Gamma(\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2})$ then the following relations hold:*

$$(\sin^2\theta_1 - \sin^2\theta_3) g(\nabla_{X_1} Y_1, X_3) = g(A_{QP_1Y_1} X_3 - A_{QY_1} P_3 X_3, X_1) + g(A_{QP_3X_3} Y_1 - A_{QX_3} P_1 Y_1, X_1), \tag{3.4}$$

$$(\sin^2\theta_2 - \sin^2\theta_3) g(\nabla_{X_2} Y_2, X_3) = g(A_{QP_2Y_2} X_3 - A_{QY_2} P_3 X_3, X_2) + g(A_{QP_3X_3} Y_2 - A_{QX_3} P_2 Y_2, X_2), \tag{3.5}$$

$$(\sin^2\theta_2 - \sin^2\theta_3) g(\nabla_{X_1} X_2, X_3) = g(A_{QP_2X_2} X_3 - A_{QX_2} P_3 X_3, X_1) + g(A_{QP_3X_3} X_2 - A_{QX_3} P_2 X_2, X_1), \tag{3.6}$$

$$(3.7)\qquad (\sin^2\theta_1 - \sin^2\theta_3)g(\nabla_{X_2}X_1, X_3) = g(A_{QP_1X_1}X_3 - A_{QX_1}P_3X_3, X_2) + g(A_{QP_3X_3}X_1 - A_{QX_3}P_1X_1, X_2),$$

for any $X_1, Y_1 \in \Gamma(\mathcal{D}^{\theta_1} \oplus \langle\xi\rangle)$, $X_2, Y_2 \in \Gamma(\mathcal{D}^{\theta_2} \oplus \langle\xi\rangle)$ *and* $X_3 \in \Gamma(\mathcal{D}^{\theta_3})$.

Proof. For any $X_1, Y_1 \in \Gamma(\mathcal{D}^{\theta_1} \oplus \langle\xi\rangle)$ and $X_3 \in \Gamma(\mathcal{D}^{\theta_3})$, we have from (2.3), (2.5) and (3.2) that

$$\begin{aligned} g(\nabla_{X_1}Y_1, X_3) =& g(\bar\nabla_{X_1}P_1Y_1, \phi X_3) + g(\bar\nabla_{X_1}QY_1, \phi X_3) \\ =& -g(\phi\bar\nabla_{X_1}P_1Y_1, X_3) + g(\bar\nabla_{X_1}QY_1, P_3X_3) + g(\bar\nabla_{X_1}QY_1, QX_3) \\ =& -g(\bar\nabla_{X_1}P_1^2Y_1, X_3) - g(\bar\nabla_{X_1}QP_1Y_1, X_3) + g((\bar\nabla_{X_1}\phi)P_1Y_1, X_3) \\ &+ g(\bar\nabla_{X_1}QY_1, P_3X_3) - g(\bar\nabla_{X_1}QX_3, \phi Y_1) + g(\bar\nabla_{X_1}QX_3, P_1Y_1) \\ =& -g(\bar\nabla_{X_1}P_1^2Y_1, X_3) - g(\bar\nabla_{X_1}QP_1Y_1, X_3) + g(\bar\nabla_{X_1}QY_1, P_3X_3) \\ &+ g(\bar\nabla_{X_1}bQX_3, Y_1) + g(\bar\nabla_{X_1}cQX_3, Y_1) + g(\bar\nabla_{X_1}QX_3, P_1Y_1). \end{aligned}$$

Using (2.7), (2.10) and (3.3), the above equation reduces to

$$\begin{aligned} g(\nabla_{X_1}Y_1, X_3) =& \cos^2\theta_1 g(\bar\nabla_{X_1}Y_1, X_3) + g(A_{QP_1Y_1}X_3, X_1) - g(A_{QY_1}P_3X_3, X_1) \\ &+ \sin^2\theta_3 g(\bar\nabla_{X_1}Y_1, X_3) + g(A_{QP_3X_3}Y_1, X_1) - g(A_{QX_3}P_1Y_1, X_1), \end{aligned}$$

from which the relation (3.4) follows.

The relations (3.5)–(3.7) follow similarly. ☐

Lemma 3.2. *Let* M *be a submanifold of* $\bar M$ *where* $TM = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \mathcal{D}^{\theta_3}$ *such that* $\xi \in \Gamma(\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2})$. *Then the following relations hold:*

$$\begin{aligned} (3.8)\qquad (\sin^2\theta_3 - \sin^2\theta_1)g(\nabla_{X_3}Y_3, X_1) =& g(A_{QP_3Y_3}X_1 - A_{QY_3}P_1X_1, X_3) \\ &+ g(A_{QP_1X_1}Y_3 - A_{QX_1}P_3Y_3, X_3) \\ &+ (\cos^2\theta_3 - \cos^2\theta_1)\eta(X_1)g(X_3, Y_3), \end{aligned}$$

$$\begin{aligned} (3.9)\qquad (\sin^2\theta_3 - \sin^2\theta_2)g(\nabla_{X_3}Y_3, X_2) =& g(A_{QP_3Y_3}X_2 - A_{QY_3}P_2X_2, X_3) \\ &+ g(A_{QP_2X_2}Y_3 - A_{QX_2}P_3Y_3, X_3) \\ &+ (\cos^2\theta_3 - \cos^2\theta_2)\eta(X_2)g(X_3, Y_3), \end{aligned}$$

for any $X_1 \in \Gamma(\mathcal{D}^{\theta_1} \oplus \langle\xi\rangle)$, $X_2 \in \Gamma(\mathcal{D}^{\theta_2} \oplus \langle\xi\rangle)$ *and* $X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$.

Proof. For any $X_1 \in \Gamma(\mathcal{D}^{\theta_1} \oplus \langle\xi\rangle)$ and $X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$, we have from (2.3), (2.5) and (3.2) that

$$\begin{aligned} g(\nabla_{X_3}Y_3, X_1) =& g(\bar\nabla_{X_3}P_3Y_3, \phi X_1) + g(\bar\nabla_{X_3}QY_3, \phi X_1) - \eta(X_1)g(X_3, Y_3) \\ =& -g(\phi\bar\nabla_{X_3}P_3Y_3, X_1) + g(\bar\nabla_{X_3}QY_3, P_1X_1) \\ &+ g(\bar\nabla_{X_3}QY_3, QX_1) - \eta(X_1)g(X_3, Y_3) \\ =& -g(\bar\nabla_{X_3}P_3^2Y_3, X_1) - g(\bar\nabla_{X_3}QP_3Y_3, X_1) + g((\bar\nabla_{X_3}\phi)P_3Y_3, X_1) \\ &+ g(\bar\nabla_{X_3}QY_3, P_1X_1) - g(\bar\nabla_{X_3}QX_1, \phi Y_3) \end{aligned}$$

$$
\begin{aligned}
&+ g(\bar{\nabla}_{X_3}QX_1, P_3Y_3) - \eta(X_1)g(X_3, Y_3)\\
=&\cos^2\theta_3 g(\bar{\nabla}_{X_3}Y_3, X_1) - \sin 2\theta_3 X_3(\theta_3)g(Y_3, X_1)\\
&+ \cos^2\theta_3\eta(X_1)g(X_3, Y_3) - g(\bar{\nabla}_{X_3}QP_3Y_3, X_1)\\
&+ g(\bar{\nabla}_{X_3}QY_3, P_1X_1) + g(\bar{\nabla}_{X_3}bQX_1, Y_3) + g(\bar{\nabla}_{X_3}cQX_1, Y_3)\\
&- g((\bar{\nabla}_{X_3}\phi)QX_1, Y_3) + g(\bar{\nabla}_{X_3}QX_1, P_3Y_3) - \eta(X_1)g(X_3, Y_3).
\end{aligned}
$$

Using (2.5), (2.7), (2.10), orthogonality of the distributions and symmetry of the shape operator, the above equation reduces to

$$
\begin{aligned}
g(\nabla_{X_3}Y_3, X_1) =& \cos^2\theta_3 g(\bar{\nabla}_{X_3}Y_3, X_1) + \cos^2\theta_3\eta(X_1)g(X_3, Y_3)\\
&+ g(A_{QP_3Y_3}X_1, X_3) - g(A_{QY_3}P_1X_1, X_3)\\
&+ \sin^2\theta_1 g(\bar{\nabla}_{X_1}Y_3, X_1) + g(A_{QP_1X_1}Y_3, X_3)\\
&- g(A_{QX_1}P_3Y_3, X_3) - \cos^2\theta_1\eta(X_1)g(X_3, Y_3).
\end{aligned}
$$

Following the same computational procedure for any $X_2 \in \Gamma(\mathcal{D}^{\theta_2} \oplus \langle\xi\rangle)$ and $X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$ we can establish the relation (3.9). And hence, the lemma is proved. $\square$

4. Warped Product Submanifolds of Kenmotsu Manifolds

In this section we study warped product submanifolds of the form $M = M_4 \times_f M_{\theta_3}$ of $\bar{M}$ where $M_4 = M_{\theta_1} \times M_{\theta_2}$ such that ξ is orthogonal to M_{θ_3}. Here M_{θ_1}, M_{θ_2} represents proper slant submanifolds of $\bar{M}$ with slant angles θ_1, θ_2, respectively and M_{θ_3} represents pointwise-slant submanifolds of $\bar{M}$ with slant function θ_3.

