Kragujevac Journal of Mathematics
Volume 51(6) (2027), Pages 1043–1062.

INVERSE COEFFICIENT PROBLEM FOR A FULL FRACTIONAL DIFFUSION-WAVE EQUATION WITH THE CAPUTO TIME-SPACE DERIVATIVE

DURDIMUROD DURDIEV[1,2] AND HALIM TURDIEV[1,2]

Abstract. The inverse coefficient problem is studied for a space-time fractional diffusion equation involving the Caputo operator in both spatial and temporal variables. The corresponding direct problem is formulated as an initial-boundary value problem with Cauchy-type initial conditions and Dirichlet boundary conditions. First, the initial-boundary value problem is considered. A solution is constructed using a bi-orthogonal system of functions consisting of Mittag-Leffler type functions derived from the spectral problem of fractional order and its adjoint. By employing the properties of the Mittag-Leffler function and the generalized singular Gronwall inequality, an a priori estimate for the solution is obtained in terms of an unknown coefficient. This estimate is then used to investigate the associated inverse problem. As an overdetermination condition, a nonlocal integral condition is imposed on the solution of the direct problem. The inverse problem is subsequently reduced to an equivalent Volterra-type integral equation. The Banach fixed-point theorem (contraction mapping principle) is applied to solve this equation. Local existence and uniqueness of the solution are established.

1. Introduction

Fractional calculus is an emerging field of mathematics that explores and applies derivatives and integrals of non-integer orders. This area of study has found diverse applications across a wide range of disciplines, including electromagnetics, quantum mechanics, plasma physics, fluid mechanics, chemical physics, mathematical biology,

Key words and phrases. Caputo fractional derivative, bi-orthogonal system, Mittag–Leffler function, inverse problem, integral equation, Fourier series, Banach fixed point theorem.
2020 *Mathematics Subject Classification.* 34A08, 34K10 , 34K29 , 34K37 , 35R11 , 35R30.
DOI
Received: July 07, 2025.
Accepted: March 09, 2026.

biomedicine, financial systems, chaos theory, elasticity, control theory, optics, and signal processing [1–8].

In the works [9–11], the initial boundary value problem for the fractional differential equation involving both time and space variables is considered. The study first focuses on the boundary value problem of determining the eigenvalue and eigenvector for the fractional differential equation with respect to the spatial variable. Then, the Cauchy problem for the time-fractional differential equation is addressed. The results include existence, uniqueness, and stability theorems for the inverse problem of recovering a spatially dependent source term, while existence and uniqueness results are also established for a temporally dependent source term.

Inverse problems for fractional differential wave and diffusion equations have not been extensively explored. The existing literature primarily addresses time-fractional derivative problems [12–14], source determination in linear problems [15–21], and coefficient-related nonlinear inverse problems [22–28] within the context of initial-boundary value problems for fractional diffusion-wave equations with various overdetermination conditions (see also the references in the aforementioned works). In [29] and [30], the authors investigated inverse problems aimed at determining space- and time-dependent source terms in time-fractional diffusion equations. They employed eigenfunction expansion techniques of non-self-adjoint spectral problems, utilizing the generalized Fourier method. Their results included theorems on existence and uniqueness, as well as stability estimates for the solutions to time-fractional diffusion and wave equations.

Inverse source problems for space-time fractional diffusion equations are considered in [31–35]. In [32], for a space-time fractional diffusion equation inverse problem of determining a temporal component in the source term from the total energy of the system is considered and the recovery of a space dependent source term from final data is discussed in [33]. Inverse problems of identifying the space and time dependent source terms for a space-time fractional diffusion equation are considered in [34]. Jia et al. [35] proved a uniqueness result for the determination of time dependent source term for a space-time fractional diffusion equation.

Inverse problems for classical integro-differential wave propagation equations have been extensively studied in the literature. Nonlinear inverse coefficient problems, often accompanied by various types of overdetermination conditions, are commonly addressed (e.g., [36–39], and references therein). In [40], the inverse problem of determining an unknown coefficient in the Cauchy problem for the fractional diffusion-wave equation was investigated. The study establishes the local existence and uniqueness of solutions, and provides estimates for conditional stability.

