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SEVERAL THEOREMS OF APPROXIMATION THEORY FOR THE q-BESSEL TRANSFORM

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Abstract. In this paper, we prove several theorems in approximation theory concerning the q-Bessel transform. A new estimate for the q-Bessel function is obtained. By applying the generalized modulus of continuity, we establish new Jackson-type inverse estimates for the generalized q-Bessel transform in the space $\mathcal{L}^2_{q,\alpha}(\mathbb{R}^+_q)$, where $q \in (0,1)$ and $\alpha > -1/2$.

1. Introduction

Let $f : \mathbb{R} \to \mathbb{C}$ a Lebesgue square-integrable function on $\mathbb{R}$, i.e., $f \in L^2(\mathbb{R})$. We define the finite differences of order $k \in \mathbb{N}$ by

$$\Delta_h^k(f;x) = (F_h - I)^k f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} F_h^i f(x), \quad x \in \mathbb{R},$$

where F_h is the Steklov operator defined by

$$F_h f(x) = 2^{-1} h^{-1} \int_{x-h}^{x+h} f(t)dt, \quad h > 0, \tag{1.1}$$

and I is the unit operator in $L^2(\mathbb{R})$.

For a given positive real number δ, the k^{th}-order generalized modulus of continuity of a function f is defined by

$$\Omega_k(f,\delta) = \sup_{0<h\leq\delta} \|\Delta_h^k(f;x)\|_{L^2(\mathbb{R})}.$$

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Let $W^{r,k}_{2,\Phi}(D)$, where $r = 0, 1, \dots$ and $k = 1, 2, \dots$, denote the class of functions $f \in L^2(\mathbb{R})$ for which the generalized partial derivatives exist in the sense of Levi:

$$\frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \dots, \frac{\partial^r f}{\partial x^r},$$

which all belong to $L^2(\mathbb{R})$ (see [21, p. 172]). These functions satisfy the estimate

$$\Omega_k(D^r f, \delta) = O\Big(\Phi(\delta^k)\Big), \quad \text{as } \delta \to 0,$$

where $D = \frac{\partial}{\partial x}$, and the iterated derivatives $D^i f$ are defined recursively by

$$D^0 f = f, \quad D^i f = D(D^{i-1} f), \quad i = 1, 2, \dots, r.$$

Here, Φ is a continuous, strictly increasing function on $[0, +\infty)$ with $\Phi(0) = 0$.

The following theorem is an analogue of Jackson's direct theorem from the classical theory of function approximation [21, Ch. 5] (see also [1, Theorem 1]).

Theorem 1.1. *It holds that*

$$\sup_{f \in W^{r,k}_{2,\Phi}(D)} \sqrt{\int_{|\lambda| \geq N} |\widehat{f}(\lambda)|^2 d\lambda} = O\left(N^{-r} \Phi\left[\left(\frac{2}{N}\right)^k\right]\right),$$

as $N \to +\infty$, where $r = 0, 1, \dots$, $k = 1, 2, \dots$, and $\widehat{f}$ stands for the Fourier transform of f.

In the case where $\Phi(t) = t^\nu$ with $\nu > 0$, Abilov et al. characterized the functions $f \in L^2(\mathbb{R})$ by the following equivalence (see [1, Theorem 2]).

Theorem 1.2. *Let $\Phi(t) = t^\nu$ with $\nu > 0$. Then, the assertions $f \in W^{r,k}_{2,t^\nu}(D)$ and*

$$\sqrt{\int_{|\lambda| \geq N} |\widehat{f}(\lambda)|^2 d\lambda} = O(N^{-r-k\nu}), \quad \text{as } N \to +\infty,$$

are equivalent, where $r = 0, 1, \dots$, $k = 1, 2, \dots$, and $0 < \nu < 2$.

Recently, considerable attention has been paid to the development of various q-analogues of Fourier analysis using tools from quantum calculus (see [4, 5, 8, 9, 12] and references therein). In the context of q-Bessel analysis, many works have focussed on the construction of q-Fourier analysis associated with the q-Hankel transform. This theory was first introduced by Koornwinder and Swarttouw [19], and later extended by Fitouhi et al. [9, 10, 13]. It is therefore natural to investigate q-analogues of classical theorems in this framework.

In this paper, building on the work of Abilov, we establish new Jackson-type inverse estimates for the q-Bessel transform for certain classes of functions characterised by a modulus of continuity and associated with the q-Bessel operator. Similar results were obtained in [2, 6, 7, 25–30]. To achieve these results, we use a generalised q-Bessel translation operator rather than the Steklov operator defined in (1.1).

2. Harmonic Analysis Associated with the q-Bessel Operator

This section provides the essential background to q-harmonic analysis related to the q-Bessel transform. We give a brief overview of the standard notions and notations of q-theory, while further details and results on q-Bessel analysis can be found in [5, 9–11, 13, 17, 18].