Now we construct an example of a non-trivial warped product submanifold M of $\bar{M}$ of the form $M_4 \times_f M_{\theta_3}$.

Example 4.1. Consider the Kenmotsu manifold $M = \mathbb{R} \times_f \mathbb{C}^7$ with the structure (ϕ, ξ, η, g) is given by

$$
\phi\bigg(\sum_{i=1}^{7}(X_i\frac{\partial}{\partial x_i} + Y_i\frac{\partial}{\partial y_i}) + Z\frac{\partial}{\partial t}\bigg) = \sum_{i=1}^{7}\bigg(X_i\frac{\partial}{\partial y_i} - Y_i\frac{\partial}{\partial x_i}\bigg),
$$

$\xi = \frac{\partial}{\partial t}$, $\eta = dt$ and $g = \eta\otimes\eta + \sum_{i=1}^{7}(dx^i \otimes dx^i + dy^i \otimes dy^i)$. Let M be a submanifold of $\bar{M}$ defined by the immersion χ as follows:

$$
\begin{aligned}
&\chi(u, v, \theta, \phi, r, s, t)\\
=&(u\cos\theta, u\sin\theta, 2u+3v, 3u+2v, v\cos\phi, v\sin\phi, 3\theta+5\phi, 5\theta+3\phi, v\cos\theta, v\sin\theta,\\
&u\cos\phi, u\sin\phi, 2r+5s, 5r+2s, t).
\end{aligned}
$$

Then the local orthonormal frame of TM is spanned by the following:

$$
\begin{aligned}
Z_1 &= \cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial y_1} + 2\frac{\partial}{\partial x_2} + 3\frac{\partial}{\partial y_2} + \cos\phi\frac{\partial}{\partial x_6} + \sin\phi\frac{\partial}{\partial y_6},\\
Z_2 &= 3\frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial y_2} + \cos\phi\frac{\partial}{\partial x_3} + \sin\phi\frac{\partial}{\partial y_3} + \cos\theta\frac{\partial}{\partial x_5} + \sin\theta\frac{\partial}{\partial y_5},
\end{aligned}
$$

$$Z_3 = -u\sin\theta\frac{\partial}{\partial x_1} + u\cos\theta\frac{\partial}{\partial y_1} + 3\frac{\partial}{\partial x_4} + 5\frac{\partial}{\partial y_4} - v\sin\theta\frac{\partial}{\partial x_5} + v\cos\theta\frac{\partial}{\partial y_5},$$
$$Z_4 = -v\sin\phi\frac{\partial}{\partial x_3} + v\cos\phi\frac{\partial}{\partial y_3} + 5\frac{\partial}{\partial x_4} + 3\frac{\partial}{\partial y_4} - u\sin\phi\frac{\partial}{\partial x_6} + u\cos\phi\frac{\partial}{\partial y_6},$$
$$Z_5 = 2\frac{\partial}{\partial x_7} + 5\frac{\partial}{\partial y_7}, \quad Z_6 = 5\frac{\partial}{\partial x_7} + 2\frac{\partial}{\partial y_7} \quad \text{and} \quad Z_7 = \frac{\partial}{\partial t}.$$

Then

$$\phi Z_1 = \cos\theta\frac{\partial}{\partial y_1} - \sin\theta\frac{\partial}{\partial x_1} + 2\frac{\partial}{\partial y_2} - 3\frac{\partial}{\partial x_2} + \cos\phi\frac{\partial}{\partial y_6} - \sin\phi\frac{\partial}{\partial x_6},$$
$$\phi Z_2 = 3\frac{\partial}{\partial y_2} - 2\frac{\partial}{\partial x_2} + \cos\phi\frac{\partial}{\partial y_3} - \sin\phi\frac{\partial}{\partial x_3} + \cos\theta\frac{\partial}{\partial y_5} - \sin\theta\frac{\partial}{\partial x_5},$$
$$\phi Z_3 = -u\sin\theta\frac{\partial}{\partial y_1} - u\cos\theta\frac{\partial}{\partial x_1} + 3\frac{\partial}{\partial y_4} - 5\frac{\partial}{\partial x_4} - v\sin\theta\frac{\partial}{\partial y_5} - v\cos\theta\frac{\partial}{\partial x_5},$$
$$\phi Z_4 = -v\sin\phi\frac{\partial}{\partial y_3} - v\cos\phi\frac{\partial}{\partial x_3} + 5\frac{\partial}{\partial y_4} - 3\frac{\partial}{\partial x_4} - u\sin\phi\frac{\partial}{\partial y_6} - u\cos\phi\frac{\partial}{\partial x_6},$$
$$\phi Z_5 = 2\frac{\partial}{\partial y_7} - 5\frac{\partial}{\partial x_7} \quad \text{and} \quad \phi Z_6 = 5\frac{\partial}{\partial y_7} - 2\frac{\partial}{\partial x_7}.$$

We take, $\mathcal{D}^{\theta_1} = \mathrm{Span}\{Z_1, Z_2\}$, $D^{\theta_2} = \mathrm{Span}\{Z_5, Z_6\}$ and $\mathcal{D}^{\theta_3} = \mathrm{Span}\{Z_3, Z_4\}$. Then it is clear that $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ are proper slant distributions with slant angles $\cos^{-1}\frac{1}{3}$ and $\cos^{-1}\frac{21}{29}$, respectively. Also, $\mathcal{D}^{\theta_3}$ is a proper pointwise slant distribution with slant function $\cos^{-1}(\frac{16}{u^2+v^2+34})$.

Clearly, $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_3}$ are integrable distributions. Let us say that M_4 and M_{θ_3} are integral submanifolds of $\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \langle\xi\rangle$ and $\mathcal{D}^{\theta_3}$, respectively. Then the metric tensor g_M of M is given by

$$\begin{aligned} g_M &= 15(du^2 + dv^2) + 29(dr^2 + ds^2) + (u^2 + v^2 + 34)(d\theta^2 + d\phi^2) \\ &= g_{M_4} + (u^2 + v^2 + 34)g_{M_{\theta_3}}. \end{aligned}$$

Thus $M = M_4 \times_f M_{\theta_3}$ is a warped product submanifold of $\bar{M}$ with the warping function $f = \sqrt{u^2 + v^2 + 34}$.

Next we obtain the following useful lemmas.

Lemma 4.1. *Let $M = M_4 \times_f M_{\theta_3}$ be a warped product submanifold of $\bar{M}$ such that $\xi \in M_4$, where $M_4 = M_{\theta_1} \times M_{\theta_2}$, $M_{\theta_1}, M_{\theta_2}$ are proper slant submanifolds and M_{θ_3} is a proper pointwise slant submanifold of $\bar{M}$, then*

$$\xi \ln f = 1, \tag{4.1}$$
$$g(h(X_1, Y_1), QX_3) = g(h(X_1, X_3), QY_1), \tag{4.2}$$
$$g(h(X_2, Y_2), QX_3) = g(h(X_2, X_3), QY_2), \tag{4.3}$$
$$g(h(X_1, X_3), QX_2) = g(h(X_1, X_2), QX_3) = g(h(X_2, X_3), QX_1), \tag{4.4}$$

for $X_1, Y_1 \in M_{\theta_1}$, $X_2, Y_2 \in M_{\theta_2}$ and $X_3, Y_3 \in M_{\theta_3}$.

Proof. The proof of (4.1) is similar as in [28].

Now, for $X_1, Y_1 \in M_{\theta_1}$ and $X_3 \in M_{\theta_3}$, we have from (2.5) and (3.3) that

$$g(h(X_1, X_3), QY_1) = -g(\bar{\nabla}_{X1} P_3 X_3, Y_1) - g(\bar{\nabla}_{X1} Q X_3, Y_1) - g(\bar{\nabla}_{X_1} X_3, P_1 Y_1). \tag{4.5}$$

Then using (1.1) in (4.5), we get (4.2).

Proceeding the same, for any $X_2, Y_2 \in M_{\theta_2}$ and $X_3 \in M_{\theta_3}$, we get (4.2).

