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k-TYPE BI-NULL CARTAN SLANT HELICES IN $\mathbb{R}_2^6$

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Abstract. In the present paper, we give the notion of k-type bi-null Cartan slant helices in $\mathbb{R}_2^6$, where $k \in \{1, 2, 3, 4, 5, 6\}$. We give the necessary and sufficient conditions for bi-null Cartan curves to be k-type slant helices in terms of their curvature functions.

1. Introduction

The notion of a slant helix is introduced by Izumiya and Takeuchi [4]. A curve γ with non-zero curvature is called a slant helix in Euclidean 3-space $\mathbb{R}^3$ if the principal normal line of γ makes a constant angle with a fixed vector in $\mathbb{R}^3$. Some characterizations of such curves were presented in $[1, 5, 6, 9]$.

Further, k-type slant helices emerged and attracted attention of researchers. Ergüt et al. ([3]) studied k-slant helices in Minkowski 3-space $\mathbb{R}_1^3$. The curves of such type were studied in Minkowski space-time $\mathbb{R}_1^4$ by some researchers in $[2, 7]$.

Bi-null Cartan curves in $\mathbb{R}_2^n$ are defined in [8]. Some characterizations of bi-null Cartan curves in terms of their curvature functions in $\mathbb{R}_2^n$ for $n \geq 6$ are also given in [8]. The necessary and the sufficient conditions for bi-null curves to be k-type slant helices in semi-Euclidean spaces $\mathbb{R}_3^6$ and $\mathbb{R}_2^5$ are given in $[10, 11]$.

On the other hand, the third named author of this paper gave the notion of bi-null Cartan curves in semi-Euclidean spaces $\mathbb{R}_2^n$ of index 2, together with the unique Cartan frame and the Cartan curvatures. He discussed some properties of bi-null Cartan curves in terms of the Cartan curvatures in the case where $n \geq 6$ ([8]). In $\mathbb{R}_2^5$ and $\mathbb{R}_3^6$, we define k-type bi-null slant helices and we give the necessary and sufficient

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conditions for bi-null curves to be k-type slant helices in terms of their curvature functions ([10, 11]).

In this paper, we give the notion of k-type bi-null Cartan slant helices in $\mathbb{R}_2^6$, where $k \in \{1,2,3,4,5,6\}$. We give the necessary and sufficient conditions for bi-null Cartan curves to be k-type slant helices in terms of their curvature functions.

2. Priliminaries

In this section, following [8], we recall the Frenet equations for bi-null Cartan curves in $\mathbb{R}_2^6$. Let $\mathbb{R}_2^6$ be the 6-dimensional semi-Euclidean space of index 2 with standard coordinate system $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and metric

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 - dx_5^2 - dx_6^2.$$

We denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathbb{R}_2^6$. Recall that a vector $v \in \mathbb{R}_2^6 \backslash \{0\}$ can be spacelike if $\langle v, v \rangle > 0$, timelike if $\langle v, v \rangle < 0$ and null (lightlike) if $\langle v, v \rangle = 0$. In particular, the vector $v = 0$ is said to be spacelike. The norm of a vector v is given by $||v|| = \sqrt{|\langle v, v \rangle|}$. Two vectors v and w are said to be orthogonal, if $\langle v, w \rangle = 0$.

We say that a curve $\gamma(t)$ in $\mathbb{R}_2^6$ is a bi-null curve if span$\{\gamma'(t), \gamma''(t)\}$ is an isotropic 2-plane for all t. That is,

$$\langle \gamma'(t), \gamma'(t) \rangle = \langle \gamma'(t), \gamma''(t) \rangle = \langle \gamma''(t), \gamma''(t) \rangle = 0,$$

and $\{\gamma'(t), \gamma''(t)\}$ are linearly independent. The condition is independent of the choice of the parameter.