Throughout this paper, we assume $q \in]0,1[$ and $\alpha > -1/2$. We introduce the following set

$$\mathbb{R}_q^+ = \{q^n,\ n \in \mathbb{Z}\}.$$

Let $a \in \mathbb{C}$, the q-shifted factorials are defined by:

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{l=0}^{n-1}(1-aq^l), \quad n = 1,2,\dots, \quad (a;q)_\infty = \prod_{l=0}^{+\infty}(1-aq^l). \tag{2.1}$$

The q-derivative of a function f is given by

$$\mathcal{D}_q f(x) = \frac{f(x)-f(qx)}{(1-q)x}, \quad \text{if } x \neq 0,$$

where $\mathcal{D}_q f(0) = f'(0)$ provided $f'(0)$ exists. Note that when f is differentiable at x, then $\mathcal{D}_q f(x)$ tends to $f'(x)$ as q tends to 1^-.

The general q-hypergeometric series is defined by

$$\begin{aligned} {}_r\phi_s\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, z\right] &= {}_r\phi_s\left(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s; q, z\right) \\ &= \sum_{n=0}^{+\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(b_1, b_2, \dots, b_s; q)_n (q;q)_n} \left[(-1)^n q^{n(n-1)/2}\right]^{1+s-r} z^n, \end{aligned} \tag{2.2}$$

where the q-shifted factorial is defined by (2.1) and

$$(a_1, a_2, \dots, a_r; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n.$$

The q-Jackson integrals from 0 to a and from 0 to $+\infty$ are defined by [14, 17]

$$\int_0^a f(x) d_q x = (1-q)a \sum_{n=0}^{+\infty} q^n f(aq^n),$$

$$\int_0^{+\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{+\infty} q^n f(q^n),$$

provided the sums converge absolutely. The q-Jackson integral in a generic interval $[a,b]$ is given by

$$\int_a^b f(x) d_q x := \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

We denote by $\mathcal{L}^p_{q,\alpha}(\mathbb{R}_q^+)$, $p \geq 1$, the set of all real functions f defined on $\mathbb{R}_q^+$ for which

$$\|f\|_{q,p,\alpha} = \left[\int_0^{+\infty} |f(x)|^p x^{2\alpha+1} d_q x\right]^{1/p} < +\infty.$$

We denote by $\mathcal{C}_{q,0}(\mathbb{R}_q^+)$, for the space of functions defined on $\mathbb{R}_q^+$ tending to 0 as $x \to +\infty$ and continuous at 0. The space $\mathcal{C}_{q,0}(\mathbb{R}_q^+)$, when equipped with the topology of uniform convergence, is a complete normed linear space with norm

$$\|f\|_{q,\infty} = \sup_{x\in\mathbb{R}_q^+} |f(x)|.$$

In [16], Ismail introduced the third q-Bessel function, defined as follows

$$\begin{aligned} J_\alpha(x;q^2) &= \frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} x^\alpha {}_1\phi_1\left(0;q^{2\alpha+2};q^2,q^2x^2\right) \\ &= \frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=0}^{+\infty} (-1)^n \frac{q^{n(n+1)}}{(q^{2\alpha+2};q^2)_n (q^2;q^2)_n} x^{2n+\alpha}. \end{aligned} \tag{2.3}$$

It satisfies (see [5]) the following.

- For all $n \in \mathbb{Z}$, $\alpha \geq -1/2$, we have

$$|J_\alpha(q^n;q^2)| \leq \frac{(-q^2;q^2)_\infty(-q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} \begin{cases} q^{n\alpha}, & \text{if } n \geq 0, \\ q^{n(n-\alpha-1)}, & \text{if } n < 0. \end{cases} \tag{2.4}$$

- For all $\sigma \in \mathbb{R}$, $x \in \mathbb{R}_q^+$, we have

$$J_\alpha(x;q^2) = o(x^{-\sigma}), \quad \text{as } x \to +\infty.$$

  In particular

$$J_\alpha(x;q^2) \to 0, \quad \text{as } x \to +\infty.$$

The function $J_\alpha(\cdot;q^2)$ is given in normalized form by

$$j_\alpha(x;q^2) = \sum_{n=0}^{+\infty} (-1)^n \frac{q^{n(n+1)}}{(q^{2\alpha+2};q^2)_n (q^2;q^2)_n} x^{2n}. \tag{2.5}$$

The following lemma was proved in [12, Lemma 3.1].

Lemma 2.1. *We have the following inequalities for the q-Bessel function:*

$$j_\alpha(x;q^2) = O(1), \quad \textit{if } x \geq 0 \textit{ and } x \in \mathbb{R}_q^+, \tag{2.6}$$

$$1 - j_\alpha(x;q^2) = O(1), \quad \textit{if } x \geq 1 \textit{ and } x \in \mathbb{R}_q^+, \tag{2.7}$$

$$1 - j_\alpha(x;q^2) = O(x^2), \quad \textit{if } x \leq 1 \textit{ and } x \in \mathbb{R}_q^+. \tag{2.8}$$

The q-integration theorem by a change of variable can be stated as follows

$$\int_a^b H\left(\frac{\lambda}{s}\right) \lambda^{2\alpha+1} d_q\lambda = s^{2\alpha+2} \int_{\frac{a}{s}}^{\frac{b}{s}} H(x) x^{2\alpha+1} d_q x, \quad \text{for all } s \in \mathbb{R}_q^+.$$

For $\lambda \in \mathbb{C}$, the function $x \mapsto j_\alpha(\lambda x, q^2)$ is a solution of the q-differential equation

$$\begin{cases} \Lambda_{q,\alpha} f(x) = -\lambda^2 f(x), \\ f(0) = 1, \end{cases}$$

where $\Lambda_{q,\alpha}$ denotes the q-Bessel operator, defined by

$$\Lambda_{q,\alpha} f(x) = \frac{f(q^{-1}x) - (1+q^{2\alpha})f(x) + q^{2\alpha} f(qx)}{x^2}, \quad x \in \mathbb{R}_q^+.$$

We now give the definition of the q-Bessel transform associated with the q-Bessel operator.