Again, for any $X_1 \in M_{\theta_1}$, $X_2 \in M_{\theta_2}$ and $X_3 \in M_{\theta_3}$ we have from (2.5) and (3.3) that

$$g(h(X_1, X_3), QX_2) = -g(\bar{\nabla}_{X_3} P_1 X_1, X_2) - g(\bar{\nabla}_{X_3} Q X_1, X_2) - g(\bar{\nabla}_{X_3} X_1, P_2 X_2). \tag{4.6}$$

Using (1.1) in (4.6), we find

$$g(h(X_1, X_3), QX_2) = g(h(X_2, X_3), QX_1). \tag{4.7}$$

Also,

$$g(h(X_1, X_2), QX_3) = -g(\bar{\nabla}_{X_1} P_2 X_2, X_3) - g(\bar{\nabla}_{X_1} P_2 X_2, X_3) - g(\bar{\nabla}_{X_1} X_2, P_3 X_3). \tag{4.8}$$

Using (1.1) in (4.8), we get

$$g(h(X_1, X_2), QX_3) = g(h(X_1, X_3), QX_2). \tag{4.9}$$

Combining (4.7) and (4.9), we obtain (4.4). This completes the proof. ☐

Lemma 4.2. *Let $M = M_4 \times_f M_{\theta_3}$ be a warped product submanifold of $\bar{M}$ such that $\xi \in M_4$, where $M_4 = M_{\theta_1} \times M_{\theta_2}$, M_{θ_1}, M_{θ_2} are proper slant submanifolds and M_{θ_3} is a proper pointwise slant submanifold of $\bar{M}$, then*

$$\begin{aligned} &g(h(X_3, X_1), QY_3) - g(h(X_3, Y_3), QX_1) \\ =&\{(X_1 \ln f) - \eta(X_1)\} g(P_3 X_3, Y_3) - (P_1 X_1 \ln f) g(X_3, Y_3), \end{aligned} \tag{4.10}$$

$$\begin{aligned} &g(h(X_3, X_2), QY_3) - g(h(X_3, Y_3), QX_2) \\ =&\{(X_2 \ln f) - \eta(X_2)\} g(P_3 X_3, Y_3) - (P_2 X_2 \ln f) g(X_3, Y_3), \end{aligned} \tag{4.11}$$

$$\begin{aligned} &g(h(X_3, Y_3), QP_1 X_1) - g(h(P_3 Y_3, X_3), QX_1) \\ &+ g(h(X_1, X_3), QP_3 Y_3) - g(h(P_1 X_1, X_3), QY_3) \\ =&(\cos^2\theta_1 - \cos^2\theta_3)[\eta(X_1) - (X_1 \ln f)] g(X_3, Y_3), \end{aligned} \tag{4.12}$$

$$\begin{aligned} &g(h(X_3, Y_3), QP_2 X_2) - g(h(P_3 Y_3, X_3), QX_2) \\ &+ g(h(X_2, X_3), QP_3 Y_3) - g(h(P_2 X_2, X_3), QY_3) \\ =&(\cos^2\theta_2 - \cos^2\theta_3)[\eta(X_2) - (X_2 \ln f)] g(X_3, Y_3), \end{aligned} \tag{4.13}$$

for $X_1 \in M_{\theta_1}$, $X_2 \in M_{\theta_2}$ and $X_3, Y_3 \in M_{\theta_3}$.

Proof. From (2.5) and (3.3), we have for $X_1 \in M_{\theta_1}$ and $X_3, Y_3 \in M_{\theta_3}$ that

$$\begin{aligned} g(h(X_3, Y_3), QX_1) =& -g(\bar{\nabla}_{X_3} X_1, P_3 Y_3) - g(\bar{\nabla}_{X_3} Q Y_3, X_1) \\ &+ \eta(X_1) g(\phi X_3, Y_3) + g(\bar{\nabla}_{X_3} P_1 X_1, Y_3). \end{aligned} \tag{4.14}$$

Using (2.7) and (1.1) in (4.14), we get (4.10). Following the same procedure, for any $X_2 \in M_{\theta_2}$ and $X_3, Y_3 \in M_{\theta_3}$ we easily obtain (4.11).

Next, replacing X_1 by P_1X_1 and Y_3 by P_3Y_3 in (4.10), respectively and then adding the obtained equations, we get (4.12). Similarly, replacing X_2 by P_2X_2 and Y_3 by P_3Y_3 in (4.11), respectively and then adding the obtained equations, we get (4.13). □

5. Characterization

We prove the following theorem.

Theorem 5.1. *Let M be a submanifold of $\bar{M}$ such that $TM = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \mathcal{D}^{\theta_3}$ with ξ orthogonal to $\mathcal{D}^{\theta_3}$, then M is locally a warped product submanifold of the form $M = M_4 \times_f M_{\theta_3}$ where $M_4 = M_{\theta_1} \times M_{\theta_2}$ if and only if*

$$\begin{aligned} &A_{QP_1X_1}Y_3 - A_{QX_1}P_3Y_3 + A_{QP_3Y_3}X_1 - A_{QY_3}P_1X_1 \\ &=(\cos^2\theta_3 - \cos^2\theta_1)[X_1\mu - \eta(X_1)]Y_3, \end{aligned} \tag{5.1}$$

$$\begin{aligned} &A_{QP_2X_2}Y_3 - A_{QX_2}P_3Y_3 + A_{QP_3Y_3}X_2 - A_{QY_3}P_2X_2 \\ &=(\cos^2\theta_3 - \cos^2\theta_2)[X_2\mu - \eta(X_2)]Y_3, \end{aligned} \tag{5.2}$$

$$\xi\mu = 1, \tag{5.3}$$

for every $X_1 \in \Gamma(\mathcal{D}^{\theta_1})$, $X_2 \in \Gamma(\mathcal{D}^{\theta_2})$, $X_3 \in \Gamma(\mathcal{D}^{\theta_3})$ and for some smooth function μ on M satisfying where $(Y_3\mu) = 0$ for any $Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$.

Proof. Let $M = M_4 \times_f M_{\theta_3}$ be a proper warped product submanifold of $\bar{M}$ such that $M_4 = M_{\theta_1} \times M_{\theta_2}$. Denote the tangent space of M_{θ_1}, M_{θ_2} and M_{θ_3} by $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_3}$ respectively. Then from (4.2) we get

$$g(A_{QP_1X_1}Y_3 - A_{QX_1}P_3Y_3 + A_{QP_3Y_3}X_1 - A_{QY_3}P_1X_1, X_1) = 0. \tag{5.4}$$

Similarly, from (4.4) we get

$$g(A_{QP_1X_1}Y_3 - A_{QX_1}P_3Y_3 + A_{QP_3Y_3}X_1 - A_{QY_3}P_1X_1, X_2) = 0. \tag{5.5}$$

So, from (5.4) and (5.5) we conclude that

$$A_{QP_1X_1}Y_3 - A_{QX_1}P_3Y_3 + A_{QP_3Y_3}X_1 - A_{QY_3}P_1X_1 \in \mathcal{D}^{\theta_3}. \tag{5.6}$$

Hence, from (4.12) and (5.6), relation (5.1) follows.

In similar way, in view of (4.3), (4.4) and (4.13) we get (5.2). The relation (5.3) is directly obtained from (4.1).

Conversely, let M be a submanifold of $\bar{M}$ such that $TM = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \mathcal{D}^{\theta_3}$ with ξ orthogonal to $\mathcal{D}^{\theta_3}$ and the conditions (5.1)–(5.3) satisfied. Then from (3.4) and (3.7), in view of (5.1), respectively we get

$$g(\nabla_{X_1}Y_1, X_3) = 0 \quad \text{and} \quad g(\nabla_{X_2}X_1, X_3) = 0, \tag{5.7}$$

and also from (3.5), (3.6) in view of (5.2), respectively we get

$$g(\nabla_{X_2}Y_2, X_3) = 0 \quad \text{and} \quad g(\nabla_{X_1}X_2, X_3) = 0. \tag{5.8}$$

Thus, from (5.7), (5.8) and the fact that $\nabla_{X_3}\xi = 0$ we conclude that $g(\nabla_E F, X_3) = 0$ for every $E, F \in \Gamma(\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \langle\xi\rangle)$. Hence the leaves of $\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \langle\xi\rangle$ are totally geodesic in M.

Now, by virtue of (3.8), (5.1) yields

$$g([X_3, Y_3], X_1) = 0, \tag{5.9}$$

and by virtue of (3.9), (5.2) yields

$$g([X_3, Y_3], X_2) = 0. \tag{5.10}$$

Hence, from (5.9), (5.10) and the fact that $h(A, \xi) = 0$, for all $A \in TM$, we conclude that

$$g([X_3, Y_3], E) = 0, \quad \text{for all } X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3}),$$

and $E \in \Gamma(\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \langle\xi\rangle)$, consequently $\mathcal{D}^{\theta_3}$ is integrable.

