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ORTHOGONAL POLYNOMIALS AND COMPLETED ZETA FUNCTION ON FUNCTION FIELDS

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Abstract. In this paper, we show that the Riemann hypothesis for function fields provides an analytic characterization in terms of the existence of a certain family of orthogonal polynomials $\{P_n(z)\}$ such that $\lim_{n\to+\infty} \frac{P_{2n}(z)}{P_{2n}(0)} = \frac{\xi_K(1/2+iz)}{\xi_K(1/2)}$, where $\xi_K(s)$ denotes the completed zeta function. Furthermore, using the fact that the Riemann hypothesis holds for function fields, we derive several explicit formulas that follow from known representations of the completed zeta function ξ_K.

1. Introduction

1.1. **Background.** The Riemann zeta function $\zeta(s)$ is a well-known function of the complex variable $s = \sigma + it$, and is defined by $\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$, which converges for $\mathrm{Re}(s) = \sigma > 1$. This function can be extended meromorphically to the entire complex plane $\mathbb{C}$. The zeta function $\zeta(s)$ is holomorphic everywhere except for a simple pole at $s = 1$. The Riemann ξ-function is defined as follows

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \tag{1.1}$$

where $\Gamma(s)$ is the gamma function. The Riemann Hypothesis (RH) asserts that all nontrivial zeros of $\zeta(s)$ lie on the critical line $\mathrm{Re}(s) = 1/2$. One of the remarkable features of the study of the Riemann hypothesis is the wide variety of its equivalent formulations in the classical setting, many of which extend naturally to the zeta function associated with a function field K of arbitrary genus over a finite field of constants [2, 5]. In the case of function fields, the analogue of the Riemann hypothesis

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was proved by Weil [8]. Hence, in this setting, our results should not be interpreted as establishing new equivalences to the Riemann hypothesis, but rather as providing an analytic characterization and a representation consistent with its established validity. This framework allows one to derive results concerning special values of L_K and ξ_K at certain points, the class number h_K, and other arithmetic invariants of function fields.

Another characterization of the Riemann Hypothesis (RH) was obtained by Cardon and Roberts [3]. They constructed a measure such that if $\{p_n(z)\}$ is the sequence of orthogonal polynomials relative to that measure, then the Riemann Hypothesis with simple zeros is true if and only if $\lim_{n\to+\infty} \frac{p_{2n}(z)}{p_{2n}(0)} = \frac{\xi(1/2+iz)}{\xi(1/2)}$ where the Riemann ξ-function is defined in (1.1).

The purpose of this paper is to study Cardon and Roberts results on function fields. As a consequence, and taking into account that (RH) holds in our case, we derive some interesting formulas that follow from known expressions of the completed zeta function ξ_K.

Let us recall the definition and some properties of the zeta function of a function field as in [6, Chapter 5, Section 5.1 and Section 5.2, 185–218]. Let K denote an algebraic function field of genus g whose constant field is the finite field $\mathbb{F}_q$. Consider the following power series denoted $Z_K(T)$ as

$$Z_K(T) = \sum_{n=0}^{+\infty} C_n T^n = \prod_{D \text{ prime}} \left(1 - T^{\deg(D)}\right)^{-1},$$

where $C_n = \#\{D \in Div(K) \mid D \geq 0,\ \deg(D) = n\}$. $Z_K(T)$ is actually a rational function

$$Z_K(T) = \frac{L_K(T)}{(1-T)(1-qT)},$$

where $L_K(T)$ factors in $\mathbb{C}[T]$ in the form

$$L_K(T) = \prod_{j=1}^{2g} (1 - \alpha_j T) \in \mathbb{Z}[T].$$

The special value $L_K(1) = \prod_{i=1}^{2g}(1 - \alpha_i)$ is the class number of K, denoted by h_K. The complex numbers $\alpha_1, \dots, \alpha_{2g}$ are algebraic integers and can be arranged so that $\alpha_j \alpha_{g+j} = q$ holds for $j = 1, \dots, g$. Since the Riemann hypothesis for function fields proved by A. Weil [8] states that the α_i, $i = 1, \dots, 2g$, have absolute value $q^{1/2}$, we may order the indices $j \in \{1, \dots, g\}$ so that $\alpha_{g+j} = \overline{\alpha_j}$. This allows us to write $\alpha_j = q^{1/2} \exp(i\theta_j)$, with $\theta_j \in [0, \pi]$.

Now, we define the (classical) zeta function ζ_K of K as follows. For $s \in \mathbb{C}$, we replace T by q^{-s} in $Z_K(T)$,

$$\zeta_K(s) := Z_K(q^{-s}) = \sum_{n=0}^{+\infty} C_n q^{-ns},$$

which converges for $\mathrm{Re}(s) > 1$. We define the following completed zeta function

$$\xi_K(s) := q^s(1-q^{-s})(1-q^{1-s})q^{(g-1)s}\zeta_K(s) = q^{gs}L_K(q^{-s}), \tag{1.2}$$

which is an entire function of order one, whose zeros coincide with the zeros of ζ_K. Moreover, ξ_K satisfies the functional equation

$$\xi_K(s) = \xi_K(1-s).$$

Let us recall that, all zeros of the zeta function ζ_K lie in the critical strip $0 \le \mathrm{Re}(s) \le 1$, and they are symmetric with respect to the real axis and the line $\mathrm{Re}(s) = 1/2$. Note that (RH) is equivalent to the statement that the zeros of ζ_K lie on the line $\mathrm{Re}(s) = 1/2$. Let $\mathbb{Z}(K)$ be the set of the zeros ρ of ζ_K, we have

$$\mathbb{Z}(K) = \left\{\frac{1}{2} \pm i\frac{\theta_j}{\log q} + i\frac{2k\pi}{\log q} \mid j \in \{1,\dots,g\},\ k \in \mathbb{Z}\right\}.$$

