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BEST PROXIMITY POINT RESULTS VIA SIMULATION FUNCTIONS IN METRIC-LIKE SPACES

G. V. V. J. RAO[1], H. K. NASHINE[2*], AND Z. KADELBURG[3]

Abstract. In this paper, we discuss the existence of best proximity points of certain mappings via simulation functions in the frame of complete metric-like spaces. Some consequences and examples are given of the obtained results.

1. Introduction

Khojasteh et al. introduced in [13] the notion of simulation function in order to unify several fixed point results obtained by various authors. These functions were later utilized by Karapinar and Khojasteh in [9] to solve some problems concerning best proximity points.

On the other hand, spaces more general than metric and fixed point and related problems in them have been lately a wide field of interest of huge number of mathematicians. Among them, metric-like spaces, introduced by Amini-Harandi in [2], took a prominent place.

In this paper, we are going to extend these investigations to best proximity points of mappings acting in complete metric-like spaces, using conditions involving simulation functions. The results will be illustrated by several examples, showing the strength of these results compared with others existing in the literature.

Key words and phrases. $\mathcal{Z}$-contraction, best proximity point, simulation function, admissible mapping.

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2. Preliminaries

Throughout the paper, $\mathbb{R}$ and $\mathbb{R}^+$, $\mathbb{R}_0^+$ will denote the set of real numbers, the set of positive real numbers and the set of nonnegative real numbers, respectively. Also, $\mathbb{N}_0$ and $\mathbb{N}$ will denote the set of nonnegative, resp. positive integers.

We shall first recall some basic definitions and some results from [1, 5, 13].

Definition 2.1 ([13])**.** A simulation function is a mapping $\zeta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t,s) < s - t$ for all $t, s > 0$;
(ζ_3) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0,\infty)$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Note that, according to the axiom (ζ_2), each simulation function ζ satisfies $\zeta(t,t) < 0$ for all $t > 0$. The family of all simulation functions will be denoted by $\mathcal{Z}$.

Example 2.1 (See, e.g., [1,5,7,13]). For $i = 1, 2, \dots, 6$, define mappings $\zeta_i : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$, as follows.

(i) $\zeta_1(t,s) = \phi_1(s) - \phi_2(t)$ for all $t, s \in \mathbb{R}_0^+$, where $\phi_1, \phi_2 : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are continuous functions, with $\phi_i(t) = 0$ if and only if $t = 0$ and $\phi_1(t) < t \leq \phi_2(t)$ for all $t > 0$.
(ii) $\zeta_2(t,s) = s - \frac{f(t,s)}{g(t,s)} t$ for all $t, s \in \mathbb{R}_0^+$, where $f, g : {\mathbb{R}_0^+}^2 \to \mathbb{R}_0^+$ are two functions, continuous with respect to each variable and such that $f(t,s) > g(t,s)$ for all $t, s > 0$.
(iii) $\zeta_3(t,s) = s - \phi(s) - t$ for all $t, s \in \mathbb{R}_0^+$, where $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous functions, with $\phi(t) = 0$ if and only if $t = 0$.
(iv) If $\varphi : \mathbb{R}_0^+ \to [0,1)$ is a function such that $\limsup_{t\to r^+} \varphi(t) < 1$ for all $r > 0$, let
$$\zeta_4(t,s) = s\varphi(s) - t, \qquad \text{for all } t, s \in \mathbb{R}_0^+.$$
(v) If $\eta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is an upper semi-continuous function such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$, let
$$\zeta_5(t,s) = \eta(s) - t, \qquad \text{for all } t, s \in \mathbb{R}_0^+.$$
(vi) If $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a function such that $\int_0^\epsilon \phi(u)\,du > \epsilon$ for each $\epsilon > 0$, let
$$\zeta_6(t,s) = s - \int_0^t \phi(u)\,du, \qquad \text{for all } t, s \in \mathbb{R}_0^+.$$

It is clear that each function ζ_i, $i = 1, 2, \dots, 6$, is a simulation function.

Definition 2.2 ([2])**.** Let X be a nonempty set, and a mapping $\sigma : X \times X \to \mathbb{R}_0^+$ is such that, for all $x, y, z \in X$,

(σ_1) $\sigma(x,y) = 0$ implies $x = y$;
(σ_2) $\sigma(x,y) = \sigma(y,x)$;
(σ_3) $\sigma(x,y) \leq \sigma(x,z) + \sigma(z,y)$.

Then (X, σ) is said to be a metric-like space.

As is well known, each partial metric space is an example of a metric-like space. The converse is not true. The following example illustrates this statement.

