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GENERALIZED VECTORIAL ALMOST PERIODICITY

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Abstract. In this paper, we introduce and analyze several new classes of generalized vectorially almost periodic functions. We also analyze Σ-almost periodic type functions and the invariance of generalized vectorial almost periodicity under the actions of convolution products.

1. Introduction and Preliminaries

The notion of almost periodicity was introduced by the Danish mathematician H. Bohr around 1924–1926 and later generalized by many others (see the research monographs [6, 9, 11, 12, 14, 15, 20, 23] and the excellent survey article [2] for further information regarding almost periodic functions and their applications). Let $(X, \|\cdot\|)$ be a complex Banach space and let $F : \mathbb{R}^n \rightarrow X$ be a continuous function, where $n \in \mathbb{N}$. Then, $F(\cdot)$ is said to be almost periodic if for each $\epsilon > 0$ there exists $W > 0$ such that for each $\mathbf{w}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{w}_0, W) \equiv \{\mathbf{t} \in \mathbb{R}^n : |\mathbf{t} - \mathbf{w}_0| \leq W\}$ such that

$$\left\| F(\mathbf{t} + \tau) - F(\mathbf{t}) \right\| \leq \epsilon, \quad \mathbf{t} \in \mathbb{R}^n,$$

here, $|\cdot - \cdot|$ denotes the Euclidean distance in $\mathbb{R}^n$.

If the function $F : \mathbb{R}^n \rightarrow X$ is Lebesgue measurable and for every non-empty compact set $K \subseteq \mathbb{R}^n$, one has $\int_K \|F(\mathbf{t})\|^p \, d\mathbf{t} < +\infty$, where $p > 0$, then $F(\cdot)$ is said to be Stepanov-p-almost periodic if for every $\epsilon > 0$ there exists $W > 0$ such that for

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each $\mathbf{w}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{w}_0, W)$ with

$$\Big\|F(\mathbf{s}+\tau+\mathbf{w})-F(\mathbf{s}+\mathbf{w})\Big\|_{L^p([0,1]^n:X)} \leq \epsilon, \quad \mathbf{s} \in \mathbb{R}^n,$$

i.e.,

$$\int_{[0,1]^n} \Big\|F(\mathbf{s}+\tau+\mathbf{w})-F(\mathbf{s}+\mathbf{w})\Big\|^p \, d\mathbf{w} \leq \epsilon^p, \quad \mathbf{s} \in \mathbb{R}^n. \tag{1.1}$$

Equivalently, the Bochner transform $\hat{F} : \mathbb{R}^n \to L^p([0,1]^n : X)$, defined by $[\hat{F}(\mathbf{t})](\mathbf{u}) := F(\mathbf{t}+\mathbf{u})$, $\mathbf{t} \in \mathbb{R}^n$, $\mathbf{u} \in [0,1]^n$, is almost periodic; $F(\cdot)$ is said to be Stepanov almost periodic if $F(\cdot)$ is Stepanov-1-almost periodic. Any Bohr almost periodic function $F(\cdot)$ has to be Stepanov-p-almost periodic for any $p > 0$.

The class of almost automorphic functions was discovered by S. Bochner in 1955 ([7]). In a joint work with S. Abbas [1], we have recently investigated vectorially Weyl almost automorphic functions. This research article aims to examine several new classes of vectorially Stepanov-p-almost periodic type functions and vectorially (equi)-Weyl-p-almost periodic type functions, where $p > 0$. The novelty of our approach is that we use the vector-valued integration in the analysis of these classes of functions; unfortunately, we cannot consider the general value of exponent $p \neq 1$ in the pure vector-valued setting. These classes of functions extend the usually considered classes of Stepanov-p-almost periodic type functions and (equi)-Weyl-p-almost periodic type functions in the scalar-valued setting as well as the usually considered classes of Stepanov-1-almost periodic type functions and (equi)-Weyl-1-almost periodic type functions in the vector-valued setting. We also analyze Σ-almost periodic type functions and the invariance of generalized vectorial almost periodicity under some convolution transforms, which is incredibly important for applications to the abstract Volterra integro-differential equations. If A is the integral generator of an an exponentially stable strongly continuous semigroup $(T(t))_{t\geq 0}$ and the function $f : \mathbb{R} \to X$ is Stepanov-p-almost periodic $(1 \leq p < +\infty)$, for instance, then the function $u : \mathbb{R} \to X$, given by

$$u(t) = \int_{-\infty}^{t} T(t-s)f(s)\,ds, \quad t \in \mathbb{R},$$

is an almost periodic solution of the abstract Cauchy problem $u'(t) = Au(t) + f(t)$, $t \in \mathbb{R}$; moreover, if the function $h : \mathbb{R} \to X$ is Stepanov-p-almost periodic $(1 \leq p < +\infty)$, the function $q : [0, +\infty) \to X$ vanishes at plus infinity and $f(t) = h(t) + q(t)$, $t \geq 0$, then the unique asymptotically almost periodic solution of the abstract Cauchy problem $u'(t) = Au(t) + f(t)$, $t \geq 0$; $u(0) = x$ is given by

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\,ds, \quad t \geq 0,$$

if $f(\cdot)$ enjoys very mild regularity assumptions. Many similar applications of this type have been given to the abstract Volterra integro-differential equations and the abstract fractional integro-differential equations (cf. [13, 14] for more details).

The structure of this research article can be shortly depicted as follows. After fixing the notation, we analyze the notion of vectorial Stepanov almost periodicity in Section 2. The class of Σ-almost periodic functions is analyzed in Subsection 2.1, and the class of vectorially Stepanov almost periodic functions in general metric is analyzed in Subsection 2.2. Vectorially Weyl almost periodic type functions are analyzed in Section 3, while the invariance of vectorial Stepanov almost periodicity and vectorial Weyl almost periodicity under the actions of convolution products is analyzed in Section 4. The last section of paper is reserved for the final conclusions; for simplicity, we will not investigate here the extensions of vectorially generalized almost periodic functions and the corresponding composition principles as well as the vectorially Weyl almost periodic type sequences.

Notation and preliminaries. We assume that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ are complex Banach spaces, $\mathcal{B}$ is a non-empty collection of non-empty subsets of X and R is a non-empty collection of sequences in $\mathbb{R}^n$. The standard assumption will be that, for every $y \in X$, there is a set $B \in \mathcal{B}$ such that $y \in B$. $L(X,Y)$ denotes the Banach space of all bounded linear operators from X into Y; $L(X,X) \equiv L(X)$ and I denotes the identity operator on Y. Define $\mathbb{N}_0 := \{0, 1, \ldots, m, \ldots\}$. If C and D are non-empty sets, then we set $\mathrm{D}^{\mathrm{C}} := \{f \mid f : \mathrm{C} \to \mathrm{D}\}$; set $0^\zeta := 0$ for $\zeta > 0$.

Let $\emptyset \neq I \subseteq \mathbb{R}^n$, $\nu : I \to (0, +\infty)$, and let $1/v(\cdot)$ be locally bounded. In this paper, we deal with the vector space $C_{b,\nu}(I : X)$ consisting of those continuous functions $u : I \to X$ such that $\|u\| := \sup_{\mathbf{t} \in I} \|\nu(\mathbf{t})u(\mathbf{t})\| < +\infty$. Then, $(C_{b,\nu}(I : X), \|\cdot\|)$ is a Banach space, as easily approved. Assuming that $\nu \equiv 1$, we also write $C_b(I : X)$ in place of $C_{b,\nu}(I : X)$.

2. Vectorial Stepanov Almost Periodicity

The class of Stepanov-p-almost periodic functions, where $p \geq 1$, was introduced by W. Stepanoff in 1926 ([22]). For example, the function

$$f(s) = \sin\left(\frac{2}{2 + \cos s + \cos(\sqrt{2}s)}\right), \quad s \in \mathbb{R},$$

is bounded, continuous and Stepanov-p-almost periodic for any finite exponent $p \geq 1$ but it is not almost periodic. If there exists $c > 0$ such that the sequence $(\lambda_k)_{k \in \mathbb{Z}}$ of real numbers satisfies $\lambda_{k+1} - \lambda_k > c$ for all $k \in \mathbb{Z}$ and $(a_k)_{k \in \mathbb{Z}}$ is a real sequence such that $\sum_{k=-\infty}^{+\infty} |a_k|^2 < +\infty$, then

$$f(s) = \sum_{k=-\infty}^{+\infty} a_k e^{i\lambda_k s}, \quad s \in \mathbb{R},$$

is a Stepanov-2-almost periodic function; cf. [20, Theorem 5.3.2, pp. 214–216]. For further information, we refer the reader to [14–16] and references quoted therein.

Now we would like to introduce the following notion.