For $T > 0$ such that T and $-T$ are not the ordinates of zeros, we denote by $N_K(T) := \#\{\rho \in \mathbb{Z}(K) \mid -T \le \mathrm{Im}(\rho) \le T\}$. In the case when T or $-T$ is the ordinate of a zero, we define $N_K(T) = N_K(T+0)$. Then, one has (see [5, page 542])

$$N_K(T) = \frac{2g\log q}{\pi}T + C_K(T), \tag{1.3}$$

where $-2g < C_K(T) \le 2g$. The function ξ_K admits the Hadamard product

$$\xi_K(s) = \prod_{\rho\in\mathbb{Z}(K)}{}^{*}\left(1-\frac{s}{\rho}\right) := \lim_{T\to+\infty}\prod_{\rho\in\mathbb{Z}(K);|\mathrm{Im}(\rho)|\le T}\left(1-\frac{s}{\rho}\right).$$

Moreover, assume that $\xi_K(1/2) \ne 0$, then

$$\xi_K(s) = \prod_{\rho\in\mathbb{Z}(K),\ \mathrm{Im}(\rho)>0}{}^{*}\left(1-\frac{s(1-s)}{\rho(1-\rho)}\right) := \lim_{T\to+\infty}\prod_{\rho\in\mathbb{Z}(K),\ 0<\mathrm{Im}(\rho)\le T}\left(1-\frac{s(1-s)}{\rho(1-\rho)}\right). \tag{1.4}$$

1.2. **Main results.** In this subsection, we give the main results of the paper.

Let us define the function $E_K(z) = \xi_K(1/2+iz)$, whose zeros lie in the strip $-1/2 < \mathrm{Im}(z) < 1/2$. This function is real for real z, $E_K(z) = E_K(-z)$, and any non-real zeros of $E_K(z)$ occur in complex conjugate pairs. For $z = x+iy$ in the region $x \ge 0$, $-1/2 \le y \le 1/2$, let $f(z)$ be analytic satisfying

$$f(z) \in \mathbb{R} \text{ for real } z, \quad \mathrm{Re}f(z) > 0, \quad |f(x+iy)| < e^{-cx}, \tag{1.5}$$

where c is a positive constant. For $T \ge 0$, let

$$F(T) = \frac{1}{2\pi i}\int_{\gamma_T}\frac{E_K'(z)}{E_K(z)}f(z)\,dz,$$

where γ_T is the positively oriented boundary of the region $0 \le x \le T$, $-1/2 \le y \le 1/2$. We recall that the zeros of $E_K(z)$ in the region $x > 0$, $0 \le y < 1/2$ as $\alpha_k + i\beta_k$, with

$\alpha_k \leq \alpha_{k+1}$. If T is not equal to any α_k, then the function $F(T)$ can be written as a finite sum

$$F(T) = \sum_{\alpha_k < T,\, \beta_k = 0} f(\alpha_k) + \sum_{\alpha_k < T,\, \beta_k > 0} \left[f(\alpha_k + i\beta_k) + f(\alpha_k - i\beta_k) \right].$$

For $T < 0$, we define $F(T) = -F(T)$.

Remark 1.1. The test function $f(z)$ is chosen so that $\mathrm{Re} f(z) > 0$ and $|f(x+iy)| < e^{-cx}$, ensuring the required analytic and spectral properties of $F(T)$. The decay condition guarantees convergence of the contour integral, while the positivity of $\mathrm{Re} f(z)$, together with the symmetry of the zeros of $E_K(z)$, implies that each term in the residue expansion of $F(T)$ is non-negative. It follows that $F(T)$ is a non-decreasing distribution function. This positivity framework is consistent with the moment problem associated with the spectral interpretation of zeros in the function field setting.

If $f(z) \equiv 1$, then $F(T) = N_K(T)$. For polynomials $p(x)$, $q(x)$ with real coefficients, we define an inner product via the Riemann-Stieltjes integral

$$\langle p(x), q(x) \rangle = \int_{-\infty}^{+\infty} p(x) q(x)\, dF(x). \tag{1.6}$$

Applying the Gram-Schmidt orthogonalization process to the polynomials $1, x, x^2, \dots$ with respect to this inner product produces a family of orthogonal polynomials $\{P_n(x)\}$, where $\deg(P_n) = n$. In this construction, it follows that $P_{2n}(x)$ is an even function and $P_{2n+1}(x)$ is an odd function.

Since the RH holds with simple zeros on function fields, we obtain the following results.

Theorem 1.1. *We assume that $\xi_K(1/2) \neq 0$ and let $\theta_j \in [0, \pi]$ with $j \in \{1, \dots, g\}$. We have*

$$\lim_{n \to +\infty} \frac{P_{2n}(z)}{P_{2n}(0)} = \prod_{j=1}^{g} \left(\frac{\theta_j^2}{\theta_j^2 - z^2 \log^2 q} \right) \frac{\xi_K(1/2 + iz)}{\xi_K(1/2)},$$

for every $z \in \mathbb{C} \backslash \{(\theta_j / \log q)^2 \mid j = 1, \dots, g\}$, where $\xi_K(s)$ is completed zeta function defined in (1.2).

We note that $\lim_{n \to +\infty} \frac{P_{2n}(z)}{P_{2n}(0)} = \lim_{n \to +\infty} \frac{P_{2n+1}(z)}{z P'_{2n+1}(0)}$. Cardon and Roberts [3] studied this approach for the classical Riemann zeta function.

The proof of Theorem 1.1 is given in Section 3.

In section 4, we give an alternative expression for $\lim_{n \to +\infty} \frac{P_{2n}(z)}{P_{2n}(0)}$.