Definition 2.1. The q-Bessel transform $\mathcal{F}_{q,\alpha}$, is defined for every function $f \in \mathcal{L}^1_{q,\alpha}(\mathbb{R}_q^+)$ by

$$\mathcal{F}_{q,\alpha}(f)(\lambda) = c_{q,\alpha} \int_0^{+\infty} f(x)\, j_\alpha(\lambda x; q^2)\, x^{2\alpha+1}\, d_q x, \quad \text{for all } \lambda \in \mathbb{R}_q^+,$$

where

$$c_{q,\alpha} = \frac{1}{1-q} \cdot \frac{(q^{2\alpha+2}; q^2)_\infty}{(q^2; q^2)_\infty}.$$

It is well-known that the q-Bessel transform satisfies the following properties (see [10, 13, 15]).

- If $f \in \mathcal{L}^1_{q,\alpha}(\mathbb{R}_q^+)$, then $\mathcal{F}_{q,\alpha}(f) \in \mathcal{C}_{q,0}(\mathbb{R}_q^+)$ and $\|\mathcal{F}_{q,\alpha}(f)\|_{q,\infty} \leq \mathcal{B}_{q,\alpha} \|f\|_{q,1,\alpha}$, where
$$\mathcal{B}_{q,\alpha} = \frac{1}{1-q} \cdot \frac{(-q^2; q^2)_\infty\, (-q^{2\alpha+2}; q^2)_\infty}{(q^2; q^2)_\infty}.$$
- For every $f \in \mathcal{L}^2_{q,\alpha}(\mathbb{R}_q^+)$, one has

$$\mathcal{F}_{q,\alpha}(\Lambda_{q,\alpha} f)(\lambda) = -\lambda^2 \mathcal{F}_{q,\alpha}(f)(\lambda), \quad \text{for all } \lambda \in \mathbb{R}_q^+. \tag{2.9}$$

Theorem 2.1 gives the Plancherel formula and inversion formula for the q-Bessel transform.

Theorem 2.1. *i) The q-Bessel transform $\mathcal{F}_{q,\alpha}$ is an isomorphism from $\mathcal{L}^2_{q,\alpha}(\mathbb{R}_q^+)$ onto itself and satisfies the q-Plancherel formula*

$$\|\mathcal{F}_{q,\alpha}(f)\|_{q,2,\alpha} = \|f\|_{q,2,\alpha}, \quad \textit{for all } f \in \mathcal{L}^2_{q,\alpha}(\mathbb{R}_q^+). \tag{2.10}$$

ii) If $f \in \mathcal{L}^1_{q,\alpha}(\mathbb{R}_q^+)$ such that $\mathcal{F}_{q,\alpha}(f) \in \mathcal{L}^1_{q,\alpha}(\mathbb{R}_q^+)$, then the q-inversion formula holds and we have

$$f(x) = c_{q,\alpha} \int_0^{+\infty} \mathcal{F}_{q,\alpha}(f)(\lambda) j_\alpha(\lambda x; q^2) \lambda^{2\alpha+1} d_q\lambda, \tag{2.11}$$

a.e. on $\mathbb{R}_q^+$.

The q-generalized translation operator associated with the q-Bessel transform, denoted by $\mathcal{T}^\alpha_{q,h}$ for $h \in \mathbb{R}_q^+$, was introduced in [13] and later corrected in [10]. It is defined using Jackson's q-integral and the q-shifted factorial as

$$\mathcal{T}^\alpha_{q,h} f(x) = \int_0^{+\infty} f(t)\, \mathcal{K}_{q,\alpha}(h, x, t)\, t^{2\alpha+1}\, d_q t,$$

where

$$\mathcal{K}_{q,\alpha}(h, x, y) = c^2_{q,\alpha} \int_0^{+\infty} j_\alpha(ht; q^2) j_\alpha(xt; q^2) j_\alpha(yt; q^2) t^{2\alpha+1} d_q t.$$

In particular, the product formula

$$\mathcal{T}^{\alpha}_{q,h} j_\alpha(x;q^2) = j_\alpha(h;q^2) j_\alpha(x;q^2)$$

holds.

The q-generalized translation operator has the following properties (see [10, 11]).