Let h^{θ_3} be the second fundamental form of M_{θ_3} in $\bar{M}$. Then for any $X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$ and $X_1 \in \Gamma(\mathcal{D}^{\theta_1})$, from (3.8), we find

$$g(h^{\theta_3}(X_3, Y_3), X_1) = -(X_1\mu)g(X_3, Y_3). \tag{5.11}$$

Similarly, for $X_2 \in \Gamma(\mathcal{D}^{\theta_2})$, from (3.9) we get

$$g(h^{\theta_3}(X_3, Y_3), X_2) = -(X_2\mu)g(X_3, Y_3). \tag{5.12}$$

Again, for any $X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$, in view of (5.3) we have

$$g(h^{\theta_3}(X_3, Y_3), \xi) = -(\xi\mu)g(X_3, Y_3). \tag{5.13}$$

Hence, from (5.11)–(5.13) we conclude that

$$g(h^{\theta}(X_3, Y_3), E) = -g(\boldsymbol{\nabla}\mu, E)g(X_3, Y_3),$$

for every $X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$ and $E \in \Gamma(\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus, \langle\xi\rangle)$. Consequently, M_{θ_3} is totally umbilical in $\bar{M}$ with mean curvature vector $H^{\theta_3} = -\boldsymbol{\nabla}\mu$.

Finally, we will show that H^{θ_3} is parallel with respect to the normal connection $\nabla^{\perp}$ of M_{θ_3} in M. We take $E \in \Gamma(\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_3} \oplus \langle\xi\rangle)$ and $X_3 \in \Gamma(\mathcal{D}^{\theta_3})$, then we have

$$g(\nabla^{\perp}_{X_3}\boldsymbol{\nabla}\mu, E) = g(\nabla_{X_3}\boldsymbol{\nabla}^{\theta_1}\mu, X_1) + g(\nabla_{X_3}\boldsymbol{\nabla}^{\theta_2}\mu, X_2) + g(\nabla_{X_3}\boldsymbol{\nabla}^{\xi}\mu, \xi),$$

where $\boldsymbol{\nabla}^{\theta_1}$, $\boldsymbol{\nabla}^{\theta_2}$ and $\boldsymbol{\nabla}^{\xi}$ are the gradient components of μ on M along $\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_2}$ and $\langle\xi\rangle$ respectively. Then by the property of Riemannian metric, the above equation reduces to

$$\begin{aligned} g(\nabla^{\perp}_{U}\boldsymbol{\nabla}\mu, E) =& X_3 g(\boldsymbol{\nabla}^{\theta_1}\mu, X_1) - g(\boldsymbol{\nabla}^{\theta_1}\mu, \nabla_{X_3}X_1) + X_3 g(\boldsymbol{\nabla}^{\theta_2}\mu, X_2) \\ &- g(\boldsymbol{\nabla}^{\theta_2}\mu, \nabla_{X_3}X_2) + X_3 g(\boldsymbol{\nabla}^{\xi}\mu, \xi) - g(\boldsymbol{\nabla}^{\xi}\mu, \nabla_{X_3}\xi) \\ =& X_3(X_1\mu) - g(\boldsymbol{\nabla}^{\theta_1}\mu, [X_3, X_1]) - g(\boldsymbol{\nabla}^{\theta_1}\mu, \nabla_{X_1}X_3) \\ &+ X_3(X_2\mu) - g(\boldsymbol{\nabla}^{\theta_2}\mu, [X_3, X_2]) - g(\boldsymbol{\nabla}^{\theta_2}\mu, \nabla_{X_2}X_3) \\ &+ X_3(\xi\mu) - g(\boldsymbol{\nabla}^{\xi}\mu, [X_3, \xi]) - g(\boldsymbol{\nabla}^{\xi}\mu, \nabla_{\xi}X_3) \\ =& X_1(X_3\mu) + g(\nabla_{X_1}\boldsymbol{\nabla}^{\theta_1}\mu, X_3) + X_2(X_3\mu) \end{aligned}$$

$$+ g(\nabla_{X_2}\boldsymbol{\nabla}^{\theta_2}\mu, X_3) + \xi(X_3\mu) - g(\nabla_\xi \boldsymbol{\nabla}^{\xi}\mu, X_3)$$
$$=0,$$

since $(X_3\mu) = 0$ for every $X_3 \in \Gamma(\mathcal{D}^{\theta_3})$ and $\nabla_{X_1}\boldsymbol{\nabla}^{\theta_1}\mu + \nabla_{X_2}\boldsymbol{\nabla}^{\theta_2}\mu + \nabla_\xi\boldsymbol{\nabla}^{\xi}\mu = \nabla_E\boldsymbol{\nabla}\mu$ is orthogonal to $\mathcal{D}^{\theta_3}$ for any $E \in \Gamma(\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \langle\xi\rangle)$ and $\boldsymbol{\nabla}\mu$ is the gradient along M_4 and M_4 is totally geodesic in $\bar{M}$. Hence, the mean curvature vector H^{θ_3} of M_{θ_3} is parallel. Thus, M_{θ_3} is an extrinsic sphere in M. Hence, by Hiepko's Theorem (see [14]), M is locally a warped product submanifold. Thus, the proof is complete. □

6. Bi-Warped Product Submanifolds

In this section we have studied bi-warped product submanifolds $M = M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$ of $\bar{M}$, where M_{θ_1}, M_{θ_2}, M_{θ_3} are pointwise slant submanifolds of $\bar{M}$ and an supporting example has been constructed. We denote $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$, $\mathcal{D}^{\theta_3}$ as the tangent spaces of M_{θ_1}, M_{θ_2}, M_{θ_3}, respectively.

Then we write

$$TM = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \mathcal{D}^{\theta_3} \oplus \langle\xi\rangle$$

and

$$T^\perp M = Q\mathcal{D}^{\theta_1} \oplus Q\mathcal{D}^{\theta_2} \oplus Q\mathcal{D}^{\theta_3}.$$

Example 6.1. Consider the Kenmotsu manifold $M = \mathbb{R} \times_f \mathbb{C}^{10}$ with the structure (ϕ, ξ, η, g) is given by

$$\phi\left(\sum_{i=1}^{10}\left(X_i\frac{\partial}{\partial x_i} + Y_i\frac{\partial}{\partial y_i}\right) + Z\frac{\partial}{\partial t}\right) = \sum_{i=1}^{10}\left(X_i\frac{\partial}{\partial y_i} - Y_i\frac{\partial}{\partial x_i}\right),$$

$\xi = \frac{\partial}{\partial t}$, $\eta = dt$ and $g = \eta \otimes \eta + \sum_{i=1}^{10}(dx^i \otimes dx^i + dy^i \otimes dy^i)$. Let M be a submanifold of $\bar{M}$ defined by the immersion χ as follows:

$$\begin{aligned}
&\chi(u, v, \theta, \phi, r, s, t)\\
=&(u\cos\theta, u\sin\theta, v\cos\phi, v\sin\phi, 3\theta + 5\phi, 5\theta + 3\phi, v\cos\theta, v\sin\theta, u\cos\phi, u\sin\phi, u\cos r,\\
&v\cos s, u\sin r, v\sin s, 3r + 2s, 2r + 3s, u\cos s, v\cos r, u\sin s, v\sin r, t).
\end{aligned}$$

Then the local orthonormal frame of TM is spanned by the following:

$$\begin{aligned}
Z_1 =&\cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial y_1} + \cos\phi\frac{\partial}{\partial x_5} + \sin\phi\frac{\partial}{\partial y_5}\\
&+ \cos r\frac{\partial}{\partial x_6} + \sin r\frac{\partial}{\partial x_7} + \cos s\frac{\partial}{\partial x_9} + \sin s\frac{\partial}{\partial x_{10}},\\
Z_2 =&\cos\phi\frac{\partial}{\partial x_2} + \sin\phi\frac{\partial}{\partial y_2} + \cos\theta\frac{\partial}{\partial x_4} + \sin\theta\frac{\partial}{\partial y_4}\\
&+ \cos s\frac{\partial}{\partial y_6} + \sin s\frac{\partial}{\partial y_7} + \cos r\frac{\partial}{\partial y_9} + \sin r\frac{\partial}{\partial y_{10}},\\
Z_3 =&- u\sin\theta\frac{\partial}{\partial x_1} + u\cos\theta\frac{\partial}{\partial y_1} + 3\frac{\partial}{\partial x_3} + 5\frac{\partial}{\partial y_3} - v\sin\theta\frac{\partial}{\partial x_4} + v\cos\theta\frac{\partial}{\partial y_4},
\end{aligned}$$