Theorem 1.2. *Assume that $\xi_K(1/2) \neq 0$. Let $z \in \mathbb{C}$ and $\theta_j \in [0, \pi]$ with $j \in \{1, \dots, g\}$. Then, we have*

$$\lim_{n \to +\infty} \frac{P_{2n}(z)}{P_{2n}(0)} = \prod_{k=1}^{+\infty} \prod_{j=1}^{g} \left(1 - \frac{(z \log q)^2}{(\pm\theta_j + 2k\pi)^2} \right).$$

Furthermore, we express the limit $\lim_{n\to+\infty} \frac{P_{2n}(z)}{P_{2n}(0)}$ in terms of the polynomial L_K (see Theorem 4.1). As an application of Theorem 4.1, we derive explicit formulas for this limit at certain points involving g, h_K and the polynomial L_K (see Corollary 4.1). Finally, we state an interesting results (see Corollary 4.2).

2. Preliminary Results

In this section, we present results needed to prove Theorem 1.1, with references to Szegö [7] and Chihara [4] for the foundational theory of orthogonal polynomials.

A function ψ is called a distribution function if it is bounded, non-decreasing, and its moments

$$\mu_n = \int_{-\infty}^{+\infty} x^n \, d\psi(x) \tag{2.1}$$

exist for $n \in \mathbb{N}$. Two distribution functions ψ_1 and ψ_2 are called substantially equal if and only if there exists a constant K such that $\psi_1(x) = \psi_2(x) + K$ at all common points of continuity. The spectrum of ψ is the set

$$\mathfrak{S}(\psi) = \{x \mid \psi(x+\delta) - \psi(x-\delta) > 0 \text{ for all } \delta > 0\}. \tag{2.2}$$

If $\mathfrak{S}(\psi)$ is infinite, then

$$\langle p(x), q(x) \rangle = \int_{-\infty}^{+\infty} p(x)q(x)\, d\psi(x), \tag{2.3}$$

defines an inner product on the space of polynomials with real coefficients. By (2.3), we orthogonalized the set of non-negative powers of x with the Gram-Schmidt process to get orthogonal polynomials $\{P_n(x)\}$ with real coefficients:

$$P_0(x) = 1, \quad P_n(x) = x^n - \sum_{k=0}^{n-1} \frac{\langle x^n, P_k \rangle}{\langle P_k, P_k \rangle} P_k(x), \quad n \geq 1.$$

Lemma 2.1. ([4, Theorems I.5.2 and I.5.3] *or* [7, Theorems 3.3.1 and 3.3.3])

i) *For each $n \geq 1$, all zeros of $P_n(x)$ are real and simple.*

ii) *The zeros of $P_n(x)$ and $P_{n+1}(x)$ interlace. Futhermore, between any two zeros of $P_n(x)$, there is at least one zero of $P_m(x)$ for $m > n$.*

Let denote the zeros of $P_n(x)$ as $y_{n,1} < y_{n,2} < \cdots < y_{n,n}$. Using the moments given in (2.1), we define a moment functional on the space of polynomials by

$$\mathcal{L}[p(x)] = \int_{-\infty}^{+\infty} p(x)\, d\psi(x) = \sum_{k=0}^{n} c_k \mu_k, \quad \text{where } p(x) = c_0 + c_1 x + \cdots + c_n x^n.$$

Using [4, Theorem I.6.1] (or [7, Theorem 3.4.1]), there exist numbers $A_{n,1}, A_{n,2}, \ldots, A_{n,n}$ such that for any polynomial $\pi(x)$ of degree at most $2n-1$

$$\mathcal{L}[\pi(x)] = \sum_{k=1}^{n} A_{n,k} \pi(y_{n,k}). \tag{2.4}$$

All the numbers $A_{n,k}$ are positive and $A_{n,1} + A_{n,2} + \cdots + A_{n,n} = \mu_0$. Equation (2.4) is known as the Gauss quadrature formula, and the numbers $A_{n,k}$ are called the Christoffel numbers. Moreover, the zeros of the polynomials $\{P_n(x)\}$ are strongly related to the spectrum $\mathfrak{S}(\psi)$. Let us define

$$\psi_n(x) = \begin{cases} 0, & \text{if } x < y_{n,1}, \\ A_{n,1} + \cdots + A_{n,p}, & \text{if } y_{n,p} \leq x < y_{n,p+1}, \quad 1 \leq p < n, \\ \mu_0, & \text{if } x \geq y_{n,n}. \end{cases}$$

Lemma 2.2. ([4, Theorem II.3.1]) *There is a subsequence of $\{\psi_n\}$ that converges on $(-\infty, +\infty)$ to a distribution function η which has an infinite spectrum and such that*

$$\mu_n = \int_{-\infty}^{+\infty} x^n \, d\psi(x) = \int_{-\infty}^{+\infty} x^n \, d\eta(x), \quad n = 0, 1, 2, \ldots$$

In general, η is not substantially equal to ψ. Distribution functions such as η, which are subsequential limits of $\{\psi_n\}$, are called natural representatives of the moment functional $\mathcal{L}$.

Lemma 2.3. ([4, Theorems II.4.1 and II.4.3] *or* [7, Theorem 3.4.2])

i) *The open interval $]y_{n,i}, y_{n,(i+1)}[$ contains at least one spectral point of the function ψ.*
ii) *Let η be a natural representative of $\mathcal{L}$, and let $s \in \mathfrak{S}(\eta)$. Then, every neighborhood of s contains a zero of $P_n(x)$ for infinitely many values of n.*

Given a sequence of moments $\{\mu_n\}$, the Hamburger moment problem consists of classifying the distribution functions ϕ that satisfy $\mu_n = \int_{-\infty}^{+\infty} x^n \, d\phi(x)$, $n \in \mathbb{N}$. We say the moment problem is determined if all solutions ϕ of the Hamburger moment problem are substantially equal. Carleman provided a sufficient condition for a moment problem to be determined.

Lemma 2.4. ([1, p. 85]) *The moment problem $\mu_n = \int_{-\infty}^{+\infty} x^n \, d\psi(x)$ is determined if*

$$\sum_{n=1}^{+\infty} \mu_{2n}^{-1/(2n)} = +\infty.$$

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 using the same argument as Cardon and Roberts [3]. Therefore, we will only include and provide details for the different points.