Definition 2.1. (i) A locally Lebesgue integrable function $F : \mathbb{R}^n \to X$ is vectorially Stepanov almost periodic if for every $\epsilon > 0$ there exists $W > 0$ such that for each $\mathbf{w}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{w}_0, W)$ with

$$\left\| \int_{[0,1]^n} \Big[F(\mathbf{s} + \tau + \mathbf{w}) - F(\mathbf{s} + \mathbf{w}) \Big] \, d\mathbf{w} \right\| \leq \epsilon, \quad \mathbf{s} \in \mathbb{R}^n. \tag{2.1}$$

(ii) A p-locally Lebesgue integrable function $F : \mathbb{R}^n \to \mathbb{C}$ is vectorially Stepanov-p-almost periodic if for every $\epsilon > 0$ there exists $W > 0$ such that for each $\mathbf{w}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{w}_0, W)$ with

$$\left| \int_{[0,1]^n} \Big[F(\mathbf{s} + \tau + \mathbf{w}) - F(\mathbf{s} + \mathbf{w}) \Big]^p \, d\mathbf{w} \right| \leq \epsilon^p, \quad \mathbf{s} \in \mathbb{R}^n. \tag{2.2}$$

It is clear that the notion of vectorial Stepanov-p-almost periodicity cannot be easily introduced if $p \neq 1$ and $X \neq \mathbb{C}$.

Immeditely from the equations (1.1), (2.1) and (2.2), it follows that any Stepanov almost periodic function is vectorially Stepanov almost periodic as well as that any scalar-valued Stepanov-p-almost periodic function is vectorially Stepanov-p-almost periodic. The relations between Stepanov-p-almost periodic functions in norm (cf. [18, Definition 11]) and vectorial Stepanov-p-almost periodic functions are quite non-trivial ($p = 1$, general X; $p > 0$ and $p \neq 1$, $X = \mathbb{C}$). Furthermore, the class of scalar-valued vectorially Stepanov-p-almost periodic functions behaves very badly if $p \neq 1$ and we will not examined this class in more detail henceforth. The use of vector-valued integration in Definition 2.1(i) may cause some unpleasant difficulties sometimes, as well; for example, it is not simple to consider the products of vectorially Stepanov almost periodic functions and the existence of Bohr-Fourier coefficients of vectorially Stepanov almost periodic functions (see, e.g., [14, Theorem 2.1.1]).

Example 2.1. On some function spaces, the Gaussian semigroup

$$(G(s)F)(y) := \Big(4\pi s\Big)^{-(n/2)} \int_{\mathbb{R}^n} F(y - z) e^{-\frac{|z|^2}{4s}} \, dz, \quad s > 0, \ f \in Y, \ y \in \mathbb{R}^n,$$

is a strongly continuous semigroup generated by the Laplacian Δ_Y acting with its maximal distributional domain in Y. Let $F(\cdot)$ be bounded and vectorially Stepanov almost periodic, and let $s0 > 0$ be fixed. Then, the function $\mathbb{R}^n \ni x \mapsto u(x, s_0) \equiv (G(s_0)F)(x) \in \mathbb{C}$ is bounded and vectorially Stepanov almost periodic, which is am essential consequence of the next calculus ($x \in \mathbb{R}^n$; $\tau \in \mathbb{R}^n$ satisfies the necessary requirements):

$$\begin{aligned}
&\left|\int_{[0,1]^n}\Big[(G(s_0)F)(v+\tau+z)-(G(s_0)F)(v+z)\Big]\,dz\right|\\
=&\left|\int_{[0,1]^n}\left[\int_{\mathbb{R}^n}F(v+\tau+u-w)e^{-\frac{|w|^2}{4s_0}}\,dw-\int_{\mathbb{R}^n}F(v+u-w)e^{-\frac{|w|^2}{4s_0}}\,dw\right]du\right|\\
=&\left|\int_{[0,1]^n}\left[\int_{\mathbb{R}^n}\Big[F(v+\tau+u-w)-F(v+u-w)\Big]e^{-\frac{|w|^2}{4s_0}}\,dw\right]du\right|\\
=&\left|\int_{\mathbb{R}^n}e^{-\frac{|w|^2}{4s_0}}\left[\int_{[0,1]^n}\Big[F(v+\tau+u-w)-F(v+u-w)\Big]\,du\right]dw\right|\\
\leq&\int_{\mathbb{R}^n}e^{-\frac{|w|^2}{4s_0}}\left|\int_{[0,1]^n}\Big[F(v+\tau+u-w)-F(v+u-w)\Big]\,du\right|dy\leq\epsilon\int_{\mathbb{R}^n}e^{-\frac{|w|^2}{4t_0}}\,dw.
\end{aligned}$$

We can also consider here vectorially Stepanov c-almost periodic functions; see Definition 2.4 below with $\rho=c\mathrm{I}$ and $c\in\mathbb{C}\setminus\{0\}$.

The proof of following result is relatively plain.

Proposition 2.1. *A locally integrable function $F:\mathbb{R}^n\to X$ is vectorially Stepanov almost periodic if and only if the function $G=\Sigma(F):\mathbb{R}^n\to X$, given by $G(\mathbf{t}):=\int_{[0,1]^n}F(\mathbf{t}+\mathbf{u})\,d\mathbf{u}$, $\mathbf{t}\in\mathbb{R}^n$, is almost periodic.*

In the one-dimensional setting, we have that $G(t)=F^{[1]}(t+1)-F^{[1]}(t)$, $t\in\mathbb{R}$, where $F^{[1]}(t):=\int_0^t F(s)\,ds$, $t\in\mathbb{R}$ is the first integral of function $F(\cdot)$. In particular, if $F^{[1]}(\cdot)$ is almost periodic, then $F(\cdot)$ is vectorially Stepanov almost periodic (see also Kadets's theorem [4, Theorem 4.6.11] and the research articles [3] by J. Andres, D. Pennequin, [8] by C. Budde, J. Kreulich, [10] by H.-S. Ding et al., and [21] by A. M. Samoilenko, S. I. Trofimchuk for further information concerning the integration of almost periodic functions). The following example justifies the introduction of notion in Definition 2.1.

Example 2.2. Let us recall that there exists a function $f:\mathbb{R}\to\mathbb{R}$ which is bounded continuous and has the feature that the function $f^{[1]}(\cdot)$ is almost periodic and the function $f(\cdot)$ is not Stepanov almost periodic; cf. [5, Example 3.2]. Since $f^{[1]}(\cdot)$ is almost periodic, we have that $f(\cdot)$ is vectorially Stepanov almost periodic; see also [15, p. 62] for more details about this function.

Proposition 2.1 implies that vectorially Stepanov almost periodic functions form a vector space as well as that any vectorially Stepanov almost periodic function $F:\mathbb{R}^n\to X$ is vectorially Stepanov bounded in the sense that

$$\sup_{\mathbf{t}\in\mathbb{R}^n}\left\|\int_{t+[0,1]^n}F(\mathbf{w})\,d\mathbf{w}\right\|<+\infty.$$

The notion of vectorial Stepanov boundedness and the usually considered Stepanov boundedness are different.

Example 2.3. Let $X := \mathbb{C}$ and $F(t) := t\sin(t^2)$, $t \in \mathbb{R}$. Then we have

$$\int_x^{x+1} \Bigl| s \cdot \sin\Bigl(s^2\Bigr)\Bigr| \, ds = \frac{1}{2}\int_{x^2}^{(x+1)^2} |\sin v| \, dv \geq c|x|, \quad x \in \mathbb{R},$$

for some $c > 0$. Hence,

$$\sup_{x\in\mathbb{R}} \Biggl[\Bigl(1+|x|\Bigr)^{-\sigma} \int_x^{x+1} \Bigl|f(s)\Bigr| \, ds\Biggr] = +\infty, \quad \sigma \in (0,1).$$

A similar argumentation shows that

$$\Biggl|\int_x^{x+1} f(s) \, du\Biggr| = \frac{1}{2}\Bigl|\cos\Bigl(x^2\Bigr) - \cos\Bigl((x+1)^2\Bigr)\Bigr| \leq 1, \quad x \in \mathbb{R},$$

which yields the required conclusion. Further on,

$$\begin{aligned}&\int_x^{x+1} [f(s+\tau) - f(s)] \, ds \\ =&\frac{1}{2}\Biggl[\Bigl(\cos\Bigl((x+\tau)^2\Bigr) - \cos\Bigl((x+\tau+1)^2\Bigr)\Bigr) - \Bigl(\cos\Bigl(x^2\Bigr) - \cos\Bigl((x+1)^2\Bigr)\Bigr)\Biggr], \quad x, \tau \in \mathbb{R},\end{aligned}$$

which shows that $f(\cdot)$ cannot be vectorially Stepanov almost periodic.