Theorem 2.2. *i) For $f \in \mathcal{L}^p_{q,\alpha}(\mathbb{R}^+_q)$, $p \geq 1$, we have $\mathcal{T}^{\alpha}_{q,h} f \in \mathcal{L}^p_{q,\alpha}(\mathbb{R}^+_q)$ and*

$$\|\mathcal{T}^{\alpha}_{q,h} f\|_{q,p,\alpha} \leq \|f\|_{q,p,\alpha}.$$

ii) For $f \in \mathcal{L}^1_{q,\alpha}(\mathbb{R}^+_q)$, we have

$$\mathcal{F}_{q,\alpha}(\mathcal{T}^{\alpha}_{q,h} f)(\lambda) = j_\alpha(\lambda h;q^2) \mathcal{F}_{q,\alpha}(f)(\lambda). \tag{2.12}$$

For every $f \in \mathcal{L}^2_{q,\alpha}(\mathbb{R}^+_q)$, we define the differences $\Delta^m_h f$ of order m, $m = 1, 2, \ldots$, with step $h > 0$, $h \in \mathbb{R}^+_q$ by:

$$\Delta^1_h f(x) = \Delta_h f(x) := \mathcal{T}^{\alpha}_{q,h} f(x) - f(x),$$
$$\Delta^m_h f(x) = \Delta_h(\Delta^{m-1}_h f(x)), \quad \text{for } m \geq 2.$$

The m^{th}-order generalized modulus of continuity of $f \in \mathcal{L}^2_{q,\alpha}(\mathbb{R}^+_q)$ is defined by

$$\omega_m(f,\delta)_{q,2,\alpha} = \sup_{0<h\leq\delta} \|\Delta^m_h f\|_{q,2,\alpha}, \quad \delta > 0.$$

Let $W^r_{q,2}(\Lambda_{q,\alpha})$, $r = 0, 1, \ldots$, denote the class of functions $f \in \mathcal{L}^2_{q,\alpha}(\mathbb{R}^+_q)$ that have generalized derivatives in the sense of distributions satisfying

$$\Lambda^r_{q,\alpha} f \in \mathcal{L}^2_{q,\alpha}(\mathbb{R}^+_q), \quad r = 1, 2, \ldots,$$

i.e.,

$$W^r_{q,2}(\Lambda_{q,\alpha}) := \left\{ f \in \mathcal{L}^2_{q,\alpha}(\mathbb{R}^+_q) : \Lambda^r_{q,\alpha} f \in \mathcal{L}^2_{q,\alpha}(\mathbb{R}^+_q), \quad r = 1, 2, \ldots \right\},$$

where

$$\Lambda^0_{q,\alpha} f = f, \quad \Lambda^r_{q,\alpha} f = \Lambda_{q,\alpha}\left(\Lambda^{r-1}_{q,\alpha} f\right), \quad r = 1, 2, \ldots$$

From (2.9), for all $f \in W^r_{q,2}(\Lambda_{q,\alpha})$, we have

$$\mathcal{F}_{q,\alpha}\left(\Lambda^r_{q,\alpha} f\right)(\lambda) = (-1)^r \lambda^{2r} \mathcal{F}_{q,\alpha}(f)(\lambda), \quad r = 1, 2, \ldots \tag{2.13}$$

3. Main Results

We now present the main results of this paper. Their proofs rely on several preliminary lemmas.

Lemma 3.1. *For $\alpha \geq -1/2$ and $x \in \mathbb{R}^+_q$, we have*

$$\sqrt{x} J_\alpha(x;q^2) = \mathcal{O}(1), \quad \textit{for } x \geq 0. \tag{3.1}$$

Moreover, we have

$$j_\alpha(x;q^2) = \mathcal{O}(x^{-\alpha-\frac{1}{2}}), \quad \textit{for } x \geq 0. \tag{3.2}$$

Proof. First, note that

$$x \in \mathbb{R}_q^+, x \geq 1 \quad \text{if and only if} \quad x = q^n, n \leq 0,$$

and

$$x \in \mathbb{R}_q^+, 0 \leq x \leq 1 \quad \text{if and only if} \quad x = q^n, n \geq 0.$$

From this and (2.4), we distinguish two cases.

For $x = q^n \in \mathbb{R}_q^+$ with $n \geq 0$, we obtain

$$|q^{\frac{n}{2}} J_\alpha(q^n; q^2)| \leq \frac{(-q^2; q^2)_\infty (-q^{2\alpha+2}; q^2)_\infty}{(q^2; q^2)_\infty} q^{n(\alpha+\frac{1}{2})}.$$

Since $n(\alpha + \frac{1}{2}) \geq 0$, it follows that $q^{n(\alpha+\frac{1}{2})} \leq 1$, and therefore we obtain

$$|q^{\frac{n}{2}} J_\alpha(q^n; q^2)| \leq \frac{(-q^2; q^2)_\infty (-q^{2\alpha+2}; q^2)_\infty}{(q^2; q^2)_\infty}.$$

Therefore, equivalently,

$$\sqrt{x} J_\alpha(x; q^2) = \mathcal{O}(1), \quad \text{for } x \in \mathbb{R}_q^+, 0 \leq x \leq 1. \tag{3.3}$$

For the case $x = q^n \in \mathbb{R}_q^+$ with $n \leq 0$, we get

$$|q^{\frac{n}{2}} J_\alpha(q^n; q^2)| \leq \frac{(-q^2; q^2)_\infty (-q^{2\alpha+2}; q^2)_\infty}{(q^2; q^2)_\infty} q^{n(n-\alpha-\frac{1}{2})}.$$