The proof follows almost exactly the same steps as in [3, Theorem 1, Section 3]. Compared to the classical case (the Riemann zeta function), we know explicitly the non trival zeros of ζ_K where are periodic modulo $2\pi i/\log q$ and this simplify the proof of our main Theorem 1.1 and yields many interesting consequences.

Lemma 3.1. *The nth moments*

$$\mu_n = \int_{-\infty}^{+\infty} x^n \, dF$$

exist, and (1.6) *defines an inner product on the space of polynomials with real coefficients.*

Proof. The proof closely follows the same approach as done in [3, Lemma 3.1].

From (1.5), we have

$$\begin{aligned}\int_0^{+\infty} x^n\,dF =& \sum_{k=1,\ \beta_k=0}^{+\infty} \alpha_k^n f(\alpha_k) + \sum_{k=1,\ \beta_k\neq 0}^{+\infty} \alpha_k^n\left[f(\alpha_k+i\beta_k)+f(\alpha_k-i\beta_k)\right]\\ \leq& 3\sum_{k=1}^{+\infty}\alpha_k^n e^{-c\alpha_k}.\end{aligned}$$

By Equation (1.3), we get the following estimate

$$N_K(T) \sim \frac{2g\log q}{\pi}T.$$

Therefore, the critical strip with $\mathrm{Im}(z) > 0$ are labelled as $\rho_k + it_k$ with $t_{k+1} \geq t_k$. Then,

$$t_k \sim \frac{\pi k}{2g\log q},$$

where $k \in \mathbb{N}$. Hence, there exist positive constants A and B such that for $k \geq 1$

$$Ak < \alpha_k < Bk.$$

Since $F(T) = -F(-T)$, we also know that $\mu_n = 0$ for odd n. When n is even, we have

$$\mu_n = \int_{-\infty}^{+\infty} x^n\,dF = 2\int_0^{+\infty} x^n\,dF \leq 6B^n\sum_{k=1}^{+\infty} k^n\exp\left(-cAk\right). \tag{3.1}$$

This shows that the moments $\mu_n = \int_{-\infty}^{+\infty} x^n\,dF$ exist for all $n \geq 0$. Thus, the inner product in (1.6) exists for any real polynomials $p(x)$ and $q(x)$. The bilinearity is clear and the measure dF has an infinite support. Therefore, Lemma 3.1 is easily proved. ☐

Lemma 3.2. *The Hamburger moment problem for the moments of the distribution function F,*

$$\mu_n = \int_{-\infty}^{+\infty} x^n\,dF$$

is determined.

Proof. The proof closely follows the argument used in [3, Lemma 3.2].

We first estimate the summation in (3.1). Let us define

$$S(n) = \sum_{k=1}^{+\infty} k^n\exp\left(-cAk\right) = \underbrace{\sum_{1\leq k\leq M+1}^{+\infty} k^n\exp\left(-cAk\right)}_{S_1(n)} + \underbrace{\sum_{k>M+1}^{+\infty} k^n\exp\left(-cAk\right)}_{S_2(n)}.$$

The function $k^n \exp(-cAk)$ has a maximum of $\left(\frac{n}{ecA}\right)^n$ when $k = \frac{n}{cA}$ yields the following bound on $S_1(n)$:

$$S_1(n) \leq M \left(\frac{n}{ecA}\right)^n. \tag{3.2}$$

Assume that M is sufficiently large so that the following conditions are satisfied:

$$k > \frac{n}{cA},\ k > 1, \quad \text{for } k \geq M \text{ and } M > \left(\frac{2(n+1)}{cA}\right)^2. \tag{3.3}$$

Condition (3.3) ensures that the function $k^n \exp(-cAk)$ decreases for $k \geq M$. Then, we have

$$\begin{aligned} S_2(n) = \sum_{k>M+1} k^n \exp(-cAk) &< \int_M^{+\infty} k^{n+1} \exp(-cAk)\, dk \\ &= \int_M^{+\infty} w^{n+1} \exp(-cAw)\, dw, \end{aligned}$$

for any positive α and w such that $w < \exp(\alpha w)/\alpha$. Putting $\alpha = \frac{cA}{2(n+1)}$, we obtain

$$S_2(n) < \left(\frac{2(n+1)}{cA}\right)^{n+1} \int_M^{+\infty} \exp\left(-\frac{cA}{2}w\right) dw = \frac{2}{cA}\left(\frac{\frac{2(n+1)}{cA}}{\exp\left(\frac{cA}{2(n+1)}M\right)}\right)^{n+1}.$$

Now, by (3.3), we have $\frac{2(n+1)}{cA} < \exp\left(\frac{cAM}{2(n+1)}\right)$, which implies

$$S_2(n) < \frac{2}{cA}. \tag{3.4}$$

From (3.2) and (3.4), we obtain

$$S(n) = S_1(n) + S_2(n) < M\left(\frac{n}{ecA}\right)^n + \frac{2}{cA}.$$

Let $M = \kappa^n$, where $\kappa > 1$. For sufficiently large n, (3.3) is satisfied. Therefore, for large even n,

$$\mu_n^{1/n} \leq (6B^n S(n))^{1/n} < \left(6B^n\left(\left(\frac{\kappa n}{ecA}\right)^n + \frac{2}{cA}\right)\right)^{1/n} < \frac{2B\kappa}{ecA}n.$$

Therefore,

$$\sum_{n=0}^{+\infty} \mu_{2n}^{-1/2n} = +\infty,$$

According to Carleman's criterion (see Lemma 2.4), it follows that the Hamburger moment problem $\mu_n = \int_{-\infty}^{+\infty} x^n\, dF$ is determined. □

Let $\{P_n(x)\}$ be the family of orthogonal polynomials obtained from Lemma 3.1 by orthogonalizing the set of non-negative powers of x according the Gram-Schmidt procedure. Since $\mu_{2k+1} = 0$ and $\mu_{2k} > 0$ for all k, it follows that each polynomial