$$\begin{aligned}
Z_4 =& -v\sin\phi\frac{\partial}{\partial x_2}+v\cos\phi\frac{\partial}{\partial y_2}+5\frac{\partial}{\partial x_3}+3\frac{\partial}{\partial y_3}-u\sin\phi\frac{\partial}{\partial x_5}+u\cos\phi\frac{\partial}{\partial y_5},\\
Z_5 =& -u\sin r\frac{\partial}{\partial x_6}+u\cos r\frac{\partial}{\partial x_7}+3\frac{\partial}{\partial x_8}+2\frac{\partial}{\partial y_8}-v\sin r\frac{\partial}{\partial y_9}+v\cos r\frac{\partial}{\partial y_{10}},\\
Z_6 =& V-Xv\sin s\frac{\partial}{\partial y_6}+v\cos s\frac{\partial}{\partial y_7}+2\frac{\partial}{\partial x_8}+3\frac{\partial}{\partial y_8}-u\sin s\frac{\partial}{\partial x_9}+u\cos s\frac{\partial}{\partial x_{10}}
\end{aligned}$$

and

$$Z_7=\frac{\partial}{\partial t}.$$

Then

$$\begin{aligned}
\phi Z_1 =& \cos\theta\frac{\partial}{\partial y_1}-\sin\theta\frac{\partial}{\partial x_1}+\cos\phi\frac{\partial}{\partial y_5}-\sin\phi\frac{\partial}{\partial x_5}\\
&+\cos r\frac{\partial}{\partial y_6}+\sin r\frac{\partial}{\partial y_7}+\cos s\frac{\partial}{\partial y_9}+\sin s\frac{\partial}{\partial y_{10}},\\
\phi Z_2 =& \cos\phi\frac{\partial}{\partial y_2}-\sin\phi\frac{\partial}{\partial x_2}+\cos\theta\frac{\partial}{\partial y_4}-\sin\theta\frac{\partial}{\partial x_4}\\
&-\cos s\frac{\partial}{\partial x_6}-\sin s\frac{\partial}{\partial x_7}-\cos r\frac{\partial}{\partial x_9}-\sin r\frac{\partial}{\partial x_{10}},\\
\phi Z_3 =& -u\sin\theta\frac{\partial}{\partial y_1}-u\cos\theta\frac{\partial}{\partial x_1}+3\frac{\partial}{\partial y_3}-5\frac{\partial}{\partial x_3}-v\sin\theta\frac{\partial}{\partial y_4}-v\cos\theta\frac{\partial}{\partial x_4},\\
\phi Z_4 =& -v\sin\phi\frac{\partial}{\partial y_2}-v\cos\phi\frac{\partial}{\partial x_2}+5\frac{\partial}{\partial y_3}-3\frac{\partial}{\partial x_3}-u\sin\phi\frac{\partial}{\partial y_5}-u\cos\phi\frac{\partial}{\partial x_5},\\
\phi Z_5 =& -u\sin r\frac{\partial}{\partial y_6}+u\cos r\frac{\partial}{\partial y_7}+3\frac{\partial}{\partial y_8}-2\frac{\partial}{\partial x_8}+v\sin r\frac{\partial}{\partial x_9}-v\cos r\frac{\partial}{\partial x_{10}},\\
\phi Z_6 =& v\sin s\frac{\partial}{\partial x_6}-v\cos s\frac{\partial}{\partial x_7}+2\frac{\partial}{\partial y_8}-3\frac{\partial}{\partial x_8}-u\sin s\frac{\partial}{\partial y_9}+u\cos s\frac{\partial}{\partial y_{10}}.
\end{aligned}$$

We take $\mathcal{D}^{\theta_1}=\mathrm{Span}\{Z_1,Z_2\}$, $\mathcal{D}^{\theta_2}=\mathrm{Span}\{Z_3,Z_4\}$ and $\mathcal{D}^{\theta_3}=\mathrm{Span}\{Z_5,Z_6\}$. Then it is clear that $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_3}$ are proper pointwise slant distributions with slant functions $\cos^{-1}\{\frac{1}{2}\cos(r-s)\}$, $\cos^{-1}(\frac{16}{u^2+v^2+34})$ and $\cos^{-1}(\frac{5}{u^2+v^2+13})$, respectively. Clearly, $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_3}$ are integrable distributions. Let us say that M_{θ_1}, M_{θ_2} and M_{θ_3} are integral submanifolds of $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_3}$, respectively. Then the metric tensor g_M of M is given by

$$\begin{aligned}
g_M &= 4(du^2+dv^2)+(u^2+v^2+34)(d\theta^2+d\phi^2)+(u^2+v^2+13)(dr^2+ds^2)\\
&= g_{M_{\theta_1}}+(u^2+v^2+34)g_{M_{\theta_2}}+(u^2+v^2+13)g_{M_{\theta_3}}.
\end{aligned}$$

Thus, $M=M_{\theta_1}\times_{f_1}M_{\theta_2}\times_{f_2}M_{\theta_3}$ is a bi-warped product submanifold of $\bar{M}$ with the warping functions $f_1=\sqrt{u^2+v^2+34}$ and $f_2=\sqrt{u^2+v^2+13}$.

Proposition 6.1 ([33])**.** *Let $M=M_{\theta_1}\times_{f_1}M_{\theta_2}\times_{f_2}M_{\theta_3}$ be a bi-warped product submanifold of $\bar{M}$. Then M is a single warped product if ξ is orthogonal to $\mathcal{D}^{\theta_1}$.*

Proposition 6.2 ([33]). *Let $M = M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$ be a bi-warped product submanifold of $\bar{M}$ such that ξ such that M is tangent to M_{θ_1}. Then*

$$\xi(\ln f_i) = 1, \quad \textit{for all } i = 1, 2. \tag{6.1}$$

Lemma 6.1. *Let $M = M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$ be a bi-warped product submanifold of $\bar{M}$ such that ξ is tangent to M_{θ_1}. Then*

$$g(h(X_1, Y_1), QX_3) = g(h(X_1, X_3), QY_1), \tag{6.2}$$

$$g(h(X_2, Y_2), QX_3) = g(h(X_1, X_3), QY_2), \tag{6.3}$$

$$g(h(X_1, X_2), QX_3) = g(h(X_1, X_3), QX_2), \tag{6.4}$$

for every $X_1, Y_1 \in \Gamma(\mathcal{D}^{\theta_1})$, $X_2, Y_2 \in \Gamma(\mathcal{D}^{\theta_2})$ and $X_3 \in \Gamma(\mathcal{D}^{\theta_3})$.

Proof. Proof is similar to the proof of Lemma 4.1. □

Lemma 6.2. *Let $M = M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$ be a bi-warped product submanifold of $\bar{M}$ such that ξ is tangent to M_{θ_1}. Then*

$$\begin{aligned} &g(h(X_2, Y_2), QX_1) - g(h(X_1, X_2), QY_2) \\ =&(P_1 X_1 \ln f_1) g(X_2, Y_2) + [X_1(\ln f_1) - \eta(X_1)] g(X_2, P_2 Y_2), \end{aligned} \tag{6.5}$$

$$\begin{aligned} &g(h(X_3, Y_3), QX_1) - g(h(X_1, X_3), QY_3) \\ =&(P_1 X_1 \ln f_2) g(X_3, Y_3) + [X_1(\ln f_2) - \eta(X_1)] g(X_3, P_3 Y_3), \end{aligned} \tag{6.6}$$

$$\begin{aligned} &g(h(X_3, Y_3), QX_2) - g(h(X_2, X_3), QY_3) \\ =&(P_2 X_2 \ln f_2) g(X_3, Y_3) + X_2(\ln f_2) g(X_3, P_3 Y_3), \end{aligned} \tag{6.7}$$

for every $X_1 \in \Gamma(\mathcal{D}^{\theta_1})$, $X_2, Y_2 \in \Gamma(\mathcal{D}^{\theta_2})$ and $X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$.

Proof. Proof is similar to the proof of Lemma 4.2. □

Lemma 6.3. *Let $M = M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$ be a bi-warped product submanifold of $\bar{M}$ such that ξ is tangent to M_{θ_1}. Then*

$$\begin{aligned} &g(h(X_1, Y_2), QP_2 X_2) - g(h(X_1, P_2 X_2), QY_2) \\ =&2\cos^2\theta_2 \{(X_1 \ln f_1) - \eta(X_1)\} g(X_2, Y_2), \end{aligned} \tag{6.8}$$

$$\begin{aligned} &g(h(X_1, X_3), QP_3 Y_3) - g(h(X_1, P_3 X_3), QY_3) \\ =&2\cos^2\theta_3 \{(X_1 \ln f_2) - \eta(X_1)\} g(X_3, Y_3), \end{aligned} \tag{6.9}$$

$$\begin{aligned} &g(h(X_2, X_3), QP_3 Y_3) - g(h(X_2, P_3 X_3), QY_3) \\ =&2\cos^2\theta_3 (X_2 \ln f_2) g(X_3, Y_3), \end{aligned} \tag{6.10}$$

for every $X_1 \in \Gamma(\mathcal{D}^{\theta_1})$, $X_2, Y_2 \in \Gamma(\mathcal{D}^{\theta_2})$ and $X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$.