Consider the time-space fractional diffusion-wave equation in the domain $\Omega_T := \{(x,t) : 0 < x < 1,\, 0 < t \le T\}$:

$$(1.1)\qquad \left(\partial_t^\alpha u\right)(x,t) - \left(\partial_x^\beta u\right)(x,t) + q(t)u(x,t) = f(x,t), \quad (x,t) \in \Omega_T,$$

where ∂_t^α and ∂_x^β denote the Caputo fractional derivatives of order $1 < \alpha, \beta < 2$ with respect to time t and space x (refer to Definitions 1 and 2 in the Preliminaries). The equation is subject to the following initial and boundary conditions:

$$u(x,t)\,|_{t=0} = \varphi(x), \quad u_t(x,t)\,|_{t=0} = \psi(x), \quad x \in [0,1], \tag{1.2}$$

$$u(0,t) = u(1,t) = 0, \quad 0 \le t \le T, \tag{1.3}$$

where $\varphi(x)$, $\psi(x)$, and $f(x,t)$ are given smooth functions.

We pose the inverse problem as follows: determine the function $q(t)$, $t > 0$, in (1.1) given that the solution of the initial-boundary problem (1.1)-(1.3) satisfies the condition

$$\int_0^1 u(x,t)\,dx = g(t), \quad 0 \le t \le T, \tag{1.4}$$

where $g(t)$ is a given function.

Assume that throughout this article the functions $\varphi(x)$, $\psi(x)$, $f(x,t)$, and $g(t)$ satisfy the following assumptions:

*A*1) $\{\varphi, \psi\} \in C^2[0,1]$, $\{\varphi^{(3)}, \psi^{(3)}\} \in L_2(0,1)$, $\varphi(0) = \varphi(1) = 0$, $\psi(0) = \psi(1) = 0$, $\varphi''(0) = \varphi''(1) = 0$, $\psi''(0) = \psi''(1) = 0$;

*A*2) $f(x,\cdot) \in C[0,T]$ and for $t \in [0,T]$, $f(\cdot,t) \in C^2[0,l]$, $f^{(3)}(\cdot,t) \in L_2(0,1)$, $f(0,t) = f(1,t) = 0$, $f_{xx}(0,t) = f_{xx}(1,t) = 0$;

*A*3) $\partial_t^\alpha g(t) \in C[0,T]$ and $|g(t)| \ge g_0 > 0$, g_0 is a given number, $\int_0^1 \varphi(x)dx = g(0)$, $\int_0^1 \psi(x)dx = g'(0)$.

The article is structured as follows. Section 2 presents the fundamental definitions and preliminary results required for the subsequent analysis. In Section 3, we establish the existence and uniqueness of the solution to the direct problem (1.1)–(1.3), along with a stability estimate for this solution. Section 4 is dedicated to solving the inverse problem (1.1)–(1.4). Finally, concluding remarks are provided at the end of the paper.

2. Preliminaries

In this section, we present some useful definitions and results that will be used in subsequent discussions.

Definition 2.1. (see [1, pp. 69–76]) The Riemann-Liouville fractional integral of order $0 < \alpha < 1$ for an integrable function $h(t) \in AC[0,T]$ is defined by

$$I_{0+,t}^\alpha h(t) = \frac{d^{-\alpha}}{dx^{-\alpha}} h(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau) d\tau, \quad t > 0.$$

Definition 2.2. (see [1, pp. 69–76]) The Caputo fractional derivative of order $n-1 < \alpha < n$ of the functions $h(t) \in AC^n[0,T]$ is defined by

$$\left(\partial_t^\alpha h\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{h^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad t > 0.$$

Let $\{\gamma_k\}_0^n$ be a set of real numbers satisfying the condition $0 < \gamma_j \le 1$, $0 \le j \le n$. We denote

$$\sigma_k = \sum_{j=0}^{k} \gamma_j - 1, \quad \mu_k = \sigma_k + 1 = \sum_{j=0}^{k} \gamma_j, \quad 0 \le k \le n,$$

and assume that

$$\frac{1}{\rho} = \sum_{j=0}^{n} \gamma_j - 1 = \sigma_n = \mu_n - 1 > 0.$$

Following M. M. Dzhrbashyan [41], we consider the integro-differential operators

$$\begin{aligned} D^{(\sigma_0)} y(x) &\equiv \frac{d^{-(1-\gamma_0)}}{dx^{-(1-\gamma_0)}} y(x), \\ D^{(\sigma_1)} y(x) &\equiv \frac{d^{-(1-\gamma_1)}}{dx^{-(1-\gamma_1)}} \cdot \frac{d^{\gamma_0}}{dx^{\gamma_0}} y(x), \\ D^{(\sigma_2)} y(x) &\equiv \frac{d^{-(1-\gamma_2)}}{dx^{-(1-\gamma_2)}} \cdot \frac{d^{\gamma_1}}{dx^{\gamma_1}} \cdot \frac{d^{\gamma_0}}{dx^{\gamma_0}} y(x). \end{aligned}$$