We say that a bi-null curve $\gamma(t)$ in $\mathbb{R}_2^6$ is parametrized by the bi-null arc if $\left\langle \gamma^{(3)}(t), \gamma^{(3)}(t) \right\rangle = 1$. If a bi-null curve $\gamma(t)$ in $\mathbb{R}_2^6$ satisfies $\left\langle \gamma^{(3)}(t), \gamma^{(3)}(t) \right\rangle \neq 0$, then by the anti-isometry of $\mathbb{R}_2^6$, we may assume that $\left\langle \gamma^{(3)}(t), \gamma^{(3)}(t) \right\rangle > 0$ and we can see that

$$u(t) = \int_{t_0}^{t} \left\langle \gamma^{(3)}(t), \gamma^{(3)}(t) \right\rangle^{1/6} dt$$

becomes the bi-null arc parameter.

Let us say that a bi-null curve $\gamma(t)$ in $\mathbb{R}_2^6$ with $\left\langle \gamma^{(3)}(t), \gamma^{(3)}(t) \right\rangle > 0$ is a bi-null Cartan curve if $\left\{\gamma'(t), \gamma''(t), \gamma^{(3)}(t), \gamma^{(4)}(t), \gamma^{(5)}(t)\right\}$ are linearly independent for any t.

For a bi-null Cartan curve $\gamma(t)$ in $\mathbb{R}_2^6$ with bi-null arc parameter t, there exists a unique pseudo-orthonormal frame $\{L_1, L_2, N_1, N_2, W_1, W_2\}$ such that

$$\begin{aligned}
\gamma' &= L_1, \quad L_1' = L_2, \quad L_2' = W_1, \\
W_1' &= -k_0 L_2 - N_2, \\
N_2' &= -k_1 L_1 - N_1 + k_0 W_1, \\
N_1' &= k_1 L_2 + k_2 W_2, \\
W_2' &= -k_2 L_1,
\end{aligned} \tag{2.1}$$

where N_1, N_2 are null, $\langle L_1, N_1\rangle = \langle L_2, N_2\rangle = 1$, $\{L_1, N_1\}$, $\{L_2, N_2\}$ and $\{W_1, W_2\}$ are mutually orthogonal, $\{W_1, W_2\}$ are orthonormal of signature $(+, +)$, and $\{L_1, L_2, N_1, N_2, W_1, W_2\}$ is positively oriented.

We say that the pseudo-orthonormal frame $\{L_1, L_2, N_1, N_2, W_1, W_2\}$ is the Cartan frame and the functions $\{k_0, k_1, k_2\}$ are the Cartan curvatures of γ.

3. k-Type Bi-Null Cartan Slant Helices

Let $\gamma(t)$ be a bi-null Cartan curve in $\mathbb{R}^6_2$ with bi-null arc parameter t and Cartan frame $\{L_1, L_2, N_1, N_2, W_1, W_2\}$. Let us set $V_1 = L_1$, $V_2 = L_2$, $V_3 = N_1$, $V_4 = N_2$, $V_5 = W_1$ and $V_6 = W_2$. Then we give the following definition.

Definition 3.1. A bi-null Cartan curve γ in $\mathbb{R}^6_2$ with Cartan frame $\{V_1, V_2, V_3, V_4, V_5, V_6\}$ is called a k-type bi-null Cartan slant helix if there exists a non-zero fixed vector $U \in \mathbb{R}^6_2$ such that the following holds

$$\langle V_k, U\rangle = \text{constant}, \quad \text{where } k \in \{1, 2, 3, 4, 5, 6\}.$$

Firstly, we consider 1-type bi-null Cartan slant helices in $\mathbb{R}^6_2$.

Theorem 3.1. *Let $\gamma(t)$ be a bi-null Cartan curve in $\mathbb{R}^6_2$ parametrized by bi-null arc t with k_0, k_1 and non-zero k_2. Then $\gamma(t)$ is a 1-type bi-null Cartan slant helix if and only if k_1 and k_2 satisfy the following*

$$k_2 - \left(\frac{k_1'}{k_2}\right)' = 0. \tag{3.1}$$

Proof. Assume that $\gamma(t)$ is a 1-type bi-null Cartan slant helix parametrized by bi-null arc t with k_0, k_1 and non-zero k_2. Then there exists a non-zero fixed vector $U \in \mathbb{R}^6_2$ such that

$$\langle L_1, U\rangle = c, \quad c \in \mathbb{R}. \tag{3.2}$$

Taking derivative of the equation (3.2) with respect to t and using equations (2.1), we get

$$\langle L_2, U\rangle = 0, \quad \langle W_1, U\rangle = 0, \quad \langle N_2, U\rangle = 0. \tag{3.3}$$