Example 2.2. Take $X = \{1, 2, 3\}$ and consider the metric-like $\sigma : X \times X \to \mathbb{R}_0^+$ given by

$$\sigma(1,1) = 0, \qquad \sigma(2,2) = 1, \qquad \sigma(3,3) = \frac{2}{3},$$
$$\sigma(2,1) = \sigma(1,2) = \frac{9}{10}, \quad \sigma(1,3) = \sigma(3,1) = \frac{7}{10}, \quad \sigma(2,3) = \sigma(3,2) = \frac{4}{5}.$$

Since $\sigma(2,2) \neq 0$, σ is not a metric and since $\sigma(2,2) > \sigma(2,1)$, σ is not a partial metric.

Every metric-like σ on X generates a topology τ_σ whose base is the family of all open σ-balls

$$\{\, B_\sigma(x,\delta) : x \in X, \delta > 0 \,\},$$

where $B_\sigma(x,\delta) = \{\, y \in X : |\sigma(x,y) - \sigma(x,x)| < \delta \,\}$, for all $x \in X$ and $\delta > 0$.

Definition 2.3 ([2])**.** Let (X,σ) be a metric-like space, let $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) $\{x_n\}$ is said to converge to x, w.r.t. τ_σ, if $\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x,x)$;
(ii) $\{x_n\}$ is called a Cauchy sequence in (X,σ) if $\lim_{n,m\to\infty} \sigma(x_n, x_m)$ exists (and is finite);
(iii) (X,σ) is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_σ to a point $x \in X$ such that

$$\lim_{n,m\to\infty} \sigma(x_n, x_m) = \lim_{n\to\infty} \sigma(x_n, x) = \sigma(x,x);$$

(iv) a function $f : X \to X$ is continuous if for any sequence $\{x_n\}$ in X such that $\sigma(x_n, x) \to \sigma(x,x)$ as $n \to \infty$, we have $\sigma(fx_n, fx) \to \sigma(fx, fx)$ as $n \to \infty$.

Note that the limit of a sequence in a metric-like space might not be unique.

Lemma 2.1 ([11])**.** *Let (X,σ) be a metric-like space. Let $\{x_n\}$ be a sequence in X such that $x_n \to x$ where $x \in X$ and $\sigma(x,x) = 0$. Then for all $y \in X$, we have*

$$\lim_{n\to\infty} \sigma(x_n, y) = \sigma(x,y).$$

Ψ will denote the family of non-decreasing functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfying the following conditions:

(i) $\psi(t) < t$, for any $t \in \mathbb{R}^+$;
(ii) ψ is continuous at 0.

Let (X, σ) be a metric-like space, and U and V be two non-empty subsets of X. Recall the following standard notation:

$$\begin{aligned} \sigma(U,V) &:= \inf\{\sigma(u,v) : u \in U, v \in V\},\\ U_0 &:= \{u \in U : \sigma(u,v) = \sigma(U,V) \text{ for some } v \in V\},\\ V_0 &:= \{v \in V : \sigma(u,v) = \sigma(U,V) \text{ for some } u \in U\}. \end{aligned}$$

Consider now a non-self mapping $T : U \to V$ and the equation $Tu = u$ $(u \in U)$. As is well known, a solution of this equation, if it exists, is called a fixed point of T. If such solution does not exist, an approximate solution $u^* \in U$ have the least possible error when $\sigma(u^*, Tu^*) = \sigma(U, V)$. In this case, u^* is called a best proximity point of the mapping $T : U \to V$.

Finally, recall the following useful notions.

Definition 2.4 ([6])**.** Let U and V be nonempty subsets of a metric-like space (X, σ), and $\alpha : U \times U \to \mathbb{R}_0^+$ be a function. We say that the mapping T is α-proximal admissible if

$$\alpha(x,y) \geq 1 \text{ and } \sigma(u,Tx) = \sigma(v,Ty) = \sigma(U,V) \;\Rightarrow\; \alpha(u,v) \geq 1,$$

for all $x, y, u, v \in X$.

If $\sigma(U, V) = 0$, then T reduces from α-proximal admissible to α-admissible.

Definition 2.5 ([8, 10])**.** Let $T : X \to X$ be a mapping and $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. We say that the mapping T is triangular weakly-α-admissible if

$$\alpha(x,y) \geq 1 \text{ and } \alpha(y,z) \geq 1 \Rightarrow \alpha(x,z) \geq 1.$$

3. Main Results

Definition 3.1. Let (X, σ) be a metric-like space, U and V be two non-empty subsets of X, $\psi \in \Psi$, $\alpha : X \times X \to \mathbb{R}_0^+$ and $\zeta \in \mathcal{Z}$. We say that $T : U \to V$ is an α-ψ-ζ-contraction if T is α-proximal admissible and

(3.1)

$$\alpha(x,y) \geq 1 \text{ and } \sigma(u,Tx) = \sigma(v,Ty) = \sigma(U,V) \;\Rightarrow\; \zeta(\alpha(x,y)\sigma(u,v), \psi(\sigma(x,y))) \geq 0,$$

for all $x, y, u, v \in U$.