Along with the operators $D_l^{\widetilde{\sigma}_k} y(x)$, $k = 0, 1, 2$, we introduce similar fractional integro-differentiation operators with finite endpoints at the point $x = l$ [41]. Using the notations

$$\widetilde{\sigma}_k = \sum_{j=0}^{k} \gamma_{2-j} - 1, \quad \widetilde{\mu}_k = \widetilde{\sigma}_k + 1 = \sum_{j=0}^{k} \gamma_{2-j},$$

and

$$\begin{aligned} D_l^{\widetilde{\sigma}_0} y(x) &\equiv \frac{d^{-(1-\gamma_2)}}{d(l-x)^{-(1-\gamma_2)}} f(x), \\ D_l^{\widetilde{\sigma}_1} y(x) &= - \frac{d^{-(1-\gamma_1)}}{d(l-x)^{-(1-\gamma_1)}} \cdot \frac{d^{\gamma_2}}{d(l-x)^{\gamma_2}} y(x), \\ D_l^{\widetilde{\sigma}_2} y(x) &= - \frac{d^{-(1-\gamma_0)}}{d(l-x)^{-(1-\gamma_0)}} \cdot \frac{d^{\gamma_1}}{d(l-x)^{\gamma_1}} \cdot \frac{d^{\gamma_2}}{d(l-x)^{\gamma_2}} y(x). \end{aligned}$$

Two parameter Mittag-Leffler (M-L) function. The two parameter M-L function $E_{\alpha,\beta}(z)$ is defined by the following series:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha, \beta, z \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\alpha)$ denotes the real part of the complex number α, $\Gamma(\cdot)$ is Euler's Gamma function. The Mittag-Leffler function has been extensively studied by numerous researchers, leading to various generalizations and applications. A particularly noteworthy contribution that consolidates many significant results on this function is the work by Kilbas et al. (see [1, pp. 42–44]).

The case $\beta = 1$ reduces to the Mittag-Leffler function of single parameter, i.e.,

$$E_{\alpha,1}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

For the Mittag-Leffler functions the following formula is valid [42, p. 64]:

$$\int_0^l x^{\beta-1} E_{\alpha,\beta}\left(\lambda x^\alpha\right)(l-x)^{\nu-1} E_{\alpha,\nu}\left(\lambda^*(l-x)^\alpha\right) dx$$

$$= \frac{E_{\alpha,\beta}(l^\alpha \lambda) - E_{\alpha,\nu}(l^\alpha \lambda^*)}{\lambda - \lambda^*} l^{\beta+\nu-1}, \quad \beta > 0, \nu > 0, \tag{2.1}$$

where λ and λ^*, $\lambda \neq \lambda^*$, are any complex parameters.

Proposition 2.1. *Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that κ is such that $\pi\alpha/2 < \kappa < \min\{\pi, \pi\alpha\}$. Then, there exists a constant $C = C(\alpha, \beta, \kappa) > 0$ such that*

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1 + |z|}, \quad \kappa \leq |\arg(z)| \leq \pi.$$

For the proof, we refer to ([1, pp. 40–45], [6, p. 35]).