The vector-valued integration can be useful in the studies of vectorial weighted ergodic components and vectorial weighted ergodic components in general metric, as well (see [15, Section 6.4] and [16, Section 5.2]).

2.1. **Σ-almost periodicity.** In the rest of this section, let us assume that $\emptyset \neq I \subseteq \mathbb{R}^n$, $P \subseteq Y^I$, $0 \in P$ and $\mathcal{P} = (P, d)$ is a pseudometric space; if $f \in P$, set $\|f\|_P := d(f, 0)$.

We recall that the notion of a (strongly) $(\mathrm{R}, \mathcal{B}, \mathcal{P}, L)$-multi-almost periodic function is introduced in [17, Definition 2.1]. Keeping in mind Proposition 2.1, we would like to introduce the following general notion.

Definition 2.2. Let $\emptyset \neq I \subseteq \mathbb{R}^n$, $\Sigma : Y^{I\times X} \to Y^{I\times X}$, and let R be an arbitrary non-empty collection of sequences in $\mathbb{R}^n$, $F : I \times X \to Y$, for each $B \in \mathcal{B}$ and $\mathbf{b} \in \mathrm{R}$ the set $L(B; \mathbf{b})$ is a collection of subsets of B, and we have the following:

$$\text{If } \ \mathbf{s} \in I, \ \mathbf{a} \in \mathrm{R} \text{ and } l \in \mathbb{N}, \text{ then we have } \mathbf{s} + \mathbf{a}(l) \in I. \tag{2.3}$$

Then $F(\cdot;\cdot)$ is said to be $(\Sigma, \mathrm{R}, \mathcal{B}, \mathcal{P}, L)$-multi-almost periodic, resp. strongly $(\Sigma, \mathrm{R}, \mathcal{B}, \mathcal{P}, L)$-multi-almost periodic in the case that $I = \mathbb{R}^n$, if $\Sigma(F)$ is $(\mathrm{R}, \mathcal{B}, \mathcal{P}, L)$-multi-almost periodic, resp. strongly $(\mathrm{R}, \mathcal{B}, \mathcal{P}, L)$-multi-almost periodic.

If for each $B' \in \mathcal{B}$ and $\mathbf{b}' \in \mathrm{R}$, we have $L(B'; \mathbf{b}') = \{B'\}$, then we omit the term "L" from the notation.

We can similarly define the notion of a (strong) $(\Sigma, \mathrm{R}_{\mathrm{X}}, \mathcal{B}, \mathcal{P})$-multi-almost periodic function (of type 1), where R_{X} is an arbitrary collection of sequences in $\mathbb{R}^n \times X$; cf. [17, Definition 2.2]. Once it is done, we can extend [17, Proposition 2.6] in the following way.

Proposition 2.2. *Let P be a vector structure with the usual operations, let P be complete and let the metric d be translation invariant. Let us assume that, for every $j \in \mathbb{N}$, the function $F_j : I \times X \to Y$ is $(\Sigma, \mathrm{R_X}, \mathcal{B}, \mathcal{P}, L)$-multi-almost periodic of type* 1 *as well as that, for every sequence which belongs to* $\mathrm{R_X}$, *any its subsequence also belongs to* $\mathrm{R_X}$. *If $F : I \times X \to Y$ and for each $B' \in \mathcal{B}$, $(\mathbf{b}; \mathbf{x}) = ((b_k; x_k)) = ((b_k^1, b_k^2, \ldots, b_k^n); x_k) \in \mathrm{R}_X,\ B' \in L(B; (\mathbf{b}; \mathbf{x}))$ and we have*

$$\lim_{(i',l')\to(+\infty,+\infty)} \sup_{y\in B'} \Bigl\| F_i\Bigl(\cdot + b_{k_{l'}}; y + x_{k_{l'}}\Bigr) - F\Bigl(\cdot + b_{k_{l'}}; y + x_{k_{l'}}\Bigr) \Bigr\|_P = 0,$$

then $F(\cdot;\cdot)$ is $(\Sigma, \mathrm{R_X}, \mathcal{B}, \mathcal{P}, L)$-multi-almost periodic of type 1, *if:*

(i) $\Sigma(f-g) = \Sigma f - \Sigma g,\ f, g \in P$;

(ii) *there exists $d > 0$ such that $\|\Sigma f\|_P \le d\|f\|_P,\ f \in P$.*

For example, this condition holds if $I = \mathbb{R}^n$, $P = C_b(\mathbb{R}^n : Y)$ and $\Sigma(\cdot)$ is given as in the formulation of Proposition 2.1. Suppose now P is a vector space, the metric d has the property of translation invariance and, for every $d \in \mathbb{C}$, one has $df \in P$, $f \in P$ and the existence of a real number $\phi(d) > 0$ such that $\|df\|_P \le \phi(d)\|f\|_P$ for all $f \in P$. If we assume that for each sequence in R $[\mathrm{R}_X]$ any its subsequence also belongs to R $[\mathrm{R}_X]$ as well as that the mapping Σ is linear, then the space consisting of all (strongly) $(\Sigma, \mathrm{R}, \mathcal{B}, \mathcal{P}, L)$-multi-almost periodic [(strongly) $(\Sigma, \mathrm{R}_X, \mathcal{B}, \mathcal{P}, L)$-multi-almost periodic] functions is a vector space; see also [17, Remark(ii), pp. 234–235].

The reader may consult [17, Definition 3.1] for the notions of a Bohr $(\mathcal{B}, I', \rho, \mathcal{P})$-almost periodic function and a $(\mathcal{B}, I', \rho, \mathcal{P})$-uniformly recurrent function. The following notion could be also introduced (we can similarly examine some other classes of (metrically)Σ-almost periodic type functions considered in [16, Chapter 4–Chapter 7] in the case that Σ is the identity mapping).

Definition 2.3. Let $\emptyset \neq I' \subseteq \mathbb{R}^n$, $\emptyset \neq I \subseteq \mathbb{R}^n$, $\Sigma : Y^{I\times X} \to Y^{I\times X}$, $F : I \times X \to Y$, ρ is a binary relation on Y, $R(\Sigma(F)) \subseteq D(\rho)$ and $I + I' \subseteq I$. Then, it is said that:

(i) $F(\cdot;\cdot)$ is Bohr $(\Sigma, \mathcal{B}, I', \rho, \mathcal{P})$-almost periodic if $\Sigma(F)$ is Bohr $(\mathcal{B}, I', \rho, \mathcal{P})$-almost periodic;

(ii) $F(\cdot;\cdot)$ is $(\Sigma, \mathcal{B}, I', \rho, \mathcal{P})$-uniformly recurrent if $\Sigma(F)$ is $(\mathcal{B}, I', \rho, \mathcal{P})$-uniformly recurrent.

The structural results established for Bohr $(\mathcal{B}, I', \rho, \mathcal{P})$-almost periodic functions and $(\mathcal{B}, I', \rho, \mathcal{P})$-uniformly recurrent functions can be straightforwardly generalized for Bohr $(\Sigma, \mathcal{B}, I', \rho, \mathcal{P})$-almost periodic functions and $(\Sigma, \mathcal{B}, I', \rho, \mathcal{P})$-uniformly recurrent functions, provided that the mapping $\Sigma(\cdot)$ has suitable properties; we can slightly extend the assertions of [17, Proposition 3.7, Corollary 3.8, Proposition 3.10] in this manner, for example. We will not examine such results here.

2.2. **Vectorially Stepanov almost periodic type functions.** We denote here the region I by Λ and the region I' by Λ'. We assume the following:

(SM1-1v): $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is Lebesgue measurable, $m(\Omega) > 0$, $\Lambda' + \Lambda \subseteq \Lambda$ and $\Lambda + \Omega \subseteq \Lambda$.

Now we introduce the following notion.

Definition 2.4. Suppose that (SM1-1v) holds, ρ is a binary relation on Y, $F : \Lambda \times X \to Y$, for each $x \in X$ the mapping $F(\cdot; x)$ is locally integrable,

$$[\Sigma(F)](\mathbf{s}; x) := \int_{\Omega} F(\mathbf{s} + \mathbf{w}; x)\, d\mathbf{w}, \quad \mathbf{s} \in \Lambda,\ x \in X, \tag{2.4}$$

and $R(\Sigma(F)) \subseteq D(\rho)$. Then, $F(\cdot;\cdot)$ is said to be vectorially Stepanov $(\mathcal{B}, \Lambda', \rho, \mathcal{P})$-almost periodic, resp. vectorially Stepanov $(\mathcal{B}, \Lambda', \rho, \mathcal{P})$-uniformly recurrent, if $F(\cdot;\cdot)$ is Bohr $(\Sigma, \mathcal{B}, \Lambda', \rho, \mathcal{P})$-almost periodic, resp. $(\Sigma, \mathcal{B}, \Lambda', \rho, \mathcal{P})$-uniformly recurrent, with $\Sigma(\cdot)$ being defined through (2.4).