Proof. By polarization of (6.5), we get

$$\begin{aligned} g(h(X_2, Y_2), QX_1) - g(h(X_1, Y_2), QZ) =&(P_1 X_1 \ln f_1) g(X_2, Y_2) \\ &+ [X_1(\ln f_1) - \eta(X_1)] g(X_2, Y_2). \end{aligned} \tag{6.11}$$

Subtracting (6.11) from (6.4), we find

$$g(h(X_1,Y_2),QX_2)-g(h(X_1,X_2),QY_2)=2[X_1(\ln f_1)-\eta(X_1)]g(X_2,P_2Y_2). \tag{6.12}$$

Replacing X_2 by P_2X_2 in (6.12), we get (6.8). Similarly, (6.9) follows from (6.6) and (6.10) follows from (6.7). □

Theorem 6.1. *Let $M=M_{\theta_1}\times_{f_1}M_{\theta_2}\times_{f_2}M_{\theta_3}$ be a bi-warped product submanifold of $\bar M$ such that ξ is tangent to M_{θ_1}. Then M can be $\mathcal{D}^{\theta_1}-\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_1}-\mathcal{D}^{\theta_3}$ mixed totally geodesic but cannot be $\mathcal{D}^{\theta_2}-\mathcal{D}^{\theta_3}$ mixed totally geodesic.*

Proof. The theorem follows from Lemma 6.3. □

7. Inequality

In this section, we establish a Chen-type inequality on a bi-warped product submanifold $M=M_{\theta_1}\times_{f_1}M_{\theta_2}\times_{f_2}M_{\theta_3}$ of $\bar M$ of dimension n such that ξ is tangent to M_{θ_1}. We take $\dim M_{\theta_1}=2p+1$, $\dim M_{\theta_2}=2q$, $\dim M_{\theta_3}=2s$ and their corresponding tangent spaces are $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_3}$, respectively. Assume that $\{e_1,e_2,\ldots,e_p,e_{p+1}=\sec\theta_1P_1e_1,\ldots,e_{2p}=\sec\theta_1P_1e_p,e_{2p+1}=\xi\}$, $\{e_{2p+2}=e_1^*,\ldots,e_{2p+q+1}=e_q^*,e_{2p+q+2}=e_{q+1}^*=\sec\theta_2P_2e_1^*,\ldots,e_{2p+2q+1}=e_{2q}^*=\sec\theta_2P_2e_q^*\}$ and $\{e_{2p+2q+2}=\hat e_1,\ldots,e_{2p+2q+s+1}=\hat e_s,e_{2p+2q+s+2}=\hat e_{s+1}=\sec\theta_3P_3\hat e_1,\ldots,e_{2p+2q+2s+1}=\hat e_{2s}=\sec\theta_3P_3\hat e_s\}$ are local orthonormal frames of $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_3}$, respectively. Then the local orthonormal frames for $Q\mathcal{D}^{\theta_1}$, $Q\mathcal{D}^{\theta_2}$, $Q\mathcal{D}^{\theta_3}$ and ν are $\{\tilde e_1=\csc\theta_1Qe_1,\ldots,\tilde e_p=\csc\theta_1Qe_p,\tilde e_{p+1}=\csc\theta_1\sec\theta_1QP_1e_1,\ldots,\tilde e_{2p}\csc\theta_1\sec\theta_1QP_1e_p\}$, $\{\tilde e_{2p+1}=\tilde e_1^*=\csc\theta_2Qe_1^*,\ldots,\tilde e_{2p+q}=\tilde e_q^*=\csc\theta_2Qe_q^*,\tilde e_{2p+q+1}=\tilde e_{q+1}^*=\csc\theta_2\sec\theta_2QP_2e_1^*,\ldots,\tilde e_{2p+2q}=\tilde e_{2q}^*=\csc\theta_2\sec\theta_2QP_2e_q^*\}$, $\{\tilde e_{2p+2q+1}=\tilde{\hat e}_1=\csc\theta_3Q\hat e_1,\ldots,\tilde e_{2p+2q+s}=\tilde{\hat e}_s=\csc\theta_3Q\hat e_s,\tilde e_{2p+2q+s+1}=\tilde{\hat e}_{s+1}=\csc\theta_3\sec\theta_3QP_3\hat e_1,\ldots,\tilde e_{2p+2q+2s}=\tilde{\hat e}_{2s}=\csc\theta_3\sec\theta_3QP_3\hat e_s\}$ and $\{\tilde e_{2p+2q+2s+1},\ldots,\tilde e_{2m+1}\}$ of dimensions $2p$, $2q$, $2s$ and $(2m+1-n-2p-2q-2s)$, respectively.

Theorem 7.1. *Let $M=M_{\theta_1}\times_{f_1}M_{\theta_2}\times_{f_2}M_{\theta_3}$ be both $\mathcal{D}^{\theta_1}-\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_1}-\mathcal{D}^{\theta_3}$ mixed totally geodesic bi-warped product submanifold of $\bar M$ such that ξ is tangent to M_{θ_1}. Then the squared norm of the second fundamental form satisfies*

$$\begin{aligned}\|h\|^2\geq&2q\csc^2\theta_1(\cos^2\theta_1+\cos^2\theta_2)(\|\boldsymbol{\nabla}\ln f_1\|^2-1)\\&+2s\csc^2\theta_1(\cos^2\theta_1+\cos^2\theta_3)(\|\boldsymbol{\nabla}\ln f_2\|^2-1),\end{aligned} \tag{7.1}$$

where $2q=\dim M_{\theta_1}$, $2s=\dim M_{\theta_3}$, $\boldsymbol{\nabla}\ln f_1$ and $\boldsymbol{\nabla}\ln f_2$ are the gradients of warping function $\ln f_1$ and $\ln f_2$ along M_{θ_1} and M_{θ_2}, respectively.

If the equality sign of (7.1) holds, then M_{θ_1} is totally geodesic and M_{θ_2}, M_{θ_3} are totally umbilical submanifolds of $\bar M$.

Proof. From the definition of h, we have

$$\|h\|^2=\sum_{i,j=1}^{2p+1}g(h(e_i,e_j),h(e_i,e_j)). \tag{7.2}$$

Now by decomposing (7.2) in our constructed frame fields, we get

$$\begin{aligned}\|h\|^2 =& \sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),\tilde{e}_r)^2 + 2\sum_{r=n+1}^{2m+1}\sum_{i=1}^{2p+1}\sum_{j=1}^{2q} g(h(e_i,e_j^*),\tilde{e}_r)^2\\ &+ 2\sum_{r=n+1}^{2m+1}\sum_{i=1}^{2p+1}\sum_{j=1}^{2s} g(h(e_i,\hat{e}_j),\tilde{e}_r)^2 + \sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{2q} g(h(e_i^*,e_j^*),\tilde{e}_r)^2\\ &+ 2\sum_{r=n+1}^{2m+1}\sum_{i=1}^{2q}\sum_{j=1}^{2s} g(h(e_i^*,\hat{e}_j),\tilde{e}_r)^2 + \sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{2s} g(h(\hat{e}_i,\hat{e}_j),\tilde{e}_r)^2.\end{aligned} \tag{7.3}$$