Lemma 2.1. (*see* [2, p. 278, Lemma 15.2]). *Let the Caputo fractional derivative $\partial_x^\beta \mu_n(x)$ exists for all $n \in \mathbb{N}$ and for every $\epsilon > 0$ the series $\sum_{n=1}^{+\infty} \mu_n(x)$, $\sum_{n=1}^{+\infty} \partial_x^\beta \mu_n(x)$ are uniformly convergent on the subinterval $[\epsilon, b]$. Then,*

$$\partial_x^\beta \left(\sum_{n=1}^{+\infty} \mu_n(x) \right) = \sum_{n=1}^{+\infty} \partial_x^\beta \mu_n(x), \quad \beta > 0, 0 < x < 1.$$

Definition 2.3. We define the weighted spaces of continuous functions ([1, pp. 4–5, 162—163]).

$$C^{\alpha,\beta}(\Omega_T) = \Big\{ u(x,t) : \partial_x^\beta u(\cdot,t) \in C(0,1);\ t \in [0,T] \text{ and } \partial_t^\alpha u(x,\cdot) \in C(0,T];\ x \in [0,1],\ 1 < \alpha, \beta < 2 \Big\}.$$

Lemma 2.2. (*see* [1, p. 76, Lemma 2.7]). *Let $\alpha > 0$, $p \geq 1$, $q \geq 1$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$.)*

(*a*) *If $\varphi(x) \in L_p(a,b)$ and $\psi(x) \in L_q(a,b)$, then*

$$\int_a^b \varphi(x) \left(I_{a+}^\alpha \psi \right)(x) dx = \int_a^b \psi(x) \left(I_{b-}^\alpha \varphi \right)(x) dx.$$

(*b*) *If $f(x) \in I_{b-}^\alpha \left(L_p(a,b)\right)$ and $g(x) \in I_{a+}^\alpha \left(L_q(a,b)\right)$, then*

$$\int_a^b f(x) \left(D_{a+}^\alpha g \right)(x) dx = \int_a^b g(x) \left(I_{b-}^\alpha f \right)(x) dx.$$

Lemma 2.3. ([43, p. 189]) *Suppose $b \geq 0$, $\alpha > 0$, $\gamma > 0$, $\alpha + \gamma > 1$ and $a(t)$ nonnegative function locally integrable on $0 \leq t < T$ and suppose $t^{\gamma-1}u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + b\int_0^t (t-s)^{\alpha-1}s^{\gamma-1}u(s)ds, \quad a.e.\ in\ (0,T).$$

Then,

$$u(t) \leq a(t)Z_{\alpha,\gamma}\left((b\Gamma(\alpha))^{\frac{1}{\alpha+\gamma-1}}\, t\right),$$

where

$$Z_{\alpha,\gamma}(t) = \sum_{m=0}^{+\infty} c_m t^{m(\alpha+\gamma-1)}, \quad c_0 = 1, \quad \frac{c_{m+1}}{c_m} = \frac{\Gamma(m(\alpha+\gamma-1)+\gamma)}{\Gamma(m(\alpha+\gamma-1)+\alpha+\gamma)},$$

for $m \geq 0$. As $t \to +\infty$,

$$Z_{\alpha,\gamma}(t) = O\left(t^{\frac{1}{2}\frac{\alpha+\gamma-1}{\alpha-\gamma}} \exp\left(\frac{\alpha+\gamma-1}{\alpha} t^{\frac{\alpha+\gamma-1}{\alpha}}\right)\right).$$

2.1. **Bi-orthogonal system.** The spectral problem corresponding to equations (1.1)-(1.3) is

$$\partial_x^\beta X(x) = \lambda X(x), \quad X(0) = X(1) = 0. \tag{2.2}$$

The eigenfunctions of the spectral problem (2.2) are

$$\{X_n(x)\}_{n=1}^{+\infty} = \{x^{\beta-1}E_{\beta,\beta}(\lambda_n x^\beta)\}_{n=1}^{+\infty} \tag{2.3}$$

corresponding to the eigenvalues λ_n which are the zeros of the function $E_{\beta,\beta}(\lambda)$ with $\operatorname{Im}(\lambda_n) > 0$.

The set of eigenfunctions $\{X_n(x)\}_{n=1}^{+\infty}$ is not orthogonal. Furthermore, we denote the inner product in $L_2(0,1)$ by $\langle \cdot, \cdot \rangle$. In order to derive the adjoint problem corresponding to the spectral problem (2.2), we proceed as follows:

$$\left\langle \partial_x^\beta X(x), \widetilde{X}(x) \right\rangle = \left\langle \frac{d}{dx} I_{0+,x}^{\beta} \frac{d}{dx} X(x), \widetilde{X}(x) \right\rangle.$$