$P_{2n}(x)$ is an even function, while each $P_{2n+1}(x)$ is an odd function. From (2.2) the spectrum of F is defined as follows

$$\mathfrak{S}(F) = \{x \mid F(x+\delta) - F(x-\delta) > 0 \text{ for all } \delta > 0\},$$

which consists of all α_k such that $\alpha_k + i\beta_k$ is a zero of $\xi_K(1/2 + iz)$. Recall that the positive values in $\mathfrak{S}(F)$ are given by $a_1 < a_2 < a_3 < \cdots$. For example, $a_1 \approx 14.134$ in the case of the Riemann zeta function. The n positive zeros of $P_{2n}(x)$ are denoted by $x_{2n,1} < x_{2n,2} < \cdots < x_{2n,n}$, and the n positive zeros of $P_{2n+1}(x)$ as $x_{2n+1,1} < x_{2n+1,2} < \cdots < x_{2n+1,n}$. Thus, we can write

$$\frac{P_{2n}(z)}{P_{2n}(0)} = \prod_{k=1}^{n} \left(1 - \frac{z^2}{x_{2n,k}^2}\right), \qquad \frac{P_{2n+1}(z)}{zP'_{2n+1}(0)} = \prod_{k=1}^{n} \left(1 - \frac{z^2}{x_{2n+1,k}^2}\right).$$

Lemma 3.3. *With the above notation, we have* $a_k = \lim_{n\to+\infty} x_{n,k}$.

Proof. The proof follows that of [3, Lemma 3.4]. □

Now, we are ready to prove our main result, Theorem 1.1. To begin, recall that the spectrum of F consists of all α_k such that $\alpha_k + i\beta_k$ is a zero of $\xi_K\,(1/2 + iz)$, and the a_k represent the positive values of $\mathfrak{S}(F)$.

Let us recall that the zeros ρ of the function ζ_K are denoted by

$$\rho = \frac{1}{2} + ia^{\pm}_{k,j}, \quad \text{where } a^{\pm}_{k,j} = \frac{\pm\theta_j + 2k\pi}{\log q}, \; j = 1, \ldots, g \text{ and } k \in \mathbb{Z}.$$

Then, we have

$$\begin{cases} a^{+}_{k,j} = \frac{\theta_j + 2k\pi}{\log q} > 0, & \text{for } k \geq 0 \textit{ and } j = 1, \ldots, g \\ \text{and} & \\ a^{-}_{k,j} = \frac{-\theta_j + 2k\pi}{\log q} > 0, & \text{for } k \geq 1 \textit{ and } j = 1, \ldots, g. \end{cases} \tag{3.5}$$

Let $\epsilon > 0$, K be any compact set of $\mathbb{C}$ and R such that $|z| < R$, for all $z \in K$. Define $M_R = \prod_{j=1}^{g} \prod_{k=1}^{+\infty} \left(1 + \frac{R^2}{(a^{\pm}_{k,j})^2}\right)$. Since $\xi_K(1/2 + iz)$ is an entire function of order 1, we know that $\sum_{j=1}^{g} \sum_{k=1}^{+\infty} (a^{\pm}_{k,j})^{-2} < +\infty$, so $M_R < +\infty$. Then, for $z \in K$

$$\left|\frac{P_{2n}(z)}{P_{2n}(0)}\right| = \left|\prod_{j=1}^{g} \prod_{k=1}^{n} \left(1 - \frac{z^2}{x_{2n,k,j}^2}\right)\right| \leq M_R.$$

Choose N sufficiently large such that $a^{\pm}_{k,j} > R$ when $k > N$ and $j = 1, \ldots, g$. For $n > N$, we obtain

$$\prod_{j=1}^{g} \prod_{k=N+1}^{+\infty} \left(1 - \frac{R^2}{(a^{\pm}_{k,j})^2}\right) < \left|\prod_{j=1}^{g} \prod_{k=N+1}^{n} \left(1 - \frac{z^2}{x_{2n,k,j}^2}\right)\right| < \prod_{j=1}^{g} \prod_{k=N+1}^{+\infty} \left(1 + \frac{R^2}{(a^{\pm}_{k,j})^2}\right)$$

and

$$\prod_{j=1}^{g} \prod_{k=N+1}^{+\infty} \left(1 - \frac{R^2}{(a^{\pm}_{k,j})^2}\right) < \left|\prod_{j=1}^{g} \prod_{k=N+1}^{n} \left(1 - \frac{z^2}{(a^{\pm}_{k,j})^2}\right)\right| < \prod_{j=1}^{g} \prod_{k=N+1}^{+\infty} \left(1 + \frac{R^2}{(a^{\pm}_{k,j})^2}\right).$$

Since $\lim_{N\to+\infty}\prod_{j=1}^{g}\prod_{k=N+1}^{+\infty}\left(1-\frac{R^2}{(a_{k,j}^{\pm})^2}\right)=\lim_{N\to+\infty}\prod_{j=1}^{g}\prod_{k=N+1}^{+\infty}\left(1+\frac{R^2}{(a_{k,j}^{\pm})^2}\right)=1$, we choose N so that

$$\left|\prod_{j=1}^{g}\prod_{k=N+1}^{n}\left(1-\frac{z^2}{x_{2n,k,j}^2}\right)-1\right|\le\frac{\epsilon}{M_R},\quad \left|\prod_{j=1}^{g}\prod_{k=N+1}^{n}\left(1-\frac{z^2}{(a_{k,j}^{\pm})^2}\right)-1\right|\le\frac{\epsilon}{M_R}$$

and $N_1>N$ such that, if $n>N_1$ and $z\in K$

$$\left|\prod_{j=1}^{g}\prod_{k=1}^{N}\left(1-\frac{z^2}{x_{2n,k,j}^2}\right)-\prod_{j=1}^{g}\prod_{k=1}^{N}\left(1-\frac{z^2}{(a_{k,j}^{\pm})^2}\right)\right|\le\epsilon.$$

Let $n>N_1$, we easily get

$$\left|\frac{P_{2n}(z)}{P_{2n}(0)}-\prod_{j=1}^{g}\prod_{k=1}^{+\infty}\left(1-\frac{z^2}{(a_{k,j}^{\pm})^2}\right)\right|\le 3\epsilon.$$