Neglecting the ν component terms of (7.3), we obtain

$$\begin{aligned}|h\|^2 \geq& \sum_{r=1}^{2p}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),\tilde{e_r})^2 + \sum_{r=1}^{2q}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),\tilde{e_r})^2\\ &+ \sum_{r=1}^{2s}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),\tilde{e}_r)^2 + 2\sum_{i,r=1}^{2p}\sum_{j=1}^{2q} g(h(e_i,e_j^*),\tilde{e}_r)^2\\ &+ 2\sum_{r,j=1}^{2q}\sum_{i=1}^{2p} g(h(e_i,e_j^*),\tilde{e}_r)^2 + 2\sum_{r=1}^{2s}\sum_{i=1}^{2p}\sum_{j=1}^{2q} g(h(e_i,e_j^*),\tilde{e}_r)^2\\ &+ 2\sum_{i,r=1}^{2p}\sum_{j=1}^{2s} g(h(e_i,\hat{e}_j),\tilde{e}_r)^2 + 2\sum_{r=1}^{2q}\sum_{i=1}^{2p}\sum_{j=1}^{2s} g(h(e_i,\hat{e}_j),\tilde{e}_r)^2\\ &+ 2\sum_{r,j=1}^{2s}\sum_{i=1}^{2p} g(h(e_i,\hat{e}_j),\tilde{e}_r)^2 + \sum_{r=1}^{2p}\sum_{i,j=1}^{2q} g(h(e_i^*,e_j^*),\tilde{e}_r)^2\\ &+ \sum_{i,j,r=1}^{2q} g(h(e_i^*,e_j^*),\tilde{e}_r)^2 + \sum_{r=1}^{2s}\sum_{i,j=1}^{2q} g(h(e_i^*,e_j^*),\tilde{e}_r)^2\\ &+ 2\sum_{r=1}^{2p}\sum_{i=1}^{2q}\sum_{j=1}^{2s} g(h(e_i^*,\hat{e}_j),\tilde{e}_r)^2 + 2\sum_{i,r=1}^{2q}\sum_{j=1}^{2s} g(h(e_i^*,\hat{e}_j),\tilde{e}_r)^2\\ &+ 2\sum_{j,r=1}^{2s}\sum_{i=1}^{2q} g(h(e_i^*,\hat{e}_j),\tilde{e}_r)^2 + \sum_{r=1}^{2p}\sum_{i,j=1}^{2s} g(h(\hat{e}_i,\hat{e}_j),\tilde{e}_r)^2\\ &+ \sum_{r=1}^{2q}\sum_{i,j=1}^{2s} g(h(\hat{e}_i,\hat{e}_j),\tilde{e}_r)^2 + \sum_{i,j,r=1}^{2s} g(h(\hat{e}_i,\hat{e}_j),\tilde{e}_r)^2.\end{aligned} \tag{7.4}$$

In view of Lemma (6.1), the second, third and thirteenth terms are equal to zero. Using the $\mathcal{D}^{\theta_1}-\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_1}-\mathcal{D}^{\theta_3}$ mixed totally geodesic condition, seventh to thirteenth terms are also equal to zero. Also we can not find any relation for $g(h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_1}),Q\mathcal{D}^{\theta_1})$, $g(h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_1}),Q\mathcal{D}^{\theta_2})$, $g(h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_2}),Q\mathcal{D}^{\theta_3})$, $g(h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_3}),Q\mathcal{D}^{\theta_3})$, $g(h(\mathcal{D}^{\theta_3},\mathcal{D}^{\theta_3}), Q\mathcal{D}^{\theta_2})$ and $g(h(\mathcal{D}^{\theta_3},\mathcal{D}^{\theta_3}),Q\mathcal{D}^{\theta_3})$, so we neglect first, eleventh, twelfth, fourteenth, fifteenth, seventeenth and eighteenth terms of (7.4) and obtain

$$
\begin{aligned}
\|h\|^2 \geq &\csc^2\theta_1 \sum_{r=1}^{p}\sum_{i,j=1}^{2q} g(h(e_i^*, e_j^*), Qe_r)^2 + \csc^2\theta_1\sec^2\theta_1 \sum_{r=1}^{p}\sum_{i,j=1}^{2q} g(h(e_i^*, P_1e_j^*), QP_1e_r)^2\\
&+ \csc^2\theta_1 \sum_{r,=1}^{p}\sum_{i,j=1}^{2s} g(h(\hat{e}_i, \hat{e}_j), Qe_r)^2 + \csc^2\theta_1\sec^2\theta_1 \sum_{r=1}^{p}\sum_{i,j=1}^{2s} g(h(\hat{e}_i, P_1\hat{e}_j), QP_1e_r)^2.
\end{aligned}
$$

By virtue of Lemma 6.2, the above relation yields

$$
\begin{aligned}
\|h\|^2 \geq &\csc^2\theta_1 \sum_{r=1}^{p}\sum_{i,j=1}^{2q} (P_1e_r \ln f_1)^2 g(e_i^*, e_j^*)^2\\
&+ csc^2\theta_1 \sum_{r=1}^{p}\sum_{i,j=1}^{2q} [(e_r \ln f_1) - \eta(e_r)]^2 g(e_i^*, P_2e_j^*)^2\\
&+ \csc^2\theta_1\cos^2\theta_1 \sum_{r=1}^{p}\sum_{i,j=1}^{2q} (e_r \ln f_1)^2 g(e_i^*, e_j^*)^2\\
&+ \csc^2\theta_1 \sum_{r,=1}^{p}\sum_{i,j=1}^{2q} (P_1e_r \ln f_1)^2 g(e_i^*, P_2e_j^*)^2\\
&+ \csc^2\theta_1 \sum_{r=1}^{p}\sum_{i,j=1}^{2s} (P_1e_r \ln f_2)^2 g(\hat{e}_i, \hat{e}_j)^2\\
&+ csc^2\theta_1 \sum_{r=1}^{p}\sum_{i,j=1}^{2q} [(e_r \ln f_2) - \eta(e_r)]^2 g(\hat{e}_i, P_3\hat{e}_j)^2\\
&+ \csc^2\theta_1\cos^2\theta_1 \sum_{r=1}^{p}\sum_{i,j=1}^{2s} (e_r \ln f_2)^2 g(\hat{e}_i, \hat{e}_j)^2\\
&+ \csc^2\theta_1 \sum_{r,=1}^{p}\sum_{i,j=1}^{2s} (P_1e_r \ln f_2)^2 g(\hat{e}_i, P_3\hat{e}_j)^2\\
= &2q\csc^2\theta_1(1+\sec^2\theta_1\cos^2\theta_2) \sum_{r=1}^{p} (P_1e_r \ln f_1)^2\\
&+ 2qcsc^2\theta_1(\cos^2\theta_1+\cos^2\theta_2) \sum_{r=1}^{p} [(e_r \ln f_1) - \eta(e_r)]^2\\
&+ 2q\csc^2\theta_1(1+\sec^2\theta_1\cos^2\theta_3) \sum_{r=1}^{p} (P_1e_r \ln f_2)^2\\
&+ 2qcsc^2\theta_1(\cos^2\theta_1+\cos^2\theta_3) \sum_{r=1}^{p} [(e_r \ln f_2) - \eta(e_r)]^2.
\end{aligned}
$$

Thus, we find

$$\|h\|^2 \geq 2q \csc^2\theta_1(\cos^2\theta_1+\cos^2\theta_2)\left(\sum_{r=1}^{2p+1}(P_1e_r\ln f_1)^2-(\xi\ln f_1)^2\right) + 2s\csc^2\theta_1(\cos^2\theta_1+\cos^2\theta_3)\left(\sum_{r=1}^{2p+1}(P_1e_r\ln f_2)^2-(\xi\ln f_2)^2\right). \tag{7.5}$$

Using (2.8) and Proposition 6.2, in (7.5), we get the inequality (7.1). If equality of (7.1) holds, for omitting ν components terms of (6.3), we get

$$h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_1})\perp\nu, \quad h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_2})\perp\nu, \quad h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_3})\perp\nu, \quad h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_3})\perp\nu.$$

Also, for neglecting terms of (7.4), we obtain $h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_1})\perp Q\mathcal{D}^{\theta_1}$, $h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_2})\perp Q\mathcal{D}^{\theta_2}$, $h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_2})\perp Q\mathcal{D}^{\theta_3}$, $h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_3})\perp Q\mathcal{D}^{\theta_2}$, $h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_3})\perp Q\mathcal{D}^{\theta_2}$, $h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_3})\perp Q\mathcal{D}^{\theta_3}$, $h(\mathcal{D}^{\theta_3},\mathcal{D}^{\theta_3})\perp Q\mathcal{D}^{\theta_2}$, $h(\mathcal{D}^{\theta_3},\mathcal{D}^{\theta_3})\perp Q\mathcal{D}^{\theta_3}$. Next, since M is both $\mathcal{D}^{\theta_1}-\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_1}-\mathcal{D}^{\theta_3}$ mixed totally geodesic, we get

$$h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_2})=0, \quad h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_3})=0. \tag{7.6}$$

Also, from Lemma 6.1 with (6.6), we get

$$h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_1})\perp Q\mathcal{D}^{\theta_2}, \quad h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_1})\perp Q\mathcal{D}^{\theta_3}, \quad h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_1})\perp Q\mathcal{D}^{\theta_2}.$$

Thus, we can say that

$$h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_1})=0, \tag{7.7}$$

$$h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_2})\subset Q\mathcal{D}^{\theta_1}, \tag{7.8}$$

$$h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_3})\subset Q\mathcal{D}^{\theta_1}, \tag{7.9}$$

$$h(\mathcal{D}^{\theta_3},\mathcal{D}^{\theta_3})\subset Q\mathcal{D}^{\theta_1}. \tag{7.10}$$

From (7.6) and (7.7), M_{θ_1} is totally geodesic in M and hence in $\bar{M}$ [5, 7]. Again, since M_{θ_2} and M_{θ_3} are totally umbilical in M [5, 7], with the fact (7.8)–(7.10), we conclude that M_{θ_2} and M_{θ_3} are totally umbilical in $\bar{M}$. Hence, the theorem is proved completely. □

8. Some Applications

As consequences of Theorem 5.1 we have the following.