If (2.3) holds, then it is said that $F(\cdot;\cdot)$ is vectorially Stepanov $(\mathrm{R}, \mathcal{B}, \mathcal{P}, L)$-multi-almost periodic, resp. strongly vectorially Stepanov $(\mathrm{R}, \mathcal{B}, \mathcal{P}, L)$-multi-almost periodic in the case that $\Lambda = \mathbb{R}^n$, if $\Sigma(F)$ is $(\mathrm{R}, \mathcal{B}, \mathcal{P}, L)$-multi-almost periodic, resp. strongly $(\mathrm{R}, \mathcal{B}, \mathcal{P}, L)$-multi-almost periodic, with $\Sigma(\cdot)$ being defined through (2.4).

If $F : \Lambda \times X \to Y$, $G : \Lambda \times X \to Y$ and for each $x \in X$ the mappings $F(\cdot; x)$ and $G(\cdot; x)$ are locally integrable, then we have $\Sigma(\alpha F + \beta G) = \alpha\Sigma(F) + \beta\Sigma(G)$ for all scalars α, $\beta \in \mathbb{C}$, so that the collection of all vectorially Stepanov $(\mathcal{B}, \Lambda', \rho, \mathcal{P})$-almost periodic functions is a linear vector space if the corresponding space of all $(\mathcal{B}, \Lambda', \rho, \mathcal{P})$-almost periodic functions is a linear vector space (we cannot expect the linearity of space of all vectorially Stepanov $(\mathcal{B}, \Lambda', \rho, \mathcal{P})$-uniformly recurrent functions in any sense; cf. [15]). Furthermore, if $\Lambda = \Lambda' = \mathbb{R}^n$, $\rho = \mathrm{I}$, R contains all sequences in $\mathbb{R}^n$ and $\mathcal{B}$ contains all compact sets in X, then the metrical Bochner criterion established in [19, Theorem 1] immediately clarifies the coincidence of the class of all vectorially Stepanov $(\mathcal{B}, \Lambda', \rho, \mathcal{P})$-almost periodic functions and the class consisting of all vectorially Stepanov $(\mathrm{R}, \mathcal{B}, \mathcal{P})$-multi-almost periodic functions, provided that the pseudometric space P has the properties stated in this result.

Proposition 2.3. *Assume* (SM1-1v) *holds,* $\rho = T \in L(Y)$, $F_k : \Lambda \times X \to Y$ *and the function* $F_k(\cdot;\cdot)$ *is vectorially Stepanov* $(\mathcal{B}, \Lambda', T, \mathcal{P})$*-almost periodic, resp. vectorially Stepanov* $(\mathcal{B}, \Lambda', T, \mathcal{P})$*-uniformly recurrent* $(k \in \mathbb{N})$*, where* $P = C_b(\Lambda : Y)$*. If* $F : \Lambda \times X \to Y$*, for each* $y \in X$ *the mapping* $F(\cdot; y)$ *is locally integrable, and for each* $B \in \mathcal{B}$*,*

$$\lim_{l \to +\infty} \sup_{\mathbf{s} \in \Lambda, y \in B} \left\| \int_{\Omega} \Big[F_l(\mathbf{s} + \mathbf{w}; x) - F(\mathbf{s} + \mathbf{w}; x) \Big]\, d\mathbf{w} \right\| = 0,$$

then $F(\cdot;\cdot)$ *is vectorially Stepanov* $(\mathcal{B}, \Lambda', T, \mathcal{P})$*-almost periodic, resp. vectorially Stepanov* $(\mathcal{B}, \Lambda', T, \mathcal{P})$*-uniformly recurrent.*

Proof. We examine the class of vectorially Stepanov $(\mathcal{B}, \Lambda', T, \mathcal{P})$-almost periodic functions, only. Let us fix $B \in \mathcal{B}$ and $\epsilon > 0$. Then there exist $k_0 \in \mathbb{N}$ and $W > 0$ such that for each $\mathbf{w}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{w}_0, W) \cap \Lambda'$ such that

$$\sup_{\mathbf{s}\in\Lambda, y\in B} \Bigl\| \int_{\mathbf{s}+\Omega} \Bigl[F_{k_0}(\mathbf{w}+\tau; y) - TF_{k_0}(\mathbf{w}; y) \Bigr]\, d\mathbf{w} \Bigr\| \leq \epsilon/3.$$

Then, for each $\mathbf{s} \in \Lambda$ and $y \in B$, we have the following:

$$\begin{aligned}
&\Bigl\| \int_{\mathbf{s}+\Omega} \Bigl[F(\mathbf{u}+\tau; y) - TF(\mathbf{u}; y) \Bigr]\, d\mathbf{u} \Bigr\| \\
&\leq \Bigl\| \int_{\mathbf{s}+\Omega} \Bigl[F(\mathbf{u}+\tau; y) - F_{k_0}(\mathbf{u}+\tau; y) \Bigr]\, d\mathbf{u} \Bigr\| + \Bigl\| \int_{\mathbf{s}+\Omega} \Bigl[F_{k_0}(\mathbf{u}+\tau; y) - TF_{k_0}(\mathbf{u}; y) \Bigr]\, d\mathbf{u} \Bigr\| \\
&\quad + \Bigl\| \int_{\mathbf{s}+\Omega} \Bigl[TF_{k_0}(\mathbf{u}; y) - TF(\mathbf{u}; y) \Bigr]\, d\mathbf{u} \Bigr\| \\
&\leq \Bigl\| \int_{\mathbf{s}+\Omega} \Bigl[F(\mathbf{u}+\tau; y) - F_{k_0}(\mathbf{u}+\tau; y) \Bigr]\, d\mathbf{u} \Bigr\| + \Bigl\| \int_{\mathbf{s}+\Omega} \Bigl[F_{k_0}(\mathbf{u}+\tau; y) - TF_{k_0}(\mathbf{u}; y) \Bigr]\, d\mathbf{u} \Bigr\| \\
&\quad + \|T\| \cdot \Bigl\| \int_{\mathbf{s}+\Omega} \Bigl[F_{k_0}(\mathbf{u}; y) - F(\mathbf{u}; y) \Bigr]\, d\mathbf{u} \Bigr\|.
\end{aligned}$$

The vectorial Stepanov $(\mathcal{B}, \Lambda', T, \mathcal{P})$-almost periodicity of $F(\cdot;\cdot)$ follows thereof. $\square$

Suppose now that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is Lebesgue measurable, $m(\Omega) > 0$ and $\Lambda + \Omega \subseteq \Lambda$. Denote by $S^1_v(\Lambda \times X : Y)$ the collection of all functions $F : \Lambda \times X \to Y$ such that for each $x \in X$ the function $F(\cdot; x)$ is locally integrable and

$$\Bigl\| F \Bigr\|_{S^1_{v,B,\Omega}} := \sup_{\mathbf{t}\in\Lambda, x\in B} \frac{1}{m(\Omega)} \Bigl\| \int_{\mathbf{t}+\Omega} F(\mathbf{s}; x)\, d\mathbf{s} \Bigr\| < +\infty, \quad B \in \mathcal{B}. \tag{2.5}$$

For functions $F : \Lambda \to Y$, we exclude the term "B"; moreover, we exclude the term "Ω" if $\Omega = [0, 1]^n$.

Let us observe the following facts.

(i) The assumption $\|F\|_{S^1_v} = 0$ does not imply $F(\mathbf{t}) = 0$ for a.e. $\mathbf{t} \in \Lambda$; for example, examine the function $F(t) = \cos(2\pi t)$, $t \in \mathbb{R}$.

(ii) If $\lambda \in \mathbb{C}$, $B \in \mathcal{B}$ and $\|F\|_{S^1_{v,B,\Omega}} < +\infty$, then $\|\lambda \cdot F\|_{S^1_{v,B,\Omega}} = |\lambda| \cdot \|F\|_{S^1_{v,B,\Omega}}$.

(iii) If $B \in \mathcal{B}$, $\|F\|_{S^1_{v,B,\Omega}} < +\infty$, $G : \Lambda \times X \to Y$ has the property that for each $y \in X$ the function $G(\cdot; y)$ is locally integrable and $\|G\|_{S^1_{v,B,\Omega}} < +\infty$, then

$$\Bigl\| F + G \Bigr\|_{S^1_{v,B,\Omega}} \leq \Bigl\| F \Bigr\|_{S^1_{v,B,\Omega}} + \Bigl\| G \Bigr\|_{S^1_{v,B,\Omega}}.$$

For each $B \in \mathcal{B}$, by $S^1_{v,B}(\Lambda \times B : Y)$ we denote the collection of all functions $F : \Lambda \times B \to Y$ such that for each $x \in B$ the function $F(\cdot; x)$ is locally integrable and (2.5) holds. Then $S^1_{v,B}(\Lambda \times B : Y)$ is a vector space and (i)-(iii) simply yield that $\| \cdot \|_{S^1_{v,B,\Omega}}$ is a seminorm on $S^1_{v,B}(\Lambda \times B : Y)$.