Definition 3.2. Let (X, σ) be a metric-like space, U and V be two non-empty subsets of X, $\alpha : X \times X \to \mathbb{R}_0^+$ and $\zeta \in \mathcal{Z}$. We say that $T : U \to V$ is an α-ζ-contraction if T is α-proximal admissible and

(3.2)

$$\alpha(x,y) \geq 1 \text{ and } \sigma(u,Tx) = \sigma(v,Ty) = \sigma(U,V) \;\Rightarrow\; \zeta(\alpha(x,y)\sigma(u,v), \sigma(x,y)) \geq 0,$$

for all $x, y, u, v \in U$.

Notice that Definition 3.2 is not a special case of Definition 3.1 since the function $\psi(t) = t$ does not belong to Ψ.

The following lemma provides a standard step in proving that the given sequence is Cauchy in a certain space.

Lemma 3.1 (See, e.g., [14]). *Let (X, σ) be a metric-like space and let $\{x_n\}$ be a sequence in X such that $\sigma(x_{n+1}, x_n)$ is non-increasing and that $\lim_{n\to\infty} \sigma(x_{n+1}, x_n) = 0$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ϵ when $k \to \infty$:*

$$\sigma(x_{m_k}, x_{n_k}),\ \sigma(x_{m_k+1}, x_{n_k+1}),\ \sigma(x_{m_k-1}, x_{n_k}),\ \sigma(x_{m_k}, x_{n_k-1}).$$

Now we present the main results of this article.

Theorem 3.1. *Let (X, σ) be a metric-like space, U and V be two non-empty subsets of X, $\alpha : X \times X \to \mathbb{R}_0^+$, $\psi \in \Psi$ and $\zeta \in \mathcal{Z}$ is non-decreasing with respect to its second argument. Suppose that $T : U \to V$ is an α-ψ-ζ-contraction and*

(1) *T is triangular weakly-α-admissible;*
(2) *U is closed with respect to the topology τ_σ;*
(3) *$T(U_0) \subset V_0$;*
(4) *there exist $x_0, x_1 \in U$ such that $\sigma(x_1, Tx_0) = \sigma(U, V)$ and $\alpha(x_0, x_1) \geq 1$;*
(5) *T is continuous.*

Then, T has a best proximity point, that is, there exists $z \in U$ such that $\sigma(z, Tz) = \sigma(U, V)$.

Proof. Take $x_0, x_1 \in U$ given as in (4). Taking (3) into account, we conclude that $Tx_1 \in V_0$ which implies that there exists $x_2 \in U$ such that $\sigma(x_2, Tx_1) = \sigma(U, V)$. Since $\alpha(x_0, x_1) \geq 1$ and T is α-proximal admissible, we conclude that $\alpha(x_1, x_2) \geq 1$. Recursively, a sequence $\{x_n\} \subset U$ can be chosen satisfying

$$\sigma(x_{n+1}, Tx_n) = \sigma(U, V) \text{ and } \alpha(x_n, x_{n+1}) \geq 1, \quad \text{for all } n \in \mathbb{N}_0. \tag{3.3}$$

If $x_k = x_{k+1}$ for some $k \in \mathbb{N}_0$, then $\sigma(x_k, Tx_k) = \sigma(x_{k+1}, Tx_k) = \sigma(U, V)$, meaning that x_k is the required best proximal point. Hence, we will further assume that

$$x_n \neq x_{n+1}, \quad \text{for all } n \in \mathbb{N}_0. \tag{3.4}$$

Using relations (3.3) and (3.4), we get that $\sigma(x_n, Tx_{n-1}) = \sigma(x_{n+1}, Tx_n) = \sigma(U, V)$, for all $n \in \mathbb{N}$. Furthermore, by (3.1)

$$\zeta(\alpha(x_{n-1}, x_n)\sigma(x_n, x_{n+1}), \psi(\sigma(x_{n-1}, x_n))) \geq 0, \quad \text{for all } n \in \mathbb{N}, \tag{3.5}$$

since $T : U \to V$ is an α-ψ-ζ-contraction. Regarding (3.4) and (ζ_2), the inequality (3.5) implies that

$$\sigma(x_n, x_{n+1}) \leq \alpha(x, y)\sigma(x_n, x_{n+1}) \leq \psi(\sigma(x_{n-1}, x_n)) < \sigma(x_{n-1}, x_n), \quad \text{for all } n \in \mathbb{N}.$$

Thus, $\{\sigma(x_n, x_{n+1})\}$ is a non-increasing sequence bounded from below and there exists $L \in \mathbb{R}_0^+$ such that $\sigma(x_n, x_{n+1}) \to L$ as $n \to \infty$. We shall prove that $L = 0$. Suppose, on the contrary, that $L > 0$. Taking the upper limit in (3.5) as $n \to \infty$, regarding (ζ_3), property (i) of $\psi \in \Psi$ and that ζ is non-decreasing with respect to the second argument, we deduce

$$\begin{aligned}0 &\leq \limsup_{n\to\infty} \zeta(\alpha(x_n, x_{n-1})\sigma(x_n, x_{n+1}), \psi(\sigma(x_n, x_{n-1})))\\ &\leq \limsup_{n\to\infty} \zeta(\alpha(x_n, x_{n-1})\sigma(x_n, x_{n+1}), \sigma(x_n, x_{n-1})) < 0,\end{aligned}$$

which is a contradiction. We conclude that $\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0$.