Then,

$$\lim_{n\to+\infty}\frac{P_{2n}(z)}{P_{2n}(0)}=\prod_{j=1}^{g}\prod_{k=1}^{+\infty}\left(1-\frac{z^2}{(a_{k,j}^{\pm})^2}\right).$$

The same argument yields to $\lim_{n\to+\infty}\frac{P_{2n+1}(z)}{zP'_{2n+1}(0)}=\prod_{j=1}^{g}\prod_{k=1}^{+\infty}\left(1-\frac{z^2}{(a_{k,j}^{\pm})^2}\right)$. On the other hand, by (1.4) we obtain

$$\xi_K\left(\frac{1}{2}+iz\right)=\xi_K\left(\frac{1}{2}\right)\prod_{\rho=1/2+ia_{k,j}^{\pm}\in\mathbb{Z}(K);\ a_{k,j}^{\pm}>0}\left(1-\frac{z^2}{(a_{k,j}^{\pm})^2}\right).\tag{3.6}$$

Using (3.5) and (3.6), and assuming that $\xi_K(\frac{1}{2})\neq 0$, we have

$$\left[\prod_{j=1}^{g}\left(1-\frac{z^2}{(a_{0,j}^{+})^2}\right)\right]\prod_{k=1}^{+\infty}\prod_{j=1}^{g}\left(1-\frac{z^2}{(a_{k,j}^{\pm})^2}\right)=\frac{\xi_K(1/2+iz)}{\xi_K(1/2)}.$$

Then,

$$\prod_{k=1}^{+\infty}\prod_{j=1}^{g}\left(1-\frac{z^2}{(a_{k,j}^{\pm})^2}\right)=\prod_{j=1}^{g}\left(1-\frac{z^2}{(a_{0,j}^{+})^2}\right)^{-1}\frac{\xi_K(1/2+iz)}{\xi_K(1/2)}.$$

Therefore, assuming that $\xi_K(1/2)\neq 0$ and letting $z\in\mathbb{C}\backslash\{(\theta_j/\log q)^2\mid j=1,\dots,g\}$, we obtain

$$\prod_{j=1}^{g}\prod_{k=1}^{+\infty}\left(1-\frac{z^2}{(a_{k,j}^{\pm})^2}\right)=\prod_{j=1}^{g}\left(\frac{\theta_j^2}{\theta_j^2-z^2\log^2 q}\right)\frac{\xi_K(1/2+iz)}{\xi_K(1/2)},$$

if and only if $\xi_K(1/2+iz)$ has simple real zeros. Thus, we deduce the results of Theorem 1.1.

4. Further Results on the Limit of $\{P_{2n}(z)/P_{2n}(0)\}$

In this section, we derive several interesting formulas related to the limit of $\frac{P_{2n}(z)}{P_{2n}(0)}$ as $n \to +\infty$.

From Equation (3.6), we obtain

$$\xi_K\left(\tfrac{1}{2}+iz\right) = \xi_K\left(\tfrac{1}{2}\right)\left[\prod_{j=1}^{g}\left(1-\frac{z^2\log^2 q}{\theta_j^2}\right)\right]\prod_{k=1}^{+\infty}\prod_{j=1}^{g}\left(1-\frac{z^2}{\left(\frac{\pm\theta_j+2k\pi}{\log q}\right)^2}\right). \tag{4.1}$$

Remark 4.1. We have $\xi_K(1/2)=0$ if and only if $\theta_j = 0$ for some $j=1,\dots,g$. In that case, we may replace ξ_K with the function $\frac{\xi_K(s)}{(s-1/2)^m}$ where m is the multiplicity of the zero of ξ_K at $s=1/2$. The functions $\xi_K(s)/(s-1/2)^m$ and ξ_K have the same nontrivial zeros (i.e., those with $\operatorname{Im}(\rho)>0$). Therefore, throughout this section, we assume $\xi_K(1/2)\neq 0$.

We now prove Theorem 1.2.

Proof of Theorem 1.2. From Theorem 1.1, we have

$$\lim_{n\to+\infty}\frac{P_{2n}(z)}{P_{2n}(0)} = \prod_{j=1}^{g}\left(\frac{\theta_j^2}{\theta_j^2 - z^2\log^2 q}\right)\frac{\xi_K(1/2+iz)}{\xi_K(1/2)}.$$

Substituting the expression of $\xi_K(1/2+iz)$ given by (4.1), we get

$$\lim_{n\to+\infty}\frac{P_{2n}(z)}{P_{2n}(0)} = \prod_{k=1}^{+\infty}\prod_{j=1}^{g}\left(1-\frac{z^2}{\left(\frac{\pm\theta_j+2k\pi}{\log q}\right)^2}\right).$$

This completes the proof of Theorem 1.2. □

The next result expresses the limit in terms of the polynomial L_K.

Theorem 4.1. *Assume that $L_K(q^{-1/2})\neq 0$, and let $\theta_j\in[0,\pi]$ for $j=1,\dots,g$. Then, for all $z\in\mathbb{C}\backslash\{(\theta_j/\log q)^2 \mid j=1,\dots,g\}$, we have*

$$\lim_{n\to+\infty}\frac{P_{2n}(z)}{P_{2n}(0)} = \prod_{j=1}^{g}\left(\frac{\theta_j^2}{\theta_j^2 - z^2\log^2 q}\right)\frac{q^{giz}L_K(q^{-(1/2+iz)})}{L_K(q^{-1/2})}.$$

Proof. From (1.2), we note that

$$\xi_K\left(\tfrac{1}{2}+iz\right) = q^{g(1/2+iz)}L_K\left(q^{-(1/2+iz)}\right). \tag{4.2}$$

Theorem 4.1 follows from (4.2) and Theorem 1.1. □

As an application of Theorem 4.1, we derive an explicit one for $\lim_{n\to+\infty}\frac{P_{2n}(z)}{P_{2n}(0)}$ at some points in terms of g, h_K and the polynomial L_K.