The next result can be compared to [20, Theorem 5.2.1, pp. 199–200].

Proposition 2.4. *Suppose that $B \in \mathcal{B}$, $(F_k(\cdot;\cdot))_{k\in\mathbb{N}}$ is a sequence of functions in $S^1_{v,B}(\Lambda \times B : Y)$, and the following holds.*

(i) *For every* $\epsilon > 0$, *there exists* $s_0 \in \mathbb{N}$ *such that, for every* $s_1, s_2 \in \mathbb{N}$ *with* $\min(s_1, s_2) \geq s_0$, *we have* $\|F_{s_1} - F_{s_2}\|_{S^1_{v,B,\Omega}} \leq \epsilon$.

(ii) *There exists* $F : \Lambda \times B \to Y$ *such that for each* $x \in B$ *the function* $F(\cdot; x)$ *is locally integrable and* $\lim_{k\to+\infty} \|F_k(\mathbf{t}; x) - F(\mathbf{t}; x)\| = 0$ *for a.e.* $\mathbf{t} \in \Lambda$.

(iii) *For each compact set* $K \subseteq \Lambda$ *and for each* $x \in B$, *there exists* $g_x \in L^1(K : Y)$ *such that* $\|F_k(\mathbf{t}; x)\| \leq g_x(\mathbf{t})$ *a.e. on* K *for all* $k \in \mathbb{N}$.

Then, $\lim_{k\to+\infty} \Big\| F_k - F \Big\|_{S^1_{v,B,\Omega}} = 0$.

Proof. If $\epsilon > 0$, $x \in B$ and $\mathbf{t} \in \Lambda$, then [4, Theorem 1.1.8] and (ii)-(iii) together imply that

$$\lim_{m\to+\infty} \int_{\mathbf{t}+\Omega} F_m(\mathbf{s}; x)\, d\mathbf{s} = \int_{\mathbf{t}+\Omega} F(\mathbf{s}; x)\, d\mathbf{s}. \tag{2.6}$$

On the other hand, (i) implies that there exists $s_0 \in \mathbb{N}$ such that, for every $s_1, s_2 \in \mathbb{N}$ with $\min(s_1, s_2) \geq s_0$, one has:

$$\Bigg\| \int_{\mathbf{r}+\Omega} \Big[F_{s_1}(\mathbf{s}; y) - F_{s_2}(\mathbf{s}; y) \Big]\, d\mathbf{s} \Bigg\| \leq \epsilon, \quad \mathbf{r} \in \Lambda,\ y \in B. \tag{2.7}$$

The required conclusion follows by letting $m \to +\infty$ in (2.7) and using (2.6) after that. $\square$

3. Vectorially Weyl almost Periodic Type Functions

If $F : \mathbb{R}^n \to X$ is Lebesgue measurable and for each bounded and closed set $K \subseteq \mathbb{R}^n$, we have $\int_K \|F(\mathbf{t})\|^p\, d\mathbf{t} < +\infty$, where $p > 0$, then $F(\cdot)$ is:

(i) equi-Weyl-p-almost periodic if, for every $\epsilon > 0$, there exist two finite real numbers $l' > 0$ and $L' > 0$ such that for each $\mathbf{w}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{w}_0, L') \cap \mathbb{R}^n$ with

$$\sup_{\mathbf{t}\in\mathbb{R}^n} \Bigg[(l')^{-n} \int_{\mathbf{t}+l'[0,1]^n} \Big\| F(\tau + \mathbf{u}) - F(\mathbf{u}) \Big\|^p\, d\mathbf{u} \Bigg] < \epsilon.$$

(ii) Weyl-p-almost periodic if, for every $\epsilon > 0$, there exists a finite real number $L' > 0$ such that for each $\mathbf{w}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{w}_0, L') \cap \mathbb{R}^n$ with

$$\limsup_{l'\to+\infty} \sup_{\mathbf{t}\in\mathbb{R}^n} \Bigg[(l')^{-n} \int_{\mathbf{t}+l'[0,1]^n} \Big\| F(\tau + \mathbf{u}) - F(\mathbf{u}) \Big\|^p\, d\mathbf{u} \Bigg] < \epsilon.$$

Now we would like to propose the following notion (we will not consider here the metrical generalizations of this definition; see [16, Section 4.3]).

Definition 3.1. Suppose that $p > 0$.

(i) A locally Lebesgue integrable function $F : \mathbb{R}^n \to X$ is said to be vectorially equi-Weyl-almost periodic if, for every $\epsilon > 0$, there exist two finite real numbers

$l'>0$ and $L'>0$ such that for each $\mathbf{w}_0\in\mathbb{R}^n$ there exists $\tau\in B(\mathbf{w}_0,L')$ with

$$\sup_{\mathbf{t}\in\mathbb{R}^n}\Biggl\|\int_{\mathbf{t}+l'[0,1]^n}\Bigl[F(\tau+\mathbf{u})-F(\mathbf{u})\Bigr]\,d\mathbf{u}\Biggr\|<\epsilon(l')^n.$$

(ii) A locally Lebesgue integrable function $F:\mathbb{R}^n\to X$ is said to be vectorially Weyl-almost periodic if, for every $\epsilon>0$, there exists a finite real number $L'>0$ such that for each $\mathbf{w}_0\in\mathbb{R}^n$ there exists $\tau\in B(\mathbf{w}_0,L')$ with

$$\limsup_{l'\to+\infty}\sup_{\mathbf{t}\in\mathbb{R}^n}\Biggl[(l')^{-n}\Biggl\|\int_{\mathbf{t}+l'[0,1]^n}\Bigl[F(\tau+\mathbf{u})-F(\mathbf{u})\Bigr]\,d\mathbf{u}\Biggr\|\Biggr]<\epsilon.$$

Any (equi-)Weyl-1-almost periodic function is clearly vectorially (equi-)Weyl-almost periodicand any vectorially Stepanov almost periodic function $F:\mathbb{R}^n\to X$ is clearly vectorially equi-Weyl-almost periodic.

Example 3.1. (see also [4, Example 4.6.5]) Suppose that $X:=c_0$. Define $f(t):=((1/k)\cos(t/k))_k$, $t\in\mathbb{R}$. Then $f:\mathbb{R}\to X$ is almost periodic but its first integral $F(t):=(\sin(t/k))_k$, $t\in\mathbb{R}$ is bounded, uniformly continuous but not Stepanov-p-almost automorphic for any exponent $p>0$; furthermore, we know that $F(\cdot)$ is vectorially Weyl almost automorphic as well as that $F(\cdot)$ is not vectorially Weyl almost automorphic of type 1 nor jointly vectorially Weyl almost automorphic (see [1]).

Let us prove that $F(\cdot)$ is vectorially Weyl-almost periodic, i.e., the vectorially Weyl-1-almost periodic. To show this, notice that ($r,s\in\mathbb{R},\ \tau\in\mathbb{R}$):

$$F\Bigl(r+s+\tau\Bigr)-F(r+s)=2\left(\sin\frac{\tau}{2k}\cdot\cos\frac{2r+2s+2\tau}{2k}\right)_k,$$

so that ($t\in\mathbb{R},\ \tau\in\mathbb{R};\ l>0$)

$$\begin{aligned}&\frac{1}{l}\int_t^{t+l}\Bigl[F(v+\tau)-F(v)\Bigr]\,dv\\ =&\frac{2}{l}\left(k\sin\frac{\tau}{2k}\cdot\left[\sin\frac{2t+2l+\tau}{2k}-\sin\frac{2t+\tau}{2k}\right]\right)_k\\ =&\frac{4}{l}\left(k\sin\frac{\tau}{2k}\cdot\cos\frac{2t+l+\tau}{2k}\cdot\sin\frac{l}{2k}\right)_k.\end{aligned}\tag{3.1}$$

This immediately implies the required statement since $|k\sin(\tau/2k)|\leq|\tau|/2$, $k\in\mathbb{N}$, and

$$\left|\cos\frac{4t+2l+2\tau}{4k}\cdot\sin\frac{l}{2k}\right|\leq 1,\quad k\in\mathbb{N}.$$