Since $n(n - \alpha - \frac{1}{2}) \geq 0$, it follows that

$$q^{n\left(n-\alpha-\frac{1}{2}\right)} \leq 1,$$

and thus we again obtain

$$|q^{\frac{n}{2}} J_\alpha(q^n; q^2)| \leq \frac{(-q^2; q^2)_\infty (-q^{2\alpha+2}; q^2)_\infty}{(q^2; q^2)_\infty}.$$

Consequently,

$$\sqrt{x} J_\alpha(x; q^2) = \mathcal{O}(1), \quad \text{for } x \in \mathbb{R}_q^+,\ x \geq 1. \tag{3.4}$$

Now, combining (3.3) and (3.4), we obtain

$$\sqrt{x} J_\alpha(x; q^2) = \mathcal{O}(1), \quad \text{for } x \geq 0, x \in \mathbb{R}_q^+,$$

which proves (3.1). To prove (3.2), we use formulas (2.3) and (2.5), which yield

$$j_\alpha(x; q^2) = \frac{(q^2; q^2)_\infty}{(q^{2\alpha+2}; q^2)_\infty} x^{-\alpha} J_\alpha(x; q^2).$$

Thus, taking into account the formula above and (3.1), we obtain the result. □

We now present an important lemma that will lead us to the main result.

Lemma 3.2. *Let $h > 0$ with $h \in \mathbb{R}_q^+$. If $f \in W_{q,2}^r(\Lambda_{q,\alpha})$, then*

$$\|\Delta_h^m(\Lambda_{q,\alpha}^r f)\|_{q,2,\alpha}^2 = \int_0^{+\infty} \lambda^{4r}|1 - j_\alpha(\lambda h, q^2)|^{2m}|\mathcal{F}_{q,\alpha}(f)(\lambda)|^2\lambda^{2\alpha+1}d_q\lambda,$$

where $r = 0, 1, 2, \ldots$

Proof. According to the formula (2.12), we obtain

$$\mathcal{F}_{q,\alpha}(\Delta_h f)(\lambda) = \mathcal{F}_{q,\alpha}(\mathcal{T}_{q,h}^\alpha f)(\lambda) - \mathcal{F}_{q,\alpha}(f)(\lambda) = (j_\alpha(\lambda h, q^2) - 1)\mathcal{F}_{q,\alpha}(f)(\lambda).$$

Using the recurrence proof with respect to m, we obtain

$$\mathcal{F}_{q,\alpha}(\Delta_h^m f)(\lambda) = (j_\alpha(\lambda h, q^2) - 1)^m\mathcal{F}_{q,\alpha}(f)(\lambda).$$

In view of formula (2.13), we get

$$\mathcal{F}_{q,\alpha}(\Delta_h^m(\Lambda_{q,\alpha}^r f))(\lambda) = (-1)^r\lambda^{2r}(j_\alpha(\lambda h, q^2) - 1)^m\mathcal{F}_{q,\alpha}(f)(\lambda).$$

Now, by appealing the q-Plancherel formula (2.10), we have the desired result. □

The following theorem is an analogue of Abilov's theorem [1, Theorem 1]. It represents a Jackson-type direct theorem from the classical theory of function approximation [21, Chapter 5].

Theorem 3.1. *For a function $f \in \mathcal{L}_{q,\alpha}^2(\mathbb{R}_q^+)$ in the class $W_{q,2}^r(\Lambda_{q,\alpha})$ there exists a fixed constant $c > 0$ such that for all $N > 0$, we have*

$$\mathcal{J}_N(f) = \mathcal{O}\left(N^{-2r}\omega_m\left(\Lambda_{q,\alpha}^r f, \frac{c}{N}\right)_{q,2,\alpha}\right),$$

where $r = 0, 1, 2, \ldots$, $m = 1, 2, \ldots$

Proof. Firstly, for all $h \in \mathbb{R}_q^+$, we can see that

$$\begin{aligned}\mathcal{J}_N^2(f) &= \int_N^{+\infty} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2\lambda^{2\alpha+1}d_q\lambda \\ &\le \int_N^{+\infty} |j_\alpha(\lambda h, q^2)| \cdot |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2\lambda^{2\alpha+1}d_q\lambda \\ &\quad + \int_N^{+\infty} |1 - j_\alpha(\lambda h, q^2)| \cdot |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2\lambda^{2\alpha+1}d_q\lambda. \end{aligned} \tag{3.5}$$

According to the formula (3.2), there exists a constant $c_1 > 0$ such that

$$|j_\alpha(\lambda h, q^2)| \le c_1(\lambda h)^{-\alpha-\frac{1}{2}}.$$

Hence,

$$\begin{aligned}\int_N^{+\infty} |j_\alpha(\lambda h, q^2)| \cdot |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2\lambda^{2\alpha+1}d_q\lambda &\le c_1 h^{-\alpha-\frac{1}{2}}\int_N^{+\infty} \lambda^{-\alpha-\frac{1}{2}}|\mathcal{F}_{q,\alpha}(f)(\lambda)|^2\lambda^{2\alpha+1}d_q\lambda \\ &\le c_1(hN)^{-\alpha-\frac{1}{2}}\mathcal{J}_N^2(f).\end{aligned}$$