Corollary 4.1. *Assume that $L_K(q^{-1/2})\neq 0$, and let $\theta_j\in[0,\pi]$ for $j=1,\dots,g$. Then, the following hold.*

i) *For $z = i/2$,*

$$\lim_{n\to+\infty} \frac{P_{2n}(i/2)}{P_{2n}(0)} = \prod_{j=1}^{g} \left(\frac{\theta_j^2}{\theta_j^2 + \frac{\log^2 q}{4}} \right) \frac{q^{-g/2} h_K}{L_K(q^{-1/2})}.$$

ii) *For $z = -i/2$,*

$$\lim_{n\to+\infty} \frac{P_{2n}(-i/2)}{P_{2n}(0)} = \prod_{j=1}^{g} \left(\frac{\theta_j^2}{\theta_j^2 + \frac{\log^2 q}{4}} \right) \frac{q^{g/2} L_K(q^{-1})}{L_K(q^{-1/2})}.$$

iii) *For $z = i$,*

$$\lim_{n\to+\infty} \frac{P_{2n}(i)}{P_{2n}(0)} = \prod_{j=1}^{g} \left(\frac{\theta_j^2}{\theta_j^2 + \log^2 q} \right) \frac{q^{-g} L_K(q^{1/2})}{L_K(q^{-1/2})}.$$

Proof. Each identity follows by substituting $z = i/2$, $-i/2$, and i into Theorem 4.1. □

Remark 4.2. Case $g = 1$ (elliptic curve). For an elliptic curve over $\mathbb{F}_q$, the L-polynomial is $L_K(T) = 1 - aT + qT^2$, where $a = q + 1 - \#E(\mathbb{F}_q)$. By the Hasse bound, $|a| \le 2\sqrt{q}$, so we can parametrize $a = 2\sqrt{q}\cos\theta$, with $\theta \in [0, \pi]$. This is a standard representation. Then, one has

$$h_K = L_K(1) = q + 1 - 2\sqrt{q}\cos\theta, \quad L_K(q^{-1/2}) = 2(1 - \cos\theta), \quad L_K(q^{-1}) = \frac{h_K}{q}$$

and $L_K(q^{1/2}) = 1 - 2q\cos\theta + q^2$. Substituting into Corollary 4.1 yields the following.

(i) For $z = i/2$ and $\theta \in]0, \pi]$:

$$\lim_{n\to+\infty} \frac{P_{2n}(i/2)}{P_{2n}(0)} = \frac{(2\theta^2)(\sqrt{q} + q^{-1/2} - 2\cos\theta)}{(4\theta^2 + \log^2 q)(1 - \cos\theta)}.$$

(ii) For $z = -i/2$ and $\theta \in]0, \pi]$:

$$\lim_{n\to+\infty} \frac{P_{2n}(-i/2)}{P_{2n}(0)} = \frac{(2\theta^2)(\sqrt{q} + q^{-1/2} - 2\cos\theta)}{(4\theta^2 + \log^2 q)(1 - \cos\theta)}.$$

(iii) For $z = i$ and $\theta \in]0, \pi]$:

$$\lim_{n\to+\infty} \frac{P_{2n}(i)}{P_{2n}(0)} = \frac{(\theta^2)(q + q^{-1} - 2\cos\theta)}{2(\theta^2 + \log^2 q)(1 - \cos\theta)}.$$

Therefore, we have $\lim_{n\to+\infty} \frac{P_{2n}(i/2)}{P_{2n}(0)} = \lim_{n\to+\infty} \frac{P_{2n}(-i/2)}{P_{2n}(0)}$.

Now, we provide some numerical verifications of the limits above for the fixed value $q = 25$ (see Figure 1 and Figure 2).

Corollary 4.2. *Assume that $L_K(q^{-1/2}) \neq 0$ and $\theta_j \neq 0$ for $j = 1, \ldots, g$. We have*

i)

$$\left[\prod_{j=1}^{g} \left(\frac{\theta_j^2 + \frac{\log^2 q}{4}}{\theta_j^2} \right) \right] \prod_{k=1}^{+\infty} \prod_{j=1}^{g} \left(1 + \frac{\log^2 q}{4(\pm\theta_j + 2k\pi)^2} \right) = \frac{q^{-g/2} h_K}{L_K(q^{-1/2})}, \tag{4.3}$$

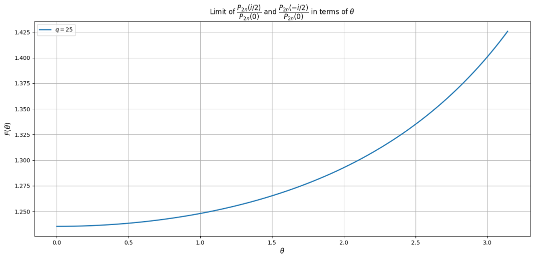

FIGURE 1.

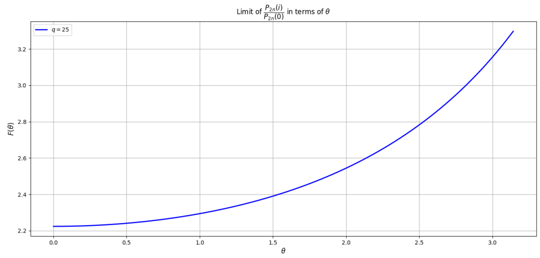

FIGURE 2.

ii)

$$\left[\prod_{j=1}^{g}\left(\frac{\theta_j^2+\log^2 q}{\theta_j^2}\right)\right]\prod_{k=1}^{+\infty}\prod_{j=1}^{g}\left(1+\frac{\log^2 q}{(\pm\theta_j+2k\pi)^2}\right)=\frac{q^{-g}L_K(q^{1/2})}{L_K(q^{-1/2})}. \tag{4.4}$$