Integration by parts and taking $\widetilde{X}(0) = \widetilde{X}(1) = 0$, we have

$$\left\langle \frac{d}{dx} I_{0+,x}^{2-\beta} \frac{d}{dx} X(x), \widetilde{X}(x) \right\rangle = \left\langle I_{0+,x}^{2-\beta} \frac{d}{dx} X(x), \frac{d}{dx}\widetilde{X}(x) \right\rangle.$$

By applying Lemma 2.2 and subsequently using integration by parts, we obtain:

$$\left\langle \partial_x^\beta X(x), \widetilde{X}(x) \right\rangle = \left\langle X(x), \frac{d}{dx} I_{1-,x}^{2-\beta} \frac{d}{dx}\widetilde{X}(x) \right\rangle.$$

Thus, the adjoint problem corresponding to the spectral problem (2.2) is given by

$$\partial_{1-,x}^\beta \widetilde{X}(x) = \lambda^* \widetilde{X}(x), \quad \widetilde{X}(0) = \widetilde{X}(1) = 0, \tag{2.4}$$

where λ^* is the complex conjugate of λ. The spectral adjoint problem (2.4) has eigenfunctions $\widetilde{X}_n(x)$ and shares the same eigenvalues as the spectral problem (2.2):

$$\{\widetilde{X}_n(x)\}_{n=1}^{+\infty} = \{(1-x)^{\beta-1} E_{\beta,\beta}(\lambda_n^*(1-x)^{\beta})\}_{n=1}^{+\infty}. \tag{2.5}$$

To advance our study, let us consider the following boundary value problem for a fractional differential equation, as investigated by Djrbashian in [41]:

$$\begin{cases} D^{\sigma_2} y(x) - (\lambda + q(x))y(x) = 0, \\ D^{(\sigma_0)} y(x) \mid_{x=0} \cos\theta_1 + D^{(\sigma_1)} y(x) \mid_{x=0} \sin\theta_1 = 0, \\ D^{(\sigma_0)} y(x) \mid_{x=1} \cos\theta_2 + D^{(\sigma_1)} y(x) \mid_{x=1} \sin\theta_2 = 0, \end{cases} \tag{2.6}$$

where $\lambda, \theta_1, \theta_2$, $\mathrm{Im}\theta_1 = \mathrm{Im}\theta_2 = 0$, are arbitrary parameters, the function $q(x)$ is Lipschitz continuous.

In addition to the problem (2.6), the following conjugate boundary value problem is also considered in [41]:

$$\begin{cases} D_{1-}^{\widetilde{\sigma_2}} y(x) - (\lambda^* + q(x))y(x) = 0, \\ D_{1-}^{\widetilde{\sigma_0}} y \mid_{x=0} \cos\theta_1 + D_{1-}^{\widetilde{\sigma_1}} y \mid_{x=0} \sin\theta_1 = 0, \\ D_{1-}^{\widetilde{\sigma_0}} y \mid_{x=1} \cos\theta_2 + D_{1-}^{\widetilde{\sigma_1}} y \mid_{x=1} \sin\theta_2 = 0. \end{cases} \tag{2.7}$$

In [41], Djrbashian constructed a biorthogonal system consisting of eigenfunctions and conjugate functions for the problems (2.6) and (2.7). He also proved that the eigenfunctions associated with these problems form a biorthogonal system.

For the systems of eigenfunctions $\{X_n(x)\}_{n=1}^{+\infty}$ and $\{\widetilde{X}_n(x)\}_{n=1}^{+\infty}$, the following statement holds.

Lemma 2.4. *The systems of functions* $\{X_n(x)\}_{n=1}^{+\infty}$ *and* $\{\widetilde{X}_n(x)\}_{n=1}^{+\infty}$ *defined by* (2.3) *and* (2.5) *are biorthonormal systems.*