We shall now prove that the sequence $\{x_n\}$ is Cauchy. Suppose that it is not. Then, there exist $\epsilon > 0$ and subsequences $\{x_{m_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$, so that $n_k > m_k > k$ and

$$\sigma(x_{m_k}, x_{n_k}) \geq \epsilon \text{ and } \sigma(x_{m_k}, x_{n_k-1}) < \epsilon. \tag{3.6}$$

By Lemma 2.1, we have

$$\lim_{k\to\infty} \sigma(x_{m_k}, x_{n_k}) = \lim_{k\to\infty} \sigma(x_{n_k-1}, x_{m_k-1}) = \epsilon.$$

Since T is triangular weakly-α-admissible, from (3.3), we get that

$$\alpha(x_n, x_m) \geq 1, \quad \text{for all } n, m \in \mathbb{N}_0 \text{ with } n > m.$$

Hence,

$$\alpha(x_{m_k}, x_{n_k}) \geq 1 \text{ and } \sigma(x_{m_k}, Tx_{m_k-1}) = \sigma(x_{n_k}, Tx_{n_k-1}) = \sigma(U, V), \quad \text{for all } k \in \mathbb{N}. \tag{3.7}$$

Since T is an α-ψ-ζ-contraction, the obtained relations (3.7) yield the following inequality:

$$0 \leq \zeta(\alpha(x_n, x_{n-1})\sigma(x_{m_k}, x_{n_k}), \psi(\sigma(x_{m_k}, x_{n_k}))), \quad \text{for all } k \in \mathbb{N}.$$

Letting $k \to \infty$, using (3.6) and (ζ_3), and regarding properties of $\psi \in \Psi$ and that ζ is non-decreasing with respect to the second argument, we obtain

$$\begin{aligned}0 &\leq \limsup_{n\to\infty} \zeta(\alpha(x_n, x_{n-1})\sigma(x_{m_k}, x_{n_k}), \psi(\sigma(x_{m_k-1}, Tx_{n_k-1})))\\ &\leq \limsup_{n\to\infty} \zeta(\alpha(x_n, x_{n-1})\sigma(x_{m_k}, x_{n_k}), \sigma(x_{m_k-1}, Tx_{n_k-1})) < 0,\end{aligned}$$

which is a contradiction. Thus, we conclude that the sequence $\{x_n\}$ is Cauchy in U.

Since U is a closed subset of a complete metric-like space (X, σ), there exists $z \in U$ such that

$$\lim_{n\to\infty} \sigma(x_n, z) = 0. \tag{3.8}$$

Since T is continuous, we deduce that

$$\lim_{n\to\infty} \sigma(Tx_n, Tz) = 0. \tag{3.9}$$

From (3.3), using the triangle inequality together with (3.8) and (3.9), we find that

$$\sigma(U, V) = \lim_{n\to\infty} \sigma(x_{n+1}, Tx_n) = \sigma(z, Tz).$$

Thus, $z \in U$ is a best proximity point of the mapping T. $\square$

The continuity hypothesis in Theorem 3.1 can be omitted if we assume the following additional condition on U:

(P) if a sequence $\{u_n\}$ in U converges to $u \in U$ and is such that $\alpha(u_n, u_{n+1}) \geq 1$ for $n \geq 1$, then there is a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ with $\alpha(u_{n(k)}, u) \geq 1$ for all k.

Theorem 3.2. *Let all the conditions of Theorem* 3.1 *hold, except that the condition* (5) *is replaced by*

(5') (P) *holds.*

Then T has a best proximity point.

Proof. As in the proof of Theorem 3.1 we conclude that there exists a sequence $\{x_n\}$ in U_0 which converges to $z \in U_0$. Using (3), we note that $Tz \in V_0$ and hence

$$\sigma(u_1, Tz) = \sigma(U, V), \quad \text{for some } u_1 \in U_0.$$

Notice that from (P), we have $\alpha(x_{n_k}, z) \geq 1$ for all $k \in \mathbb{N}$. Since T is α-proximal admissible and

$$\sigma(u_1, Tz) = \sigma(x_{n_k+1}, Tx_{n_k}) = \sigma(U, V), \tag{3.10}$$

we obtain that $\alpha(x_{n_k+1}, u_1) \geq 1$ for all $k \in \mathbb{N}$ and

$$\zeta(\alpha(x_{n_k+1}, u_1)\sigma(u_1, x_{n_k+1}), \psi(\sigma(z, x_{n_k}))) \geq 0.$$