Choosing $c>0$ such that the constant $c_2 = 1 - c_1 c^{-\alpha-\frac{1}{2}}$ is positive and setting $h = c/N$ in inequality (3.5). Then, we have

$$c_2 \mathcal{J}_N^2(f) \leq \int_N^{+\infty} |1 - j_\alpha(\lambda h, q^2)| \cdot |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2 \lambda^{2\alpha+1} d_q\lambda. \tag{3.6}$$

Furthermore, by Hölder's inequality, the second term in (3.6) satisfies

$$\begin{aligned}
&\int_N^{+\infty} |1 - j_\alpha(\lambda h, q^2)| \cdot |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2 \lambda^{2\alpha+1} d_q\lambda \\
&= \int_N^{+\infty} |1 - j_\alpha(\lambda h, q^2)| \left(|\mathcal{F}_{q,\alpha}(f)(\lambda)| \lambda^{\alpha+\frac{1}{2}}\right)^{2-\frac{1}{m}} \left(|\mathcal{F}_{q,\alpha}(f)(\lambda)| \lambda^{\alpha+\frac{1}{2}}\right)^{\frac{1}{m}} d_q\lambda \\
&\leq \left(\int_N^{+\infty} |1 - j_\alpha(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2 \lambda^{2\alpha+1} d_q\lambda\right)^{\frac{1}{2m}} (\mathcal{J}_N(f))^{\frac{2m-1}{m}} \\
&= \left(\int_N^{+\infty} \frac{1}{\lambda^{4r}} \lambda^{4r} |1 - j_\alpha(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2 \lambda^{2\alpha+1} d_q\lambda\right)^{\frac{1}{2m}} (\mathcal{J}_N(f))^{\frac{2m-1}{m}} \\
&\leq N^{\frac{-2r}{m}} (\mathcal{J}_N(f))^{\frac{2m-1}{m}} \left(\int_N^{+\infty} \lambda^{4r} |1 - j_\alpha(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2 \lambda^{2\alpha+1} d_q\lambda\right)^{\frac{1}{2m}}.
\end{aligned}$$

From Lemma 3.2, we conclude that

$$\int_N^{+\infty} \lambda^{4r} |1 - j_\alpha(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2 \lambda^{2\alpha+1} d_q\lambda \leq \|\Delta_h^m(\Lambda_{q,\alpha}^r f)\|_{q,2,\alpha}^2.$$

Therefore,

$$\int_N^{+\infty} |1 - j_\alpha(\lambda h, q^2)| \cdot |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2 \lambda^{2\alpha+1} d_q\lambda \leq N^{\frac{-2r}{m}} (\mathcal{J}_N(f))^{\frac{2m-1}{m}} \|\Delta_h^m(\Lambda_{q,\alpha}^r f)\|_{q,2,\alpha}^{\frac{1}{m}}.$$

For $h = c/N$, we obtain

$$c_2 \mathcal{J}_N^2(f) \leq N^{\frac{-2r}{m}} (\mathcal{J}_N(f))^{\frac{2m-1}{m}} \omega_m^{1/m}\left(\Lambda_{q,\alpha}^r f, \frac{c}{N}\right)_{q,2,\alpha}.$$

Consequently, by raising both sides to the power m and simplifying by $(\mathcal{J}_N(f))^{2m}$, we finally obtain

$$\mathcal{J}_N(f) \leq c_2^{-m} N^{-2r} \omega_m \left(\Lambda_{q,\alpha}^r f, \frac{c}{N}\right)_{q,2,\alpha}.$$

Therefore,

$$\mathcal{J}_N(f) = \mathcal{O}\left(N^{-2r} \omega_m \left(\Lambda_{q,\alpha}^r f, \frac{c}{N}\right)_{q,2,\alpha}\right).$$

This completes the proof of Theorem 3.1. □

Theorems 3.2 and 3.3 are analogues of the classical inverse theorems of approximation theory due to Stechkin and Timan (see [22, 23]).

Theorem 3.2. *Let $f \in \mathcal{L}_{q,\alpha}^2(\mathbb{R}_q^+)$. Then, for all $N > 0$,*

$$\omega_m\left(f, \frac{1}{N}\right)_{q,2,\alpha} = \mathcal{O}\left(N^{-2m}\left(\sum_{l=0}^{N-1} (l+1)^{4m-1} \mathcal{J}_l^2(f)\right)^{\frac{1}{2}}\right).$$

Proof. From Lemma 3.2, we have

$$\|\Delta_h^m f\|_{q,2,\alpha}^2 = \int_0^{+\infty} |1 - j_\alpha(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2 \lambda^{2\alpha+1} d_q\lambda.$$

This integral is divided into two:

$$\int_0^{+\infty} = \int_0^N + \int_N^{+\infty} = I_1 + I_2,$$

where $N = [\frac{1}{h}]$. We estimate them separately. From (2.6), we have the estimate

$$I_2 \leq c_3 \int_N^{+\infty} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2 \lambda^{2\alpha+1} d_q\lambda = c_3 \mathcal{J}_N^2(f). \tag{3.7}$$