Then using (3.3), we can write U as follows

$$U = \lambda_1 L_1 + cN_1 + \lambda_2 W_2, \tag{3.4}$$

where λ_1 and λ_2 are differentiable functions. Taking derivative of the equation (3.4) with respect to t and using equations (2.1), we have

$$0 = (\lambda_1' - \lambda_2 k_2) L_1 + (\lambda_1 + ck_1) L_2 + (ck_2 + \lambda_2') W_2,$$

which implies that

$$\begin{cases} \lambda_1' - \lambda_2 k_2 = 0, \\ \lambda_1 + ck_1 = 0, \\ ck_2 + \lambda_2' = 0. \end{cases} \tag{3.5}$$

From (3.5), we find that $c \neq 0$ and

$$\lambda_1 = -ck_1, \quad \lambda_2 = -c\frac{k_1'}{k_2}, \quad k_2 - \left(\frac{k_1'}{k_2}\right)' = 0.$$

Conversely, assume that k_1 and k_2 satisfy

$$k_2 - \left(\frac{k_1'}{k_2}\right)' = 0.$$

For $c \neq 0$, choosing the vector U as

$$U = -ck_1 L_1 + cN_1 - (ck_1'/k_2)\, W_2, \tag{3.6}$$

we get $U' = 0$ and $\langle L_1, U\rangle = c$ (constant). Thus, $\gamma(t)$ is a 1-type bi-null Cartan slant helix. □

Example 3.1. The following curvature functions satisfy (3.1):
(i) $k_1 = t^2/2$, $k_2 = 1$;
(ii) $k_1 = 2t^2$, $k_2 = -2$.

Corollary 3.1. *The axis of a* 1*-type bi-null Cartan slant helix* $\gamma(t)$ *in* $\mathbb{R}_2^6$ *with* k_0, k_1 *and non-zero* k_2*, is given by*

$$U = -ck_1 L_1 + cN_1 - (ck_1'/k_2)\, W_2, \tag{3.7}$$

where $c \in \mathbb{R}/\{0\}$.

Corollary 3.2. *There exists no* 1*-type bi-null Cartan slant helix* $\gamma(t)$ *in* $\mathbb{R}_2^6$ *with* k_0, k_1 *and non-zero* k_2*, whose axis* U *satisfies* $\langle L_1, U\rangle = 0$.

Corollary 3.3. *Let* $\gamma(t)$ *be a bi-null Cartan curve in* $\mathbb{R}_2^6$ *parametrized by bi-null arc* t *with* k_0, k_1 *and non-zero* k_2. *Assume that* $k_1' \neq 0$. *Then* $\gamma(t)$ *is a* 1*-type bi-null Cartan slant helix if and only if* k_1 *and* k_2 *satisfy that*

$$-2k_1 + \left(\frac{k_1'}{k_2}\right)^2 = \textit{constant}. \tag{3.8}$$

Proof. Assume that $\gamma(t)$ is a 1-type bi-null Cartan slant helix in $\mathbb{R}_2^6$ parametrized by bi-null arc t with k_0, k_1 and non-zero k_2. Using Corollary 3.1 and that U is constant, we obtain

$$-2k_1 + \left(\frac{k_1'}{k_2}\right)^2 = \text{constant}.$$

Conversely, assume that the relation (3.8) holds. Then taking derivative of the equation (3.8) with respect to t, we get

$$k_2 - \left(\frac{k_1'}{k_2}\right)' = 0.$$

Thus from Theorem 3.1, we find that $\gamma(t)$ is a 1-type bi-null Cartan slant helix. □

Corollary 3.4. *Let $\gamma(t)$ be a 1-type bi-null Cartan slant helix in $\mathbb{R}^6_2$ parametrized by bi-null arc t with k_0, k_1 and non-zero k_2. Then we get*

$$k_1 = \frac{1}{2}\left(\int k_2(t)\,dt + a\right)^2 + b,$$

where $a, b \in \mathbb{R}$.