Proof. To demonstrate that the system of eigenfunctions $\{X_n(x)\}_{n=1}^{+\infty}$ and $\{\widetilde{X}_n(x)\}_{n=1}^{+\infty}$ is biorthogonal, we utilize the fact that the eigenfunctions corresponding to problems (2.2) and (2.4) form a biorthogonal system. If we choose the parameters as $\gamma_0 = 1$, $\gamma_1 = \beta - 1$, $\gamma_2 = 1$, then $\sigma_3 = \gamma_0 + \gamma_1 + \gamma_2 = \beta$, $\beta \in (1, 2)$ and the operator $D^{(\sigma_2)} y(x)$:

$$\begin{aligned} D^{(\sigma_2)} y(x) =& \frac{d^{\beta-1}}{dx^{\beta-1}} \cdot \frac{d^{\gamma_0}}{dx^{\gamma_0}} y(x) = \frac{d^{\beta-2}}{dx^{\beta-2}} \cdot \frac{d^{\gamma_0+1}}{dx^{\gamma_0+1}} y(x) \\ =& \frac{d^{-(2-\beta)}}{dx^{-(2-\beta)}} \cdot \frac{d^{\gamma_0+1}}{dx^{\gamma_0+1}} y(x) = \frac{1}{\Gamma(2-\beta)} \int_0^x \frac{y''(t)}{(x-t)^{\beta-1}} dt = \partial_x^{\beta} y(x). \end{aligned}$$

By selecting the parameters for problems (2.6) and (2.7) as $\gamma_0 = 1$, $\gamma_1 = \beta - 1$, $\gamma_2 = 1$, and $\theta_1 = \theta_2 = 0$, with $q(x) \equiv 0$, we obtain problem (2.2) and problem (2.4) from (2.6) and (2.7), respectively. Using the biorthogonality of the problems (2.6) and (2.7), we show the biorthogonality of the problems (2.2) and (2.4). According to relation (2.1) and the eigenfunctions (2.3), (2.5), we get:

$$\int_0^1 x^{\beta-1}E_{\beta,\beta}\left(\lambda_n x^\beta\right)(1-x)^{\nu-1}E_{\beta,\beta}\left(\lambda_n^*(1-x)^\beta\right)dx = \frac{E_{\beta,\beta}(\lambda_n)-E_{\beta,\beta}(\lambda_n^*)}{\lambda_n-\lambda_n^*}.$$

Since λ and λ^* are the eigenvalues of problems (2.1), (2.4) respectively, $\lambda_n \neq \lambda_n^*$:

$$\int_0^1 X_n(x)\widetilde{X}_n(x)dx = \int_0^1 x^{\beta-1}E_{\beta,\beta}\left(\lambda_n x^\beta\right)(1-x)^{\nu-1}E_{\beta,\beta}\left(\lambda_n^*(1-x)^\beta\right)dx = 0.$$

Thus, the system of eigenfunctions of problems (2.1), (2.4) and the system of eigenfunctions of the problem conjugate to it form a biorthogonal system.

Now, consider the scalar product of the functions $\{X_n(x)\}_{n=1}^{+\infty}$ and $\{\widetilde{X}_n(x)\}_{n=1}^{+\infty}$ in $L_2(0,1)$:

$$\begin{aligned}\int_0^1 X_n(x)\widetilde{X}_n(x)dx &= \int_0^1 x^{\beta-1}E_{\beta,\beta}\left(\lambda_n x^\beta\right)(1-x)^{\beta-1}E_{\beta,\beta}\left(\lambda_n^*(1-x)^\beta\right)dx\\ &= \sum_{j=0}^{+\infty}\sum_{m=0}^{+\infty}\frac{\lambda_n^j\left(\lambda_n^*\right)^m}{\Gamma(j\beta+\beta)\Gamma(m\beta+\beta)}\int_0^1 x^{j\beta+\beta-1}(1-x)^{m\beta+\beta-1}dx\\ &= \sum_{j=0}^{+\infty}\sum_{m=0}^{+\infty}\frac{\lambda_n^j\left(\lambda_n^*\right)^m}{\Gamma\left((m+j)\beta)+2\beta\right)} = \sum_{j=0}^{+\infty}\sum_{k=j}^{+\infty}\frac{\lambda_n^j\left(\lambda_n^*\right)^{k-j}}{\Gamma(k\beta+2\beta)}\\ &= \sum_{k=0}^{+\infty}\frac{\left(\lambda_n^*\right)^k}{\Gamma(k\beta+2\beta)}\sum_{j=0}^{k}\left(\frac{\lambda_n}{\lambda_n^*}\right)^j = \frac{1}{\lambda_n-\lambda_n^*}\sum_{k=0}^{+\infty}\frac{\left(\lambda_n^{k+1}-\left(\lambda_n^*\right)^{k+1}\right)}{\Gamma(k\beta+2\beta)}.\end{aligned}$$