Now, we estimate $\mathcal{I}_1$. From relation (2.8), we have

$$\begin{aligned} I_1 &\leq c_4 h^{4m} \int_0^N \lambda^{4m} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2 \lambda^{2\alpha+1} d_q\lambda \\ &= c_4 h^{4m} \sum_{l=0}^{N-1} \int_l^{l+1} \lambda^{4m} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2 \lambda^{2\alpha+1} d_q\lambda \\ &\leq c_4 h^{4m} \sum_{l=0}^{N-1} a_l (\mathcal{J}_l^2(f) - \mathcal{J}_{l+1}^2(f)), \end{aligned}$$

where $a_l = (l+1)^{4m}$ and

$$\mathcal{J}_l^2(f) = \int_l^{+\infty} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2 \lambda^{2\alpha+1} d_q\lambda.$$

For all integers $M \geq 1$, the Abel transformation shows

$$\begin{aligned} \sum_{l=0}^{M} a_l (\mathcal{J}_l^2(f) - \mathcal{J}_{l+1}^2(f)) &= a_0 \mathcal{J}_0^2(f) - a_M \mathcal{J}_{M+1}^2(f) + \sum_{l=1}^{M} (a_l - a_{l-1}) \mathcal{J}_l^2(f) \\ &\leq \mathcal{J}_0^2(f) + \sum_{l=1}^{M} (a_l - a_{l-1}) \mathcal{J}_l^2(f), \end{aligned}$$

because $a_M \mathcal{J}_{M+1}^2(f) \geq 0$. Hence,

$$I_1 \leq c_4 h^{4m} \left(\mathcal{J}_0^2(f) + \sum_{l=1}^{N-1} (a_l - a_{l-1}) \mathcal{J}_l^2(f) \right).$$

Moreover by the finite increments theorem, we have

$$a_l - a_{l-1} \leq 4m(l+1)^{4m-1}.$$

Then,

$$I_1 \leq c_4 N^{-4m} \left(\mathcal{J}_0^2(f) + 4m \sum_{l=1}^{N-1} (l+1)^{4m-1} \mathcal{J}_l^2(f) \right),$$

since $N \leq \frac{1}{h}$. Combining the estimates for I_1 and I_2 gives

$$\|\Delta_h^m f\|_{q,2,\alpha}^2 = \mathcal{O}\left(N^{-4m} \sum_{l=0}^{N-1} (l+1)^{4m-1} \mathcal{J}_l^2(f) \right),$$

which implies

$$\omega_m\left(f,\frac{1}{N}\right)_{q,2,\alpha}=\mathcal{O}\left(N^{-2m}\left(\sum_{l=0}^{N-1}(l+1)^{4m-1}\mathcal{J}_l^2(f)\right)^{\frac{1}{2}}\right),$$

and this completes the proof. □

Theorem 3.3. *Let $f\in\mathcal{L}^2_{q,\alpha}(\mathbb{R}_q^+)$. If the series*

$$\sum_{l=1}^{+\infty}l^{2r-1}\mathcal{J}_l(f),\quad r=1,2,\dots$$

converges, then $f\in W^r_{q,2}(\Lambda_{q,\alpha})$ and, for all $N>0$,

$$\omega_m\left(\Lambda^r_{q,\alpha}f,\frac{1}{N}\right)_{q,2,\alpha}=\mathcal{O}\left(N^{-4m}\sum_{l=0}^{N-1}(l+1)^{4r+4m-1}\mathcal{J}_l^2(f)\right)^{\frac{1}{2}}+\mathcal{O}\left(\sum_{1=[\frac{N}{2}]}^{+\infty}l^{2r-1}\mathcal{J}_l(f)\right).$$

Proof. Let $f\in\mathcal{L}^2_{q,\alpha}(\mathbb{R}_q^+)$. By (2.13) and Plancherel formula (2.10), we have

$$\begin{aligned}\|\Lambda^r_{q,\alpha}f\|^2_{q,2,\alpha}&=\int_0^{+\infty}\lambda^{4r}|\mathcal{F}_{q,\alpha}(f)(\lambda)|^2\lambda^{2\alpha+1}d_q\lambda\\&=\sum_{l=0}^{+\infty}\int_l^{l+1}\lambda^{4r}|\mathcal{F}_{q,\alpha}(f)(\lambda)|^2\lambda^{2\alpha+1}d_q\lambda\\&=\sum_{l=0}^{+\infty}(l+1)^{4r}(\mathcal{J}_l^2(f)-\mathcal{J}_{l+1}^2(f)).\end{aligned}$$

Using an Abel transformation, we obtain

$$\|\Lambda^r_{q,\alpha}f\|^2_{q,2,\alpha}\le\mathcal{J}_0^2(f)+4r\sum_{l=1}^{+\infty}(l+1)^{4r-1}\mathcal{J}_l^2(f).$$

From the inequality $l+1\le 2l$, we conclude

$$\|\Lambda^r_{q,\alpha}f\|^2_{q,2,\alpha}\le c_5\left(\mathcal{J}_0^2(f)+\sum_{l=1}^{+\infty}l^{4r-1}\mathcal{J}_l^2(f)\right).$$

Hence,

$$\|\Lambda^r_{q,\alpha}f\|_{q,2,\alpha}=\mathcal{O}\left(\sum_{l=1}^{+\infty}l^{2r-1}\mathcal{J}_l(f)\right).$$

Since the series

$$\sum_{l=1}^{+\infty}l^{2r-1}\mathcal{J}_l(f),\quad r=1,2,\dots$$

converges, we see that $f\in W^r_{q,2}(\Lambda_{q,\alpha})$.