Secondly, we consider 2-type bi-null Cartan slant helices in $\mathbb{R}^6_2$.

Theorem 3.2. *Let $\gamma(t)$ be a bi-null Cartan curve in $\mathbb{R}^6_2$ parametrized by bi-null arc t with k_0, k_1 and non-zero k_2. Then $\gamma(t)$ is a 2-type bi-null Cartan slant helix if and only if k_0, k_1 and k_2 satisfy*

$$k_2(ct+a) + \left[\frac{1}{k_2}\left(ck_0'' - 2ck_1 - k_1'(ct+a)\right)\right]' = 0, \tag{3.9}$$

where $a, c \in \mathbb{R}$ and $(a,c) \neq (0,0)$.

Proof. Assume that $\gamma(t)$ is a 2-type bi-null Cartan slant helix parametrized by bi-null arc t with k_0, k_1 and non-zero k_2. Then there exists a non-zero fixed vector $U \in \mathbb{R}^6_2$ such that

$$\langle L_2, U\rangle = c, \quad c \in \mathbb{R}. \tag{3.10}$$

Taking derivative of the equation (3.10) with respect to t and using equations (2.1), we get $\langle W_1, U\rangle = 0$. Then we can write the vector U as follows

$$U = \lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 N_1 + cN_2 + \lambda_4 W_2, \tag{3.11}$$

where λ_i is a differentiable function ($i \in \{1,2,3,4\}$). Differentiating the equation (3.11) with respect to t and using equations (2.1), we get

$$\begin{aligned} 0 = &\left(\lambda_1' - ck_1 - \lambda_4 k_2\right) L_1 + \left(\lambda_1 + \lambda_2' + \lambda_3 k_1\right) L_2 + \left(\lambda_3' - c\right) N_1 \\ &+ \left(\lambda_2 + ck_0\right) W_1 + \left(\lambda_3 k_2 + \lambda_4'\right) W_2, \end{aligned}$$

which implies that

$$\left\{ \begin{aligned} \lambda_1' - ck_1 - \lambda_4 k_2 &= 0, \\ \lambda_1 + \lambda_2' + \lambda_3 k_1 &= 0, \\ \lambda_3' - c &= 0, \\ \lambda_2 + ck_0 &= 0, \\ \lambda_3 k_2 + \lambda_4' &= 0. \end{aligned} \right. \tag{3.12}$$

Solving (3.12), we get

$$\begin{aligned} \lambda_1 &= ck_0' - k_1(ct+a), \quad \lambda_2 = -ck_0, \quad \lambda_3 = ct + a, \\ \lambda_4 &= \frac{1}{k_2}\left(ck_0'' - 2ck_1 - k_1'(ct+a)\right) \end{aligned}$$

and

$$k_2(ct+a) + \left[\frac{1}{k_2}\left(ck_0'' - 2ck_1 - k_1'(ct+a)\right)\right]' = 0,$$

where $a, c \in \mathbb{R}$ and $(a, c) \neq (0, 0)$.

Conversely, assume that the following relation holds

$$k_2 (ct + a) + \left[\frac{1}{k_2} \left(ck_0'' - 2ck_1 - k_1' (ct + a)\right)\right]' = 0,$$

where $a, c \in \mathbb{R}$ and $(a, c) \neq (0, 0)$. Then choosing the vector U as follows

$$\begin{aligned} U = & \left(ck_0' - k_1 (ct + a)\right) L_1 - ck_0 L_2 + (ct + a) N_1 + cN_2 \\ & + \frac{1}{k_2} \left(ck_0'' - 2ck_1 - k_1' (ct + a)\right) W_2, \end{aligned}$$

we get $U' = 0$ and $\langle L_2, U\rangle = c$ (constant). Thus, $\gamma(t)$ is a 2-type bi-null Cartan slant helix. □

Example 3.2. The following curvature functions satisfy (3.9):

(*i*) $c = 0$, $a = 1$, $k_0 = t^2$, $k_1 = t^2/2$, $k_2 = 1$;

(*ii*) $c = 1$, $a = 0$, $k_0 = t$, $k_1 = t^2/8$, $k_2 = 1$.