Then, (ζ_2) implies that

$$\sigma(u_1, x_{n_k+1}) \leq \alpha(x_{n_k+1}, u_1)\sigma(u_1, x_{n_k+1}) \leq \psi(\sigma(z, x_{n_k})) < \sigma(z, x_{n_k})$$

and so $\lim_{k\to\infty} \sigma(u_1, x_{n_k+1}) \to 0$. Thus, $u_1 = z$ and by (3.10) we have $\sigma(z, Tz) = \sigma(U, V)$. ☐

Theorem 3.3. *Let (X, σ) be a metric-like space, U and V be two non-empty subsets of X, $\zeta \in \mathcal{Z}$ and $\alpha : X \times X \to \mathbb{R}_0^+$. Suppose that $T : U \to V$ is an α-ζ-contraction and that conditions* (1)-(4) *of Theorem* 3.1 *are satisfied, as well as*

$(5'')$ *T is continuous or* (P) *holds.*

Then, T has a best proximity point.

Proof. By following the lines in the proof of Theorem 3.1, we easily construct a sequence $\{x_n\}$ in U which converges to some $z \in U$, moreover

$$\lim_{n\to\infty} \sigma(x_n, z) = 0. \tag{3.11}$$

Suppose first that T is continuous. Then

$$\lim_{n\to\infty} \sigma(Tx_n, Tz) = 0. \tag{3.12}$$

From (3.3), the triangle inequality together with (3.11) and (3.12) imply

$$\sigma(U, V) = \lim_{n\to\infty} \sigma(x_{n+1}, Tx_n) = \sigma(z, Tz).$$

In other words, $z \in U$ is a best proximity of the mapping T.

Suppose now that (P) holds. Regarding (3), we note that $Tz \in V_0$ and hence

$$\sigma(u_1, Tz) = \sigma(U, V), \quad \text{for some } u_1 \in U_0.$$

Notice that from (P), we have $\alpha(x_{n_k}, z) \geq 1$ for all $k \in \mathbb{N}$. Since T is α-proximal admissible, and

$$\sigma(u_1, Tz) = \sigma(x_{n_k+1}, Tx_{n_k}) = \sigma(U, V),$$

we get that $\alpha(x_{n_k+1}, u_1) \geq 1$ for all $k \in \mathbb{N}$ and

$$\zeta(\alpha(x_{n_k+1}, u_1)\sigma(u_1, x_{n_k+1}), \sigma(z, x_{n_k})) \geq 0. \tag{3.13}$$

Then, (ζ_2) implies that $\sigma(u_1, x_{n_k+1}) \leq \alpha(x_{n_k+1}, u_1)\sigma(u_1, x_{n_k+1}) \leq \sigma(z, x_{n_k})$ and so

$$\lim_{k\to\infty} \sigma(u_1, x_{n_k+1}) \to 0.$$

Thus, $u_1 = z$ and by (3.13) we have $\sigma(z, Tz) = \sigma(U, V)$ and the proof is completed. □

Notice that Theorem 3.3 cannot be obtained by combining Theorems 3.1 and 3.2, since the function $\psi(t) = t$ does not belong to Ψ. Furthermore, in Theorems 3.1 and 3.2, we have an additional condition that ζ is non-decreasing in its second argument.

Definition 3.3. Let (X, σ) be a metric-like space, U and V be two non-empty subsets of X, $\alpha : X \times X \to \mathbb{R}_0^+$ and $\zeta \in \mathcal{Z}$. We say that $T : U \to V$ is a generalized α-ζ-contraction if T is α-proximal admissible and

$$\alpha(x, y) \geq 1 \text{ and } \sigma(u, Tx) = \sigma(v, Ty) = \sigma(U, V) \;\Rightarrow\; \zeta(\alpha(x, y)\sigma(u, v), r(x, y)) \geq 0, \tag{3.14}$$

for all $x, y, u, v \in U$ with $x \neq y$, where

$$r(x, y) = \max\left\{\sigma(x, y), \frac{\sigma(x, u)\sigma(y, v)}{\sigma(x, y)}\right\}.$$

Theorem 3.4. *Let (X, σ) be a metric-like space, U and V be two non-empty subsets of X and $\alpha : X \times X \to \mathbb{R}_0^+$, $\zeta \in \mathcal{Z}$. Suppose that $T : U \to V$ is a generalized α-ζ-contraction and conditions* (1)-(5) *of Theorem* 3.1 *are satisfied. Then T has a best proximity point.*

Proof. As in the proof of Theorem 3.1, we can construct a sequence $\{x_n\}$ in X satisfying conditions (3.3) and (3.4). Combining these relations with (3.14), we get that $\sigma(x_n, Tx_{n-1}) = \sigma(x_{n+1}, Tx_n) = \sigma(U, V)$ for all $n \in \mathbb{N}$ and

$$\zeta(\alpha(x_{n-1}, x_n)\sigma(x_n, x_{n+1}), r(x_{n-1}, x_n)) \geq 0, \quad \text{for all } n \in \mathbb{N}.$$