On the other hand, it follows from Lemma 3.2 that

$$\|\Delta_h^m(\Lambda^r_{q,\alpha}f)\|^2_{q,2,\alpha}=\int_0^{+\infty}\lambda^{4r}|1-j_\alpha(\lambda h,q^2)|^{2m}|\mathcal{F}_{q,\alpha}(f)(\lambda)|^2\lambda^{2\alpha+1}d_q\lambda.$$

This integral is divided into two:

$$\int_0^{+\infty} = \int_0^N + \int_N^{+\infty} = K_1 + K_2,$$

where $N = [\frac{1}{h}]$. We estimate them separately.

By rearranging terms analogous to summation by parts and proceeding as with I_1, we obtain

$$\begin{aligned}
K_1 &\leq c_6 h^{4m} \int_0^N \lambda^{4(r+m)} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2 \lambda^{2\alpha+1} d_q\lambda \\
&= c_6 h^{4m} \sum_{l=0}^{N-1} \int_l^{l+1} \lambda^{4(r+m)} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2 \lambda^{2\alpha+1} d_q\lambda \\
&\leq c_6 h^{4m} \sum_{l=0}^{N-1} (l+1)^{4(r+m)} (\mathcal{J}_l^2(f) - \mathcal{J}_{l+1}^2(f)) \\
&\leq c_6 N^{-4m} \sum_{l=0}^{N-1} (l+1)^{4r+4m-1} \mathcal{J}_l^2(f).
\end{aligned}$$

Now we estimate K_2, by relation (2.6), we obtain

$$\begin{aligned}
K_2 &= \int_N^{+\infty} \lambda^{4r} |1 - j_\alpha(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2 \lambda^{2\alpha+1} d_q\lambda \\
&= \mathcal{O}\left(\int_N^{+\infty} \lambda^{4r} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2 \lambda^{2\alpha+1} d_q\lambda \right) \\
&= \mathcal{O}\left(\sum_{m=1}^{+\infty} \int_{2^{m-1}N}^{2^m N} \lambda^{4r} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2 \lambda^{2\alpha+1} d_q\lambda \right) \\
&= \mathcal{O}\left(\sum_{m=1}^{+\infty} (2^m N)^{4r} \int_{2^{m-1}N}^{2^m N} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^2 \lambda^{2\alpha+1} d_q\lambda \right) \\
&= \mathcal{O}\left(\sum_{m=1}^{+\infty} (2^m N)^{4r} \mathcal{J}_{2^{m-1}N}^2(f) \right),
\end{aligned}$$

i.e.,

$$(K_2)^{\frac{1}{2}} = \mathcal{O}\left(\sum_{m=1}^{+\infty} (2^m N)^{2r} \mathcal{J}_{2^{m-1}N}(f) \right).$$

Taking account of the fact that

$$\begin{aligned}
2^{4r} \sum_{l=2^{m-2}N+1}^{2^{m-1}N} l^{2r-1} \mathcal{J}_l(f) &\geq 2^{4r} (2^{m-2}N)^{2r-1} \mathcal{J}_{2^{m-1}N}(f) 2^{m-2} N \\
&= (2^m N)^{2r} \mathcal{J}_{2^{m-1}N}(f),
\end{aligned}$$

we obtain the estimate

$$(K_2)^{\frac{1}{2}} = \mathcal{O}\left(\sum_{m=1}^{+\infty} \sum_{l=2^{m-2}N+1}^{2^{m-1}N} l^{2r-1} \mathcal{J}_l(f) \right) = \mathcal{O}\left(\sum_{1=[\frac{N}{2}]}^{+\infty} l^{2r-1} \mathcal{J}_l(f) \right).$$

Combining the estimates for K_1 and K_2 gives

$$\|\Delta_h^m(\Lambda_{q,\alpha}^r f)\|_{q,2,\alpha} = \mathcal{O}\left(N^{-4m}\sum_{l=0}^{N-1}(l+1)^{4r+4m-1}\mathcal{J}_l^2(f)\right)^{\frac{1}{2}} + \mathcal{O}\left(\sum_{1=[\frac{N}{2}]}^{+\infty} l^{2r-1}\mathcal{J}_l(f)\right),$$

which implies that

$$\omega_m\left(\Lambda_{q,\alpha}^r f, \frac{1}{N}\right)_{q,2,\alpha} = \mathcal{O}\left(N^{-4m}\sum_{l=0}^{N-1}(l+1)^{4r+4m-1}\mathcal{J}_l^2(f)\right)^{\frac{1}{2}} + \mathcal{O}\left(\sum_{1=[\frac{N}{2}]}^{+\infty} l^{2r-1}\mathcal{J}_l(f)\right).$$

□

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