Corollary 3.5. *The axis of a 2-type bi-null Cartan slant helix $\gamma(t)$ in $\mathbb{R}^6_2$ with k_0, k_1 and $k_2 \neq 0$, is given by*

$$\begin{aligned} U = & \left(ck_0' - k_1 (ct + a)\right) L_1 - ck_0 L_2 + (ct + a) N_1 + cN_2 \\ & + \frac{1}{k_2} \left(ck_0'' - 2ck_1 - k_1' (ct + a)\right) W_2, \end{aligned}$$

where $a, c \in \mathbb{R}$ and $(a, c) \neq (0, 0)$.

Corollary 3.6. *Let $\gamma(t)$ be a bi-null Cartan curve in $\mathbb{R}^6_2$ parametrized by bi-null arc t with k_0, k_1 and $k_2 \neq 0$. If $\gamma(t)$ is a 1-type bi-null Cartan slant helix then $\gamma(t)$ is a 2-type bi-null Cartan slant helix whose principal normal L_2 is orthogonal to the axis U of the helix.*

Thirdly, we consider 3-type bi-null Cartan slant helices in $\mathbb{R}^6_2$.

Theorem 3.3. *Let $\gamma(t)$ be a bi-null Cartan curve in $\mathbb{R}^6_2$ parametrized by bi-null arc t with k_0, k_1 and non-zero k_2. Then $\gamma(t)$ is a 3-type bi-null Cartan slant helix if and only if k_0, k_1 and k_2 satisfy*

$$\lambda_2^{(4)} + 2k_0\lambda_2'' + k_0'\lambda_2' - \lambda_2 k_1 = c \tag{3.13}$$

and

$$k_2\lambda_2 - \left(\frac{k_1\lambda_2'}{k_2}\right)' = 0, \tag{3.14}$$

where $c \in \mathbb{R}$ and λ_2 is a differentiable function which is not identically zero.

Proof. Assume that $\gamma(t)$ is a 3-type bi-null Cartan slant helix parametrized by bi-null arc t with k_0, k_1 and non-zero k_2. Then there exists a non-zero fixed vector $U \in \mathbb{R}^6_2$ such that

$$\langle N_1, U\rangle = c, \quad c \in \mathbb{R}. \tag{3.15}$$

Then we can write U as follows

$$U = cL_1 + \lambda_1 L_2 + \lambda_2 N_1 + \lambda_3 N_2 + \lambda_4 W_1 + \lambda_5 W_2, \tag{3.16}$$

where λ_i is a differentiable function. Taking derivative of the equation (3.16) with respect to t and using equations (2.1), we get

$$\begin{aligned} 0 = &\left(-k_1\lambda_3 - \lambda_5 k_2\right) L_1 + \left(c + \lambda_1' + \lambda_2 k_1 - \lambda_4 k_0\right) L_2 + \left(\lambda_2' - \lambda_3\right) N_1 \\ &+ \left(\lambda_3' - \lambda_4\right) N_2 + \left(\lambda_1 + k_0\lambda_3 + \lambda_4'\right) W_1 + \left(k_2\lambda_2 + \lambda_5'\right) W_2, \end{aligned}$$

which implies that

$$\left\{ \begin{aligned} -k_1\lambda_3 - \lambda_5 k_2 &= 0, \\ c + \lambda_1' + \lambda_2 k_1 - \lambda_4 k_0 &= 0, \\ \lambda_2' - \lambda_3 &= 0, \\ \lambda_3' - \lambda_4 &= 0, \\ \lambda_1 + k_0\lambda_3 + \lambda_4' &= 0, \\ k_2\lambda_2 + \lambda_5' &= 0. \end{aligned} \right. \tag{3.17}$$

By (3.17) , λ_2 cannot be identically zero. Solving (3.17), we get

$$\lambda_2^{(4)} + 2k_0\lambda_2'' + k_0'\lambda_2' - \lambda_2 k_1 = c$$

and

$$k_2\lambda_2 - \left(\frac{k_1\lambda_2'}{k_2}\right)' = 0,$$

where $c \in \mathbb{R}$.