Let us prove now that $F(\cdot)$ is not Weyl-p-almost periodic for any $p>0$. Suppose that $0<\epsilon<32^{-1}\cdot(\cos(7\pi/16)\cdot\sqrt{2}/2)^p$. Then, there exists $L>0$ such that, for every $t_0\in\mathbb{R}$, we can find $\tau\in[t_0-L,t_0+L]$ such that there exists $l_0(\tau)>0$ with

$$\int_0^l\left\|\left(\cos\frac{2x+\tau}{2k}\cdot\sin\frac{\tau}{2k}\right)_k\right\|^p\,dx\leq\epsilon\cdot l,\quad l\geq l_0(\tau).\tag{3.2}$$

Suppose that $k_0 \in \mathbb{N}$ is sufficiently large and $2k_0\pi/4 \leq \tau < (2k_0+2)\pi/4$. Then $\pi/4 \leq \tau/2k_0 < 3\pi/8$ and therefore $\sin(\tau/2k_0) \geq \sqrt{2}/2$. Suppose, further, that $x_0/k_0 = \pi/16$; then $\cos((2x+\tau)/2k_0) \geq \cos(7\pi/16)$, $x \in [2m\pi k_0, 2m\pi k_0 + x_0]$ $(m \in \mathbb{N}_0)$ and taking into account (3.2) with with $l = 2m\pi k_0 + x_0$, where m is sufficiently large, we get

$$\begin{aligned} &\epsilon\Big(2m\pi k_0 + x_0\Big) \\ &\geq \int_0^{2m\pi k_0+x_0} \left\| \left(\cos\frac{2x+\tau}{2k} \cdot \sin\frac{\tau}{2k} \right)_k \right\|^p dx \\ &\geq \int_0^{2m\pi k_0+x_0} \left| \cos\frac{2x+\tau}{2k_0} \cdot \sin\frac{\tau}{2k_0} \right|^p dx \geq (m+1)x_0 \left(\frac{\sqrt{2}}{2} \cdot \cos\frac{7\pi}{16} \right)^p, \quad l \geq l_0(\tau). \end{aligned}$$

Dividing by m and letting $m \to +\infty$, we get

$$2\pi\epsilon\Big(k_0/x_0\Big) = 32\epsilon \geq \left(\frac{\sqrt{2}}{2} \cdot \cos\frac{7\pi}{16} \right)^p,$$

which is a contradiction.

Now we will prove that $F(\cdot)$ is not vectorially equi-Weyl-almost periodic. Suppose that $0 < \epsilon < \sqrt{2}/2$. Then there exist $l > 0$ and $L > 0$ such that, for every $t_0 \in \mathbb{R}$, there exists $\tau \in [t_0 - L, t_0 + L]$ such that

$$\left\| \frac{4}{l} \left(k \sin\frac{\tau}{2k} \cdot \cos\frac{2t+l+\tau}{2k} \cdot \sin\frac{l}{2k} \right)_k \right\| \leq \epsilon, \quad t \in \mathbb{R}; \tag{3.3}$$

see (3.1). Furthermore, there exists $k_0(\epsilon, l) \in \mathbb{N}$ such that

$$4k \sin\frac{l}{2k} \geq l, \quad k \geq k_0(\epsilon, l). \tag{3.4}$$

After that, take any $t_0 > L + \pi k_0(\epsilon, l)/2$. Then there exists $k_0 \geq k_0(\epsilon, l)$ such that $2k_0\pi/4 \leq \tau < (2k_0+2)\pi/4$. Then, as above, $\pi/4 \leq \tau/2k_0 < 3\pi/8$ and $\sin(\tau/2k_0) \geq \sqrt{2}/2$. Since (3.3) holds for all $t \in \mathbb{R}$, we can take $t = -(l+\tau)/2$ in order to see that $\cos((2t+l+\tau)/2k_0) = 1$ so that (3.4) and the above estimates enable one to see that the right hand side of (3.3) is greater than or equal to $\sqrt{2}/2$, which is a contradiction.

Let us finally observe that, for every $l > 0$, we have

$$\sup_{r \in \mathbb{R}} \frac{1}{l} \left\| \int_r^{r+l} F(s)\, ds \right\| = 1, \tag{3.5}$$

so that the vectorial Weyl seminorm of $F(\cdot)$, defined through

$$\|F\|_{W,v} := \lim_{l \to +\infty} \sup_{r \in \mathbb{R}} \frac{1}{l} \left\| \int_r^{r+l} F(s)\, ds \right\|,$$

exists and equals 1. In order to see that (3.5) holds, notice that a simple computation shows that

$$\frac{1}{l} \left\| \int_r^{r+l} F(s)\, ds \right\| = \sup_{k \in \mathbb{N}} \left[\frac{2k}{l} \cdot \left| \sin\frac{2r+l}{2k} \cdot \sin\frac{l}{2k} \right| \right], \quad r \in \mathbb{R}.$$

Therefore, for every $l > 0$,

$$\begin{aligned}\frac{1}{l}\left\|\int_t^{t+l} F(s)\,ds\right\| &\geq \sup_{t\in(\pi\mathbb{Z}-l)/2}\ \sup_{k\in\mathbb{N}}\left[\frac{2k}{l}\cdot\left|\sin\frac{2t+l}{2k}\cdot\sin\frac{l}{2k}\right|\right]\\ &\geq \sup_{k\in\mathbb{N}}\left[\frac{2k}{l}\cdot\left|\sin\frac{l}{2k}\right|\right]=1,\end{aligned}$$

since $\lim_{k\to+\infty}(2k/l)\cdot|\sin(l/2k)|=1$.

If $F(\cdot)$ is locally integrable, then the existence of vectorial Weyl seminorm of $F(\cdot)$ in $[0,+\infty]$ cannot be proved with the help of the argumentation given on [15, pp. 375–376]. Moreover, this seminorm does not necessarily exists.

Example 3.2. Define the function $F:\mathbb{R}\to\mathbb{R}$ by $F(t):=-2k$ if $t\in[2k,2k+1)$ for some $k\in\mathbb{Z}$ and $F(t):=2k+2$ if $t\in[2k+1,2k+2)$ for some $k\in\mathbb{Z}$. Then,

$$\limsup_{l\to+\infty}\sup_{t\in\mathbb{R}}\frac{1}{l}\left|\int_t^{t+l}F(s)\,ds\right|=+\infty\quad\text{and}\quad\liminf_{l\to+\infty}\sup_{t\in\mathbb{R}}\frac{1}{l}\left|\int_t^{t+l}F(s)\,ds\right|=1,$$

and therefore, $\|F\|_{W,v}$ does not exist in $[0,+\infty]$. In order to see this, it suffices to show that

$$\sup_{t\in\mathbb{R}}\left|\int_t^{t+2k}F(s)\,ds\right|=2k\quad\text{and}\quad\sup_{t\in\mathbb{R}}\left|\int_t^{t+2k+1}F(s)\,ds\right|=+\infty,\quad k\in\mathbb{N}.\tag{3.6}$$

By our construction, we have $\int_{2m+1}^{2m+1+2k}F(s)\,ds=0$ for all $m,\ k\in\mathbb{Z}$. Hence,

$$\sup_{t\in\mathbb{R}}\left|\int_t^{t+2k+1}F(s)\,ds\right|\geq\sup_{t\in 2\mathbb{Z}+1}\left|\int_t^{t+2k+1}F(s)\,ds\right|.$$

If $t=2m+1$ for some $m\in\mathbb{Z}$, then we have

$$\int_t^{t+2k+1}F(s)\,ds=\int_{2m+2k+1}^{2m+2k+2}F(s)ds=2m+2k+2,$$

which simply implies the second equality in (3.6). On the other hand, we have

$$\sup_{t\in\mathbb{R}}\left|\int_t^{t+2k}F(s)\,ds\right|\geq\int_0^{2k}F(s)\,ds=2k,\quad k\in\mathbb{N}.\tag{3.7}$$

If $t\in[2m,2m+1)$ for $m\in\mathbb{Z}$, then one has

$$\begin{aligned}\int_t^{t+2k}F(s)\,ds=&\int_t^{2m+1}F(s)\,ds-\int_{t+2k}^{2m+2k+1}F(s)\,ds\\ =&(-2)(2m+1-t)m+(2m+1-t)(2m+2k)=2(2m+1-t)k,\end{aligned}$$

so that

$$\left|\int_t^{t+2k}F(s)\,ds\right|\leq 2k.\tag{3.8}$$

Similarly, if $t \in [2m+1, 2m+2)$ for an integer $m \in \mathbb{Z}$, then one has

$$\int_t^{t+2k} F(s)\, ds = \int_t^{2m+1} F(s)\, ds + \int_{2m+2k+1}^{t+2k} F(s)\, ds$$
$$= (2m+1-t)(2m+2) - (2m+1-t)(2m+2k+2) = 2(2m+1-t)k,$$

so that (3.8) again holds. By (3.7)-(3.8), we deduce the first equality in (3.6).