Here,

$$\begin{aligned} r(x_{n-1}, x_n) &= \max\left\{\frac{\sigma(x_{n-1}, x_n)\sigma(x_n, x_{n+1})}{\sigma(x_{n-1}, x_n)}, \sigma(x_{n-1}, x_n)\right\} \\ &= \max\left\{\sigma(x_n, x_{n+1}), \sigma(x_{n-1}, x_n)\right\}. \end{aligned}$$

Suppose that for some $n \in \mathbb{N}$

$$\max\left\{\sigma(x_n, x_{n+1}), \sigma(x_{n-1}, x_n)\right\} = \sigma(x_n, x_{n+1}).$$

Since $\sigma(x_n, x_{n+1}) > 0$, using the property (2) of the simulation function, we obtain

$$\zeta(\alpha(x_{n-1}, x_n)\sigma(x_n, x_{n+1}), \sigma(x_n, x_{n+1})) < 0,$$

which is a contradiction. It follows that $r(x_{n-1}, x_n) = \sigma(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$, implying that

$$\zeta(\alpha(x_{n-1}, x_n)\sigma(x_n, x_{n+1}), \sigma(x_{n-1}, x_n)) \geq 0, \quad \text{for all } n \in \mathbb{N}. \tag{3.15}$$

Using (ζ_2), the inequality (3.15) yields that

$$\sigma(x_n, x_{n+1}) \leq \sigma(x_{n-1}, x_n), \quad \text{for all } n \in \mathbb{N}.$$

Hence, $\{\sigma(x_n, x_{n+1})\}$ is a non-increasing sequence, bounded from below, converging to some $L \geq 0$. Suppose that $L > 0$. Taking the upper limit as $n \to \infty$ in (3.15), using (ζ_3), we get

$$0 \leq \limsup_{n\to\infty} \zeta(\alpha(x_n, x_{n-1})\sigma(x_n, x_{n+1}), \psi(\sigma(x_{n-1}, x_n))) < 0,$$

which is a contradiction. Hence, we conclude that $\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0$.

In order to prove that $\{x_n\}$ is a Cauchy sequence, suppose the contrary. Then, as in the proof of Theorem 3.1, there exist $\epsilon > 0$ and subsequences $\{x_{m_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$, so that for $n_k > m_k > k$ we have

$$\sigma(x_{m_k}, x_{n_k}) \geq \epsilon \text{ and } \sigma(x_{m_k}, x_{n_k-1}) < \epsilon.$$

Also, in the same way, the following inequalities hold:

$$\begin{aligned} \lim_{k\to\infty} \sigma(x_{m_k}, x_{n_k}) &= \lim_{k\to\infty} \sigma(x_{n_k-1}, x_{m_k-1}) = \epsilon, \\ \lim_{k\to\infty} \sigma(x_{m_k-1}, x_{n_k}) &= \lim_{k\to\infty} \sigma(x_{n_k-1}, x_{m_k}) = \epsilon. \end{aligned} \tag{3.16}$$

Since T is triangular weakly-α-admissible, we derive that

$$\alpha(x_n, x_m) \geq 1, \quad \text{for all } n, m \in \mathbb{N}_0 \text{ with } n > m.$$

Thus, we have

$$\alpha(x_{m_k}, x_{n_k}) \geq 1 \text{ and } \sigma(x_{m_k}, Tx_{m_k-1}) = \sigma(x_{n_k}, Tx_{n_k-1}) = \sigma(U, V), \tag{3.17}$$

for all $k \in \mathbb{N}$. Since T is a generalized α-ζ-contraction, the obtained relations (3.17) imply

$$0 \leq \zeta(\alpha(x_{m_k-1}, x_{n_k-1})\sigma(x_{m_k}, x_{n_k}), r(x_{m_k-1}, x_{n_k-1})), \quad \text{for all } k \in \mathbb{N}.$$

Since

$$r(x_{m_k-1}, x_{n_k-1}) = \max\left\{\frac{\sigma(x_{m_k-1}, x_{m_k})\sigma(x_{n_k-1}, x_{n_k})}{\sigma(x_{m_k-1}, x_{n_k-1})}, \sigma(x_{m_k-1}, x_{n_k-1})\right\}, \tag{3.18}$$

taking limits of both sides of (3.18), we conclude that $\lim_{k\to\infty} r(x_{m_k-1}, x_{n_k-1}) = \epsilon$. Letting $k \to \infty$ and keeping (3.16) and (ζ_3) in mind, we get

$$0 \leq \limsup_{n\to\infty} \zeta(\alpha(x_{m_k-1}, x_{n_k-1})\sigma(x_{m_k}, x_{n_k}), r(x_{m_k-1}, x_{n_k-1})) < 0,$$

which is a contradiction. Thus, we conclude that the sequence $\{x_n\}$ is Cauchy in U.