1. If we take $\dim M_{\theta_2}=0$ and replace θ_3 by θ_2, then M changes to a warped product pointwise bi-slant submanifold of the form $M_{\theta_1}\times_f M_{\theta_2}$, studied in [17]. In this case Theorem 5.1 of this paper takes the following form (Theorem 5.1 of [17]).

Let M be a proper pointwise bi-slant submanifold of $\bar{M}$ such that $\xi\in\Gamma(\mathcal{D}^{\theta_1})$, then M is locally a warped product submanifold of the form $M_{\theta_1}\times_f M_{\theta_2}$ if and only if

$$\begin{aligned}&A_{QP_1X_1}Y_2-A_{QX_1}P_2Y_2+A_{QP_2Y_2}X_1-A_{QY_2}P_1X_1\\ =&(\cos^2\theta_2-\cos^2\theta_1)[(X_1\mu)-\eta(X_1)]Y_2,\end{aligned}$$

for any $X_1 \in \Gamma(\mathcal{D}^{\theta_1})$, $X_2 \in \Gamma(\mathcal{D}^{\theta_2})$, for some smooth function μ on M satisfying $(Y\mu) = 0$, for any $W \in \Gamma(\mathcal{D}^{\theta_2})$. Thus, Theorem 5.1 of this paper is a generalisation of Theorem 5.1 of [17].

2. If we take $\theta_1 = 0$, $\theta_2 = constant = \theta$, $\theta_3 = \frac{\pi}{2}$, then M changes to a warped product skew CR-submanifold of the form $M_1 \times_f M_\perp$, where $M_1 = M_T \times M_\theta$, studied in [28]. In this case Theorem 5.1 of this paper takes the following form (Theorem 5.3 of [28]).

Let M be a proper skew CR-submanifold of $\bar{M}$, then M is locally a $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic warped product submanifold of the form $M_1 \times_f M_\perp$, where $M_1 = M_T \times M_\theta$ if and only if

(i) $A_{\phi Z}X \in \Gamma(\mathcal{D}^\perp)$ for any $X \in \Gamma(\mathcal{D}^T \oplus \mathcal{D}^\theta) \oplus \{\xi\}$ and $Z \in \Gamma(\mathcal{D}^\perp)$;

(ii) for any $X_1 \in \Gamma(\mathcal{D}^T)$, $X_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, $A_{\phi Z}X_1 = -(\phi X_1\mu)$, $A_\phi ZX_2 = 0$, $A_{QX_2Z} = (P_2X_2\mu)Z$, $(\xi\mu) = 1$,

for some smooth function μ on M satisfying $(V\mu) = 0$, for any $V \in \Gamma(\mathcal{D}^\perp)$. Thus, Theorem 5.1 of this paper is a generalization of Theorem 5.3 of [28].

3. If we take $\theta_1 = \frac{\pi}{2}$, $\theta_2 = constant = \theta$, $\theta_3 = 0$, then M changes to a warped product skew CR-submanifold of the form $M_2 \times_f M_T$, where $M_2 = M_\perp \times M_\theta$, studied in [19]. In this case Theorem 5.1 of this paper takes the following form (Theorem 5.1 of [19]).

Let M be a proper skew CR-submanifold of $\bar{M}$, then M is locally a warped product submanifold of the form $M_2 \times_f M_T$, where $M_2 = M_\perp \times M_\theta$ if and only if

(i) $A_\phi ZX = \{\eta(Z) - (Z\mu)\}\phi X$;

(ii) $A_{QUX} = \{\eta(U) - (U\mu)\}\phi X + (P_2U\mu)X$;

(iii) $(\xi\mu) = 1$,

for any $X \in \Gamma(\mathcal{D}^T)$, $U \in \Gamma(\mathcal{D}^\theta)$, $Z \in \Gamma(\mathcal{D}^\perp)$, for some smooth function μ on M satisfying $(Y\mu) = 0$, for any $Y \in \Gamma(\mathcal{D}^T)$. Thus, Theorem 5.1 of this paper is a generalisation of Theorem 5.1 of [19].

4. If we take $\theta_1 = 0$, $\theta_2 = \frac{\pi}{2}$ and $\theta_3 = \theta$ then M changes to a warped product submanifold of the form $M_3 \times_f M_\theta$, where $M_3 = M_T \times M_\perp$, studied in [18]. In this case Theorem 5.1 of this paper takes the following form (Theorem 5.1 of [18]).

Let M be a submanifold of a Kenmotsu manifold $\bar{M}$ such that $TM = \mathcal{D}^T \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta$ with ξ is orthogonal to M_θ. Then M is locally a warped product submanifold of the form $M = M_3 \times_f M_\theta$, where $M_3 = M_T \times M_\perp$, if and only if the following relations hold:

(i) $A_{QV}\phi X - A_{QPV}X = \sin^2\theta[(X\mu) - \eta(X)]V$;

(ii) $A_{\phi Z}PV - A_{QPV}Z = -\cos^2\theta[(Z\mu) - \eta(Z)]V$;

(iii) $(\xi\mu) = 1$,

for every $X \in \Gamma(\mathcal{D}^T)$, $Z \in \Gamma(\mathcal{D}^\perp)$ and $V \in \Gamma(\mathcal{D}^\theta)$ and $(V\mu) = 0$ for some function μ on M satisfying $(W\mu) = 0$, for any $W \in \Gamma(\mathcal{D}^\theta)$. Thus, Theorem 5.1 of this paper is a generalisation of Theorem 5.1 of [18].

As consequences of Theorem 7.1, we have the following.

1. If we consider $\theta_1 = constant$, $\theta_2 = 0$, $\theta_3 = \frac{\pi}{2}$, then the submanifold M changes to bi-warped product submanifold of the form $M_\theta \times_{f_1} M_T \times_{f_2} M_\perp$, studied in [33]. In this case Theorem 7.1 of this paper takes the following form.

Let $M = M_\theta \times_{f_1} M_T \times_{f_2} M_\perp$ be a bi-warped product submanifold of $\bar{M}$ such that ξ is tangent to M_θ, then the squared norm of the second fundamental form satisfies

$$\|h\|^2 \geq 2q \csc^2 \theta(1 + \cos^2 \theta)(\|\boldsymbol{\nabla} \ln f_1\|^2 - 1) + 2s \cot^2 \theta(\|\boldsymbol{\nabla} \ln f_2\|^2 - 1),$$

where $2q = \dim M_T$, $2s = \dim M_\perp$, $\boldsymbol{\nabla} \ln f_1$ and $\boldsymbol{\nabla} \ln f_2$ are the gradients of warping function $\ln f_1$ and $\ln f_2$ along M_T and $M_\perp$, respectively.

If the equality sign holds, then M_θ is totally geodesic and M_T, $, M_\perp$ are totally umbilical submanifold of $\bar{M}$. Taking $\dim M_T = 2q = m_1$ and $\dim M_\perp = 2s = m_2$, we see that this statement coincides with the statement of Theorem 6 of [33]. Thus, Theorem 7.1 of this paper is a generalisation of Theorem 6 of [33].

2. If we consider $\dim M_{\theta_2} = 0$, then the submanifold M changes into warped product pointwise bi-slant submanifold of the form $M_{\theta_1} \times_f M_{\theta_2}$ studied in [17]. In this case Theorem 7.1 of this paper takes the following form.

Let $M = M_{\theta_1} \times_f M_{\theta_2}$ be a warped product pointwise bi-slant submanifold of $\bar{M}$ such that ξ is tangent to M_{θ_1}, then the squared norm of the second fundamental form satisfies

$$\|h\|^2 \geq 2q \csc^2 \theta_1(\cos^2 \theta_1 + \cos^2 \theta_2)(\|\boldsymbol{\nabla} \ln f\|^2 - 1),$$

where $2q = \dim M_{\theta_2}$, $\boldsymbol{\nabla} \ln f$ is the gradient of warping function $\ln f$ along M_{θ_1}. If the equality sign holds, then M_{θ_1} is totally geodesic and M_{θ_2} is totally umbilical submanifold of $\bar{M}$. Thus, we see that this statement coincides with the statement of Theorem 6.1 of [19]. Hence Theorem 7.1 of this paper is a generalization of Theorem 6.1 of [17].

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