The uniform convergence of a sequence of locally integrable functions implies the convergence of this sequence in the vectorial Weyl seminorm; furthermore, the standard evidence shows that, if the sequence of vectorially (equi)-Weyl-almost periodic functions converges in the vectorial Weyl seminorm, then the limit function is likewise vectorially (equi) Weyl-almost periodic.

It is not so simple to deduce a proper analogue of Proposition 2.1 for vectorially Weyl almost periodic type functions. After clarifying this fact, we will generalize the notion introduced in Definition 3.1 as follows (cf. also [16, Definition 3.1.1–Definition 3.1.6]).

Definition 3.2. Suppose that $F : \Lambda \times X \to Y$, ρ is a binary relation on Y and the following condition holds.

(WM1-1v): $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is Lebesgue measurable, $m(\Omega) > 0$, $\Lambda' + \Lambda \subseteq \Lambda$ and $\Lambda + l\Omega \subseteq \Lambda$ for $l > 0$.

(i) It is said that $F(\cdot;\cdot)$ is vectorially equi-Weyl-$(\mathcal{B}, \Lambda', \rho, \Omega)$-almost periodic if, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exist two finite real numbers $l' > 0$ and $L' > 0$ such that for each $\mathbf{w}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{w}_0, L') \cap \Lambda'$ such that for each $\mathbf{t} \in \Lambda$ and $x \in B$ there exists an integrable mapping $y_{\mathbf{t};x} : \mathbf{t} + l'\Omega \to Y$ such that $y_{\mathbf{t};x}(\mathbf{u}) \in \rho(F(\mathbf{u}; x))$ for a.e. $\mathbf{u} \in \mathbf{t} + l'\Omega$ and

$$\sup_{\mathbf{t}\in\Lambda; x\in B} \left\| \int_{\mathbf{t}+l'\Omega} \Big[F(\tau + \mathbf{u}; x) - y_{\mathbf{t};x}(\mathbf{u}) \Big]\, d\mathbf{u} \right\| < \epsilon (l')^n. \tag{3.9}$$

(ii) It is said that $F(\cdot;\cdot)$ is vectorially Weyl-$(\mathcal{B}, \Lambda', \rho, \Omega)$-almost periodic if, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L' > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{w}_0, L') \cap \Lambda'$ such that for each $\mathbf{t} \in \Lambda$ and $x \in B$ there exists an integrable mapping $y_{\mathbf{t};x} : \mathbf{t} + l'\Omega \to Y$ such that $y_{\mathbf{t};x}(\mathbf{u}) \in \rho(F(\mathbf{u}; x))$ for a.e. $\mathbf{u} \in \mathbf{t} + l'\Omega$ and

$$\limsup_{l'\to+\infty} \sup_{\mathbf{t}\in\Lambda; x\in B} \left[(l')^{-n} \left\| \int_{\mathbf{t}+l'\Omega} \Big[F(\tau + \mathbf{u}; x) - y_{\mathbf{t};x}(\mathbf{u}) \Big]\, d\mathbf{u} \right\| \right] < \epsilon.$$

We have the following result.

Proposition 3.1. (*cf. also* [16, pp. 23–24]) *Suppose that* $m \in \mathbb{N}$, $F : \Lambda \times X \to Y$, $\rho = T \in L(Y)$ *and* (WM1-1v) *holds. If* $F(\cdot;\cdot)$ *is vectorially* (*equi-*) *Weyl-*$(\mathcal{B}, \Lambda', T, \Omega)$-*almost periodic, then* $F(\cdot;\cdot)$ *is vectorially* (*equi-*) *Weyl-*$(\mathcal{B}, m\Lambda', T^m, \Omega)$-*almost periodic, as well.*

Proof. We examine the class of vectorially equi-Weyl-$(\mathcal{B}, \Lambda', T, \Omega)$-almost periodic functions, only. Let $\epsilon > 0$ and $B \in \mathcal{B}$ be fixed. Then there exist two finite real numbers $l' > 0$ and $L' > 0$ such that for each $\mathbf{w}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{w}_0, L') \cap \Lambda'$ such that for each $\mathbf{t} \in \Lambda$ and $x \in B$ we have that (3.9) holds with $y_{\mathbf{t};x} = TF(\mathbf{u}; x)$ for a.e. $\mathbf{u} \in \mathbf{t} + l'\Omega$. Let $\mathbf{t} \in \Lambda$ and $x \in B$ be fixed, and let $\mathbf{w}_0 \in \Lambda'$ and $\tau \in B(\mathbf{w}_0, L') \cap \Lambda'$ be as above. Then, it is clear that $s\Lambda' + \Lambda \subseteq \Lambda$, $s \in \mathbb{N}$ and

$$F(\mathbf{u} + m\tau; x) - T^m F(\mathbf{u}; x) = \sum_{j=0}^{m-1} T^j \Big[F(\mathbf{u} + (m-j)\tau; x) - TF(\mathbf{u} + (m-j-1)\tau; x) \Big],$$

for any $\mathbf{u} \in \mathbf{t} + l'\Omega$. This implies

$$\begin{aligned} &\Big\| \int_{\mathbf{t}+l'\Omega} \Big[F(\mathbf{u} + m\tau; x) - T^m F(\mathbf{u}; x) \Big]\, d\mathbf{u} \Big\| \\ &\times \sum_{j=0}^{m-1} \|T\|^j \cdot \Big\| \int_{\mathbf{t}+l'\Omega} \Big[F(\mathbf{u} + (m-j)\tau; x) - T^m F(\mathbf{u} + (m-j-1)\tau; x) \Big]\, d\mathbf{u} \Big\| \\ = &\sum_{j=0}^{m-1} \|T\|^j \cdot \Big\| \int_{\mathbf{t}+(m-j-1)\tau+l'\Omega} \Big[F(\mathbf{u} + \tau; x) - T^m F(\mathbf{u}; x) \Big]\, d\mathbf{u} \Big\| \le \sum_{j=0}^{m-1} \|T\|^j \epsilon. \end{aligned}$$

The required conclusion follows from this estimate. □

We close this section by observing that Proposition 3.1 can be also formulated for the corresponding classes of vectorially Stepanov almost periodic functions.

4. Invariance of Vectorial Stepanov Almost Periodicity and Vectorial Weyl Almost Periodicity under the Actions of Convolution Products

We open this section by clarifying the following result.

Proposition 4.1. *Assume $\nu : \mathbb{R} \to (0, +\infty)$, $1/v(\cdot)$ is locally bounded and there exists a Lebesgue measurable function $\psi : \mathbb{R} \to (0, +\infty)$ such that $\nu(x + y) \le \psi(x)\nu(y)$ for all $x, y \in \mathbb{R}$. Suppose further that $f : \mathbb{R} \to X$ is a bounded, vectorially $(\Lambda', T, \mathcal{P})$-almost periodic function, resp. a bounded, vectorially $(\Lambda', T, \mathcal{P})$-uniformly recurrent function, where $\rho = T \in L(X)$ and $P = C_{b,\nu}(\mathbb{R} : X)$. If $(R(t))_{t>0}$ is any strongly continuous operator family in $L(X)$ such that $R(t)T = TR(t)$ for all $t > 0$ and $\int_0^{+\infty} \|R(r)\| \cdot (1 + \psi(r))\, dr < +\infty$, then the function $F : \mathbb{R} \to X$, given by*

$$F(t) := \int_{-\infty}^{t} R(t - s) f(s)\, ds, \quad t \in \mathbb{R}, \tag{4.1}$$

is bounded, continuous and vectorially $(\Lambda', T, \mathcal{P})$-almost periodic, resp. bounded, continuous and vectorially $(\Lambda', T, \mathcal{P})$-uniformly recurrent.