The final step of the proof is the same as for Theorem 3.1. □

4. Corollaries and Examples

Using Example 2.1, it is possible to get a number of consequences of our main results by choosing the simulation function ζ and $\alpha(x,y)$ in a proper way. We skip making such a list of corollaries since they seem clear. We just state the following one as a sample

Corollary 4.1. *Let (X,σ) be a metric-like space, U and V be two non-empty subsets of X and $\alpha : X \times X \to \mathbb{R}_0^+$, $\psi \in \Psi$. Suppose that $T : U \to V$ is a given α-proximal admissible mapping such that*

$$\alpha(x,y) \geq 1 \text{ and } \sigma(u,Tx) = \sigma(v,Ty) = \sigma(U,V) \;\Rightarrow\; \alpha(x,y)\sigma(u,v) \leq \psi(\sigma(x,y))),$$

for all $x,y,u,v \in U$. Suppose also

(a) *T is triangular weakly-α-admissible;*
(b) *U is closed with respect to the topology induced by τ_σ;*
(c) *$T(U_0) \subset V_0$;*
(d) *there exist $x_0, x_1 \in U$ such that $\sigma(x_1, Tx_0) = \sigma(U,V)$ and $\alpha(x_0,x_1) \geq 1$;*
(e) *T is continuous or* (P) *holds.*

Then, T has a best proximity point.

In particular, if the given space (X,σ) is also endowed with a partial order $\preceq$, by taking

$$\alpha(x,y) \geq 1 \Leftrightarrow x \succeq y,$$

one can get standard variations of the given results in a partially ordered space.

The following illustrative examples show how our results can be used for certain mappings acting in metric-like spaces.

Example 4.1. Consider $X = \{a,b,c,d\}$ equipped with $\sigma : X \times X \to \mathbb{R}_0^+$ defined by

$$\sigma(a,a) = \frac{1}{2}, \quad \sigma(b,b) = 0, \quad \sigma(c,c) = 2, \quad \sigma(d,d) = \frac{1}{3}, \quad \sigma(a,b) = 3,$$
$$\sigma(a,c) = \frac{5}{2}, \quad \sigma(a,d) = \frac{3}{2} \quad \sigma(b,c) = 2, \quad \sigma(b,d) = \frac{3}{2}, \quad \sigma(c,d) = \frac{5}{2},$$

and $\sigma(x,y) = \sigma(y,x)$ for $x,y \in X$. It is clear that (X,σ) is a complete metric-like space. Take $U = \{b,c\}$ and $V = \{c,d\}$. Consider the mapping $T : U \to V$ defined by $Tb = d$, and $Tc = c$. Remark that $\sigma(U,V) = \sigma(b,d) = \frac{3}{2}$. Also, $U_0 = \{b\}$ and $V_0 = \{d\}$. Note that $T(U_0) \subseteq V_0$. Take $\psi(t) = \frac{5}{6}t$, and $\zeta(t,s) = \frac{3}{4}s - t$ for all $t,s \geq 0$. Define $\alpha : X \times X \to \mathbb{R}_0^+$ by

$$\alpha(x,y) = \begin{cases} 1, & x,y \in U, \\ 0, & \text{otherwise.} \end{cases}$$

Let $x, y, u, v \in U$ be such that

$$\alpha(x,y) \geq 1 \text{ and } \sigma(u,Tx) = \sigma(v,Ty) = \sigma(U,V) = \frac{3}{2}.$$

Then, necessarily, we have $x = y = u = v = b$. So, $\alpha(u,v) \geq 1$, that is, T is α-proximal admissible.

We need to prove that T is an α-ψ-ζ contraction. By the previous conclusion, the only case to be checked is when $x = y = u = v = b$. Then we have

$$\zeta(\alpha(b,b)\sigma(b,b), \psi(\sigma(b,b))) = \zeta(1 \cdot 0, \psi(0)) = 0.$$

Thus, all the conditions of Theorem 3.1 are satisfied. So T has a best proximity point (which is $z = b$). On the other hand, e.g., Corollary 2.2 (with $k = 2$) of [4] is not applicable for the standard metric.

Example 4.2. Consider the set $X = \{a, b, c, d\}$ equipped with the following complete metric-like σ:

$$\sigma(a,a) = \sigma(b,b) = \frac{1}{4}, \quad \sigma(c,c) = \sigma(d,d) = 2,$$
$$\sigma(a,b) = \sigma(c,d) = \frac{1}{2}, \quad \sigma(a,c) = \sigma(b,d) = 1, \quad \sigma(a,d) = \sigma(b,c) = \frac{3}{2},$$

and $\sigma(x,y) = \sigma(y,x)$ for all $x, y \in X$. Let $U = \{a, b\}$ and $V = \{c, d\}$; then $\sigma(U,V) = 1$, $U_0 = U$ and $V_0 = V$. Consider, further, the mappings $T : U \to V$ given by $Ta = c$, $Tb = c$, $\alpha : X \times X \to [0, +\infty)$ given by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x, y \in U, \\ 0, & \text{otherwise,} \end{cases}$$