Proof. From Theorem 1.2 evaluated at $z=i/2$, we obtain

$$\lim_{n\to+\infty}\frac{P_{2n}(i/2)}{P_{2n}(0)}=\prod_{k=1}^{+\infty}\prod_{j=1}^{g}\left(1+\frac{\log^2 q}{4(\pm\theta_j+2k\pi)^2}\right). \tag{4.5}$$

Combining (4.5) with Corollary 4.1 (i), we observe that all expressions involved are convergent. Hence, by the uniqueness of the limit, we get

$$\prod_{j=1}^{g}\left(\frac{\theta_j^2}{\theta_j^2+\frac{\log^2 q}{4}}\right)\frac{q^{-g/2}h_K}{L_K(q^{-1/2})}=\prod_{k=1}^{+\infty}\prod_{j=1}^{g}\left(1+\frac{\log^2 q}{4(\pm\theta_j+2k\pi)^2}\right),$$

which leads to

$$\left[\prod_{j=1}^{g}\left(\frac{\theta_j^2+\frac{\log^2 q}{4}}{\theta_j^2}\right)\right]\prod_{k=1}^{+\infty}\prod_{j=1}^{g}\left(1+\frac{\log^2 q}{4(\pm\theta_j+2k\pi)^2}\right)=\frac{q^{-g/2}h_K}{L_K(q^{-1/2})}.$$

This proves Equation (4.3).

Similarly, by Theorem 1.2 for $z=i$, we have

$$\lim_{n\to+\infty}\frac{P_{2n}(i)}{P_{2n}(0)}=\prod_{k=1}^{+\infty}\prod_{j=1}^{g}\left(1+\frac{\log^2 q}{(\pm\theta_j+2k\pi)^2}\right). \tag{4.6}$$

Combining (4.6) with Corollary 4.1 (iii), we observe that all expressions involved are convergent. Hence, by the uniqueness of the limit, the two expressions can be identified and we obtain

$$\prod_{j=1}^{g}\left(\frac{\theta_j^2}{\theta_j^2+\log^2 q}\right)\frac{q^{-g}L_K(q^{1/2})}{L_K(q^{-1/2})}=\prod_{k=1}^{+\infty}\prod_{j=1}^{g}\left(1+\frac{\log^2 q}{(\pm\theta_j+2k\pi)^2}\right).$$

Therefore, we have (4.4). □

Let us treat the case $g = 1$ which corresponds to the function field of an elliptic curve. Then, we get

$$\lim_{n\to+\infty} \frac{P_{2n,\theta}(z)}{P_{2n,\theta}(0)} = \prod_{k=1}^{+\infty} \left(1 - \frac{(z\log q)^2}{(\theta+2k\pi)^2}\right)\left(1 - \frac{(z\log q)^2}{(2k\pi-\theta)^2}\right), \quad \text{where } \theta \in [0,\pi]. \tag{4.7}$$

Corollary 4.3. (i) *If $\theta = 0$, then we have*

$$\lim_{n\to+\infty} \frac{P_{2n,0}(z)}{P_{2n,0}(0)} = \frac{4\sin^2\left(\frac{z\log q}{2}\right)}{(z\log q)^2}.$$

(ii) *If $\theta = \pi/2$, then we have*

$$\lim_{n\to+\infty} \frac{P_{2n,\pi/2}(z)}{P_{2n,\pi/2}(0)} = \frac{\cos(z\log q)}{1 - \frac{4(z\log q)^2}{\pi^2}}.$$

(iii) *If $\theta = \pi$, then we have*

$$\lim_{n\to+\infty} \frac{P_{2n,\pi}(z)}{P_{2n,\pi}(0)} = \frac{\pi^2}{\pi^2 - (z\log q)^2}\cos^2\left(\frac{z\log q}{2}\right).$$

Proof. The proof of Corollary 4.3 follows from (4.7) for $\theta = 0$, $\pi/2$ and π. □

Remark 4.3. From Corollary 4.3, we derive explicit formulas for the limit of $\frac{P_{2n,\theta}(z)}{P_{2n,\theta}(0)}$ as $n \to +\infty$, for specific values of θ in each case at some points.

i) If $\theta = 0$ and $z = \pi/\log q$, then we obtain

$$\lim_{n\to+\infty} \frac{P_{2n,0}(\pi/\log q)}{P_{2n,0}(0)} = \frac{4}{\pi^2}.$$

ii) If $\theta = \pi$ and $z = 2\pi/\log q$, then we have

$$\lim_{n\to+\infty} \frac{P_{2n,\pi}(2\pi/\log q)}{P_{2n,\pi}(0)} = \frac{-1}{3}.$$

5. Concluding Remarks

Some perspectives arising from this work are as follows.

i) Since the (RH) holds on function fields and the zeros of the completed zeta function are explicitly known, it is possible to construct a family of orthogonal polynomials $\{P_n(z)\}$.

ii) An interesting problem is to determine the behavior of the error term in the approximation

$$\frac{P_{2n}(z)}{P_{2n}(0)} - \prod_{j=1}^{g}\left(\frac{\theta_j^2}{\theta_j^2 - z^2\log^2 q}\right)\frac{\xi_K(1/2+iz)}{\xi_K(1/2)}.$$

Let us note that this quantity tends to zero since the (RH) holds for function fields. However, the precise rate of convergence remains unknown. This question is closely related to delicate problems in the asymptotic theory of orthogonal polynomials.

Classical results such as Rakhmanov's theorem, the theory of Mhaskar-Saff weights, and the work of Van Assche on varying recurrence coefficients provide important tools for studying convergence rates and zero distributions under regularity assumptions on the underlying measure. In the present setting, however, the spectral measure is discrete and periodic, which places it outside the standard continuous frameworks usually considered in these theories. The lattice-type structure of the zeros may produce additional oscillatory effects, making a precise estimate of the convergence rate a non-trivial problem. It would therefore be of particular interest to investigate how rapidly the zeros of the orthogonal polynomials approach their limiting distribution in this arithmetic setting.

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