Conversely, assume that the following relation holds

$$\lambda_2^{(4)} + 2k_0\lambda_2'' + k_0'\lambda_2' - \lambda_2 k_1 = c$$

and

$$k_2\lambda_2 - \left(\frac{k_1\lambda_2'}{k_2}\right)' = 0,$$

where $c \in \mathbb{R}$ and λ_2 is a differentiable function which is not identically zero. Then choosing the vector U as follows

$$U = cL_1 - \left(k_0\lambda_2' + \lambda_2^{(3)}\right) L_2 + \lambda_2 N_1 + \lambda_2' N_2 + \lambda_2'' W_1 - \frac{k_1}{k_2}\lambda_2' W_2,$$

we get $U' = 0$ and $\langle N_1, U\rangle = c$ (constant). Thus, $\gamma(t)$ is a 3-type bi-null Cartan slant helix. □

Example 3.3. The following curvature functions satisfy the equations (3.13) and (3.14):

(i) $\lambda_2 = t$, $k_0 = -t + (t^4/8)$, $k_1 = t^2/2$, $k_2 = 1$, $c = -1$;

(ii) $\lambda_2 = t^2$, $k_0 = t^6/128$, $k_1 = t^4/8$, $k_2 = t$, $c = 0$.

Corollary 3.7. *The axis of a* 3*-type bi-null Cartan slant helix* $\gamma(t)$ *in* $\mathbb{R}^6_2$ *with* k_0, k_1 *and* $k_2 \neq 0$, *is given by*

$$U = cL_1 - \left(k_0\lambda_2' + \lambda_2^{(3)}\right) L_2 + \lambda_2 N_1 + \lambda_2' N_2 + \lambda_2'' W_1 - \frac{k_1}{k_2}\lambda_2' W_2,$$

where $c \in \mathbb{R}$ and λ_2 is a differentiable function that is not identically zero.

Let us consider 4-type bi-null Cartan slant helices in $\mathbb{R}_2^6$. In the following, we omit the proofs.

Theorem 3.4. *Let $\gamma(t)$ be a bi-null Cartan curve in $\mathbb{R}_2^6$ parametrized by bi-null arc t with k_0, k_1 and non-zero k_2. Then $\gamma(t)$ is a 4-type bi-null Cartan slant helix if and only if k_0, k_1 and k_2 satisfy*

$$k_2\lambda_2 + \left[\frac{1}{k_2}\left(k_0\lambda_2^{(3)} + k_0'\lambda_2'' - k_1'\lambda_2 - 2k_1\lambda_2'\right)\right]' = 0 \tag{3.18}$$

and

$$c + k_0\lambda_2' + \lambda_2^{(3)} = 0, \tag{3.19}$$

where $c \in \mathbb{R}$ and λ_2 is a differentiable function which is not identically zero.

Example 3.4. The following curvature functions satisfy the equations (3.18) and (3.19):
(i) $\lambda_2 = t$, $k_0 = -c$, $k_1 = t^2/8$, $k_2 = 1$;
(ii) $\lambda_2 = 1$, $k_0 = t$, $k_1 = t^2/2$, $k_2 = 1$, $c = 0$.

Corollary 3.8. *The axis of a 4-type bi-null Cartan slant helix $\gamma(t)$ in $\mathbb{R}_2^6$ with k_0, k_1 and $k_2 \neq 0$, is given by*

$$\begin{aligned} U = &\left(k_0\lambda_2'' - k_1\lambda_2\right)L_1 + cL_2 + \lambda_2 N_1 + \lambda_2' N_2 + \lambda_2'' W_1 \\ &+ \frac{1}{k_2}\left(k_0\lambda_2^{(3)} + k_0'\lambda_2'' - k_1'\lambda_2 - 2k_1\lambda_2'\right)W_2, \end{aligned}$$

where $c \in \mathbb{R}$ and λ_2 is a differentiable function that is not identically zero.

Let us consider 5-type bi-null Cartan slant helices in $\mathbb{R}_2^6$. In the following, we omit the proofs.