and $\zeta \in \mathcal{Z}$ given by $\zeta(t,s) = s - \frac{2+t}{1+t}t$. Let us check that the mapping T is a generalized α-ζ-contraction. Let $x, y, u, v \in U$ be such that $x \neq y$, $\alpha(x,y) \geq 1$, $\sigma(u,Tx) = \sigma(v,Ty) = 1$. Then it must be $u = v = a$ and either $x = a$, $y = b$ or $x = b$, $y = a$. In both cases, it is $\alpha(u,v) \geq 1$. In order to check condition (3.14), it is enough to consider the case $x = a$, $y = b$, $u = v = a$ (the other is treated symmetrically). Then,

$$\begin{aligned}\zeta(\alpha(x,y)\sigma(u,v), r(x,y)) &= \zeta\left(1 \cdot \frac{1}{4}, \max\left\{\frac{1}{2}, \frac{\frac{1}{4} \cdot \frac{1}{2}}{\frac{1}{2}}\right\}\right) = \zeta\left(\frac{1}{4}, \frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{2 + \frac{1}{4}}{1 + \frac{1}{4}} \cdot \frac{1}{4} = \frac{1}{20} > 0,\end{aligned}$$

and the condition is satisfied. All other conditions of Theorem 3.4 are fulfilled, hence, we conclude that the mapping T has a best proximity point (which is $z = a$).

5. Application to Best Proximity Results on a Metric-like Space with a Graph

Throughout this section, (X,σ) will denote a metric-like space and $G=(V(G),E(G))$ will be a directed graph such that its set of vertices $V(G)=X$ and the set of edges $E(G)$ contains all loops, i.e., $\Delta:=\{(x;x):x\in X\}\subseteq E(G)$. We need in the sequel the following hypothesis:

(P_G) if a sequence $\{u_n\}$ in X converges to $u\in A$ such that $(u_n,u_{n+1})\in E(G)$, then there is a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ with $(u_{n(k)},u)\in E(G)$ for all k.

Definition 5.1. Let U and V be two non-empty subsets of X and $\alpha: X\times X\to\mathbb{R}_0^+$. We say that $T:U\to V$ is a G-proximal mapping if

$$\left.\begin{array}{c}(x,y)\in E(G),\ \ \alpha(x,y)\geq 1,\\ \sigma(u,Tx)=\sigma(v,Ty)=\sigma(U,V)\end{array}\right\}\Rightarrow (u,v)\in E(G), \tag{5.1}$$

for all $x,y,u,v\in U$.

Definition 5.2 ([8,10])**.** Let U and V be two non-empty subsets of X, let $T:U\to V$ be a mapping and $\alpha: X\times X\to\mathbb{R}_0^+$ be a function. We say that T is triangular weakly-G-admissible if

$$\alpha(x,y)\in E(G) \text{ and } \alpha(y,z)\in E(G)\ \Rightarrow\ \alpha(x,z)\in E(G).$$

Corollary 5.1. *Let U and V be two non-empty subsets of X and $\psi\in\Psi$. Suppose that $T:U\to V$ is a mapping such that*

$$\sigma(Tx,Ty)\leq\psi(\sigma(x,y)),$$

for all $x,y\in U$ such that $(x,y)\in E(G)$. Suppose also:

(a) *T is triangular weakly-G-admissible;*
(b) *$T(U_0)\subset V_0$;*
(c) *there exist $x_0,x_1\in U$ such that $\sigma(x_1,Tx_0)=\sigma(U,V)$ and $(x_0,x_1)\in E(G)$;*
(d) *T is continuous or (R_G) holds.*

Then, T has a best proximity point.

Proof. It suffices to consider $\alpha: X\times X\to\mathbb{R}_0^+$ such that

$$\alpha(x,y)=\begin{cases}1, & \text{if } (x,y)\in E(G),\\ 0, & \text{if not.}\end{cases}$$

All the hypotheses of Corollary 4.1 are satisfied. □

In this way, we can derive all results and consequences of the paper [15], extending them to partially ordered metric-like spaces. Similarly, we can extend the frame of several other existing results from, e.g., [3, 10, 12, 16].

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[1]Department of Mathematics,
ICFAI University,
Raipur, Chhattisgarh-490042, INDIA
Email address: gvvjagan1@gmail.com

[2]Department of Mathematics,
School of Advanced Sciences,
Vellore Institute of Technology,
Vellore-632014, TN, INDIA
Email address: drhknashine@gmail.com, hemant.nashine@vit.ac.in

[3]Faculty of Mathematics,
University of Belgrade,
Studentski trg 16, 11000 Beograd, Serbia
Email address: kadelbur@matf.bg.ac.rs

*Corresponding author