Proof. Suppose that $\epsilon > 0$. Then there exists $l' > 0$ such that, for every $w_0 \in \Lambda'$ there exists $\tau \in [w_0 - l', w_0 + l'] \cap \Lambda'$ such that

$$\Big\| \Big[\int_0^1 f(t + \tau + s)\, ds - T \int_0^1 f(t + s)\, ds \Big] \cdot \nu(t) \Big\| \le \epsilon, \quad t \in \mathbb{R}. \tag{4.2}$$

Clearly, $F(\cdot)$ is well-defined, bounded and continuous and $F(t)=\int_0^{+\infty}R(r)f(t-r)\,dr$ for all $t\in\mathbb{R}$. Furthermore, we can use (4.2), the Fubini theorem and the prescribed assumptions to show that:

$$
\begin{aligned}
&\Bigg\|\Bigg[\int_0^1 F(t+\tau+s)\,ds-T\int_0^1 F(t+s)\,ds\Bigg]\cdot\nu(t)\Bigg\|\\
=&\Bigg\|\Bigg[\int_0^1\int_0^{+\infty}R(r)f(t+\tau+s-r)\,dr\,ds-T\int_0^1\int_0^{+\infty}R(r)f(t+s-r)\,dr\,ds\Bigg]\cdot\nu(t)\Bigg\|\\
=&\Bigg\|\Bigg[\int_0^1\int_0^{+\infty}R(r)\big[f(t+\tau+s-r)\,dr\,ds-Tf(t+s-r)\big]\,dr\,ds\Bigg]\cdot\nu(t)\Bigg\|\\
=&\Bigg\|\Bigg[\int_0^{+\infty}R(r)\Bigg[\int_0^1\big[f(t+\tau+s-r)\,dr\,ds-Tf(t+s-r)\big]\,ds\Bigg]\,dr\cdot\nu(t)\Bigg]\Bigg\|\\
\leq&\int_0^{+\infty}\|R(r)\|\cdot\Bigg\|\int_0^1\big[f(t+\tau+s-r)\,dr\,ds-Tf(t+s-r)\big]\,ds\Bigg\|\,dr\cdot\nu(t)\\
\leq&\int_0^{+\infty}\|R(r)\|\cdot\frac{\epsilon}{\nu(t-r)}\cdot\nu(t)\,dr\leq\epsilon\int_0^{+\infty}\|R(r)\|\psi(r)\,dr.
\end{aligned}
$$

This simply implies the desired conclusion. □

The statement of [14, Proposition 2.6.11] cannot be reconsidered for the vectorial Stepanov almost periodicity. Concerning the vectorial Weyl almost periodicity, we will clarify the following result; cf. also the proof of [14, Proposition 2.11.1 (i)].

Proposition 4.2. *Suppose that $(R(t))_{t>0}\subseteq L(X)$ is a strongly continuous operator family satisfying that $\int_0^{+\infty}\|R(s)\|\,ds<+\infty$, $T\in L(X)$ and $R(t)T=TR(t)$ for all $t>0$. If $f:\mathbb{R}\to X$ is bounded and vectorially (equi-) Weyl-$(\Lambda',T,\mathcal{P})$-almost periodic, where $P=C_b(\mathbb{R}:X)$, then the function $F(\cdot)$, given by (4.1), is bounded, continuous and vectorially (equi-) Weyl-$(\Lambda',T,\mathcal{P})$-almost periodic.*

The statement of [14, Theorem 2.11.4] cannot be reconsidered for the vectorial Weyl almost periodicity, unfortunately. Let us finally note that we can employ different pivot spaces X and Y in the formulations of Proposition 4.1 and Proposition 4.2, provided that $T=c\mathrm{I}$, where $c\in\mathbb{C}\setminus\{0\}$.

5. Conclusions and Final Remarks

In this research article, we have introduced and analyzed several new classes of vectorial Stepanov-p-almost periodic type functions and vectorial (equi)-Weyl-p-almost periodic type functions, where $p>0$. In contrast with the usual research studies of generalized almost periodic functions, we have used the vector-valued integration in our approach. We have also analyzed Σ-almost periodic type functions and the invariance of vectorial Stepanov almost periodicity and vectorial Weyl almost periodicity under the actions of convolution products.

The notion introduced in Definition 2.4 (ii) can be extended in the following way (the metrical generalizations of this definition can be also introduced; see [16, Section 4.2] for more details).

Definition 5.1. Suppose that (SM1-1v) holds, $p > 0$, ρ is a binary relation on Y, $F : \Lambda \times X \to Y$ and $R(F) \subseteq D(\rho)$. The function $F(\cdot;\cdot)$ is said to be vectorially Stepanov $(\mathcal{B}, \Lambda', \rho, p)$-almost periodic if for each $x \in X$ the mapping $F(\cdot; x)$ is p-locally integrable as well as, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists $l' > 0$ such that for each $\mathbf{w}_0 \in \Lambda$ there exists a point $\tau \in B(\mathbf{w}_0, l') \cap \Lambda'$ such that for each $\mathbf{t} \in \Lambda$ and $x \in B$ there exists a p-locally integrable function $y_{\mathbf{t};x}(\cdot)$ on Ω such that $y_{\mathbf{t};x}(\mathbf{u}) \in \rho(F(\mathbf{t} + \mathbf{u}; x))$ for a.e. $\mathbf{u} \in \Omega$ and

$$\left| \int_{\Omega} \Big[F(\mathbf{t} + \tau + \mathbf{u}; x) - y_{\mathbf{t};x}(\mathbf{u}) \Big]^p \, d\mathbf{u} \right| \leq \epsilon^p, \quad \mathbf{t} \in \Lambda, \ x \in B.$$

The notion of a vectorially Stepanov $(\mathcal{B}, \Lambda', \rho, p)$-uniformly recurrent function can be similarly introduced in the scalar-valued setting. Let us finally note that we can also consider the class of scalar-valued vectorially (equi-)Weyl-p-almost periodic functions and the class of scalar-valued vectorially (equi-)Weyl-$(\mathcal{B}, \Lambda', \rho, p)$-almost periodic functions following our approaches from Definition 3.1 and Definition 3.2 as well as many other classes of generalized vectorially almost periodic functions. More details will appear somewhere else.

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References

[1] S. Abbas and M. Kostić, *Metrical Weyl almost automorphy and applications,* Adv. Pure Appl. Math. (accepted). `https://www.openscience.fr/Metrical-Weyl-almost-automorphy-and-applications`
[2] J. Andres, A. M. Bersani and R. F. Grande, *Hierarchy of almost-periodic function spaces,* Rend. Mat. Appl. **2**6(7) (2006), 121–188.
[3] J. Andres and D. Pennequin, *On the nonexistence of purely Stepanov almost-periodic solutions of ordinary differential equations,* Proc. Amer. Math. Soc. **140** (2012), 2825–2834.
[4] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics, Vol. **96**, Birkhäuser, Basel, 2001.
[5] B. Basit and H. Güenzler, *Spectral criteria for solutions of evolution equations and comments on reduced spectra,* Far East J. Math. Sci. (FJMS) **65** (2012), 273–288.
[6] A. S. Besicovitch, *Almost Periodic Functions,* Dover Publ., New York, 1954.
[7] S. Bochner, *Curvature and Betti numbers in real and complex vector bundles,* Rend. Semin. Mat. Univ. Politec. Torino **15** (1955–1956).
[8] C. Budde and J. Kreulich, *On splittings and integration of almost periodic functions with and without geometry,* Semigroup Forum **108** (2024), 335–364.
[9] T. Diagana, *Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces,* Springer-Verlag, New York, 2013.
[10] H.-S. Ding, W-G. Jian, N. V. Minh and G. M. N'Guérékata, *Kadets type and Loomis type theorems for asymptotically almost periodic functions,* J. Diff. Equ. **373** (2023), 389–410.
[11] A. M. Fink, *Almost Periodic Differential Equations,* Springer-Verlag, Berlin, 1974.
[12] G. M. N'Guérékata, *Almost Automorphic and Almost Periodic Functions in Abstract Spaces,* Kluwer Acad. Publ, Dordrecht, 2001.
[13] M. Kostić, *Abstract Volterra Integro-Differential Equations,* CRC Press, Boca Raton, Fl., 2015.

[14] M. Kostić, *Almost Periodic and Almost Automorphic Type Solutions to Integro-Differential Equations,* W. de Gruyter, Berlin, 2019.
[15] M. Kostić, *Selected Topics in Almost Periodicity,* W. de Gruyter, Berlin, 2022.
[16] M. Kostić, *Metrical Almost Periodicity and Applications to Integro-Differential Equations,* W. de Gruyter, Berlin, 2023.
[17] M. Kostić, *Metrical almost periodicity and applications,* Ann. Polon. Math. **129** (2022), 219–254.
[18] M. Kostić and W.-S. Du, *Stepanov almost periodic type functions and applications to abstract impulsive Volterra integro-differential inclusions,* Fractal Fract. (2023), Article ID 736. `https://doi.org/10.3390/fractalfract7100736`
[19] M. Kostić, V. E. Fedorov and H. C. Koyuncuoğlu, *Metrical Bochner criterion and metrical Stepanov almost periodicity,* Chelj. Phy. Math. J. **9** (2024), 90–100.
[20] M. Levitan, *Almost Periodic Functions,* G.I.T.T.L., Moscow, 1953 (in Russian).
[21] A. M. Samoilenko and S. I. Trofimchuk, *Unbounded functions with almost periodic differences,* Ukrainian Math. J. **43** (1992), 1306–1309.
[22] W. Stepanoff, *Über einige Verallgemeinerungen der fast periodischen Funktionen,* Math. Ann. **95** (1926), 473–498.
[23] S. Zaidman, *Almost-Periodic Functions in Abstract Spaces,* Pitman Research Notes in Math, Vol. **126**, Pitman, Boston, 1985.

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