Theorem 3.5. *Let $\gamma(t)$ be a bi-null Cartan curve in $\mathbb{R}_2^6$ parametrized by bi-null arc t with k_0, k_1 and non-zero k_2. Then $\gamma(t)$ is a 5-type bi-null Cartan slant helix if and only if k_0, k_1 and k_2 satisfy*

$$\lambda' + k_2\left(c\frac{t^2}{2} + a_1 t + a_2\right) = 0,$$

where

$$\lambda = \frac{1}{k_2}\left(3ck_0' + \left(k_0'' - 2k_1\right)\left(ct + a_1\right) - k_1'\left(\frac{ct^2}{2} + a_1 t + a_2\right)\right),$$

$c, a_1, a_2 \in \mathbb{R}$ and $(c, a_1, a_2) \neq (0,0,0)$.

Example 3.5. The following curvature functions satisfy the above equations:
(i) $c = a_1 = 0$, $a_2 = 1$, $k_0 = t^2$, $k_1 = -t^2/2$, $k_2 = 1$;
(ii) $c = a_2 = 0$, $a_1 = 1$, $k_0 = 1$, $k_1 = (t^2/8) + (1/2t^2)$, $k_2 = 1$.

Corollary 3.9. *The axis of a* 5*-type bi-null Cartan slant helix* $\gamma(t)$ *in* $\mathbb{R}^6_2$*, with* k_0*,* k_1 *and* $k_2 \neq 0$*, is given by*

$$\begin{aligned}U = &\left(2ck_0 + k_0'(ct+a_1) - k_1\left(\frac{ct^2}{2} + a_1t + a_2\right)\right)L_1 - k_0(ct+a_1)L_2\\&+\left(\frac{ct^2}{2} + a_1t + a_2\right)N_1 + (ct+a_1)N_2 + cW_1\\&+\frac{1}{k_2}\left(3ck_0' + (k_0'' - 2k_1)(ct+a_1) - k_1'\left(\frac{ct^2}{2} + a_1t + a_2\right)\right)W_2,\end{aligned}$$

where $c, a_1, a_2 \in \mathbb{R}$ *and* $(c, a_1, a_2) \neq (0,0,0)$*.*

Corollary 3.10. *Let* $\gamma(t)$ *be a bi-null Cartan curve in* $\mathbb{R}^6_2$ *parametrized by bi-null arc* t *with* k_0*,* k_1 *and* $k_2 \neq 0$*. If* $\gamma(t)$ *is a* 2*-type bi-null Cartan slant helix then* $\gamma(t)$ *is a* 5*-type bi-null Cartan slant helix such that* W_1 *is orthogonal to the axis* U *of the helix.*

Lastly, we consider 6-type bi-null Cartan slant helices in $\mathbb{R}^6_2$.

Theorem 3.6. *Let* $\gamma(t)$ *be a bi-null Cartan curve in* $\mathbb{R}^6_2$ *parametrized by bi-null arc* t *with* k_0*,* k_1*,* k_2*. If* $\gamma(t)$ *is a* 6*-type bi-null Cartan slant helix, then it lies in some hyperplane of index* 2*.*

Proof. Assume that $\gamma(t)$ is a 6-type bi-null Cartan slant helix parametrized by bi-null arc t with k_0, k_1, k_2. Then there exists a non-zero fixed vector $U \in \mathbb{R}^6_2$ such that

$$\langle W_2, U\rangle = c, \quad c \in \mathbb{R}. \tag{3.20}$$

Taking derivative of the equation (3.20) with respect to t and using equations (2.1), we get

$$k_2\langle L_1, U\rangle = 0. \tag{3.21}$$

Assume that $k_2 \neq 0$. Then we have

$$\langle L_1, U\rangle = \langle L_2, U\rangle = \langle W_1, U\rangle = \langle N_2, U\rangle = \langle N_1, U\rangle = \langle W_2, U\rangle = 0,$$

and U is zero, which is a contradiction. Thus, $k_2 = 0$ which means that $\gamma(t)$ lies in some hyperplane of index 2. □

Remark 3.1. Any bi-null Cartan curve in $\mathbb{R}^6_2$ with $k_2 = 0$ is 6-type bi-null Cartan helix, since W_2 is a constant vector, so it trivially makes constant angle with any fixed direction.

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