For $\lambda_n = \lambda_n^*$

$$\int_0^1 X_n(x)\widetilde{X}_n(x)dx = \int_0^1 x^{\beta-1}E_{\beta,\beta}\left(\lambda_n x^\beta\right)(1-x)^{\beta-1}E_{\beta,\beta}\left(\lambda_n^*(1-x)^\beta\right)dx = \frac{1}{\Gamma(2\beta)}.$$

If we take the system as

$$Y_n(x) = \Gamma(2\beta)\widetilde{X}_n(x), \tag{2.8}$$

then we obtain

$$\int_0^1 X_n(x)Y_m(x)dx = \begin{cases}0, & m\neq n,\\ 1, & m=n.\end{cases}$$

□

Let us now verify that the systems given by equations (2.3) and (2.8) constitute a complete system in $L_2(0,1)$.

Lemma 2.5. *The systems of functions $X_n(x)$ and $Y_n(x)$ are closed in $L_2(0,1)$.*

Proof. Let us first prove that the system (2.3) is complete in $L_2(0,1)$: from this, the closedness of the system (2.3) in $L_2(0,1)$ follows immediately. Then, we obtain that the same system (2.8) is a closed system in $L_2(0,1)$. Then, there exists a nontrivial function $\mu(x) \in L_2(0,1)$ that is orthogonal to all functions of system (2.3). Since $\mu(x)$ is orthogonal to $X_n(x)$, then we will write $\mu(x)$ as a convergent series in $L_2(0,1)$

$$\mu(x) = \sum_{k=1}^{+\infty} d_k Y_k(x). \tag{2.9}$$

Moreover, $\mu(x)$ is assumed to be orthogonal to all functions of the form $X_n(x)$, therefore, by multiplying the series (2.9) by $X_n(x)$ and integrating along $(0,1)$, we obtain

$$0 = \int_0^1 \mu(x) X_n(x) dx = \int_0^1 \sum_{k=1}^{+\infty} d_k Y_k(x) X_n(x) dx = d_n.$$

From this it follows that $d_n = 0$, $n = 1,2,\ldots$. Therefore, from (2.9), we obtain $\mu(x) = 0$ on the interval $[0,1]$. Thus, the system (2.3) is complete in $L_2(0,1)$.

We prove the completeness of the system (2.8). Let there exists a nontrivial function $\nu(x)$ in $L_2(0,1)$, that is orthogonal to all functions of system (2.8). Since the function $\nu(x)$ is orthogonal to the system $Y_n(x)$, it can be represented in $L_2(0,1)$ as a series, i.e

$$\nu(x) = \sum_{k=1}^{+\infty} \tilde{d}_k X_k(x). \tag{2.10}$$

$\nu(x)$ is assumed to be orthogonal to all functions of the form $Y_n(x)$, therefore, by multiplying the series (2.10) by $Y_n(x)$ and integrating along $(0,1)$, we obtain

$$0 = \int_0^1 \nu(x) X_n(x) dx = \int_0^1 \sum_{k=1}^{+\infty} \tilde{d}_k Y_k(x) X_n(x) dx = \tilde{d}_n.$$

It follows from this that $\tilde{d}_n = 0$, $n = 1,2,\ldots$. Therefore, from (2.10) we obtain that $\nu(x) = 0$ is on the interval $[0,1]$. Thus, system (2.8) is complete in $L_2(0,1)$. □

Lemma 2.6. *The systems of functions* (2.3) *and* (2.5) *are minimal in* $L_2(0,1)$.

The proof of Lemma 2.6 immediately follows from the existence of a biorthonormal system established in Lemma 2.4.

Lemma 2.7 ([9])**.** *The eigenvalues λ_n, that are the zeros of the function $E_{\beta,\beta}(\lambda)$ with $Im(\lambda_n) > 0$, satisfy the following relations.*

1) $|\lambda_k| < |\lambda_{k+1}|$, *for* $k \geq 1$.

2) *For n large enough and* $\arg(\lambda_n) > \frac{\beta\pi}{2}$, *we have* $|e^{\lambda_n t}| < 1$ *and* $|\lambda_n| = O(n^\beta)$, $1 < \beta < 2$.

In the following section, we will address the solution of the initial-boundary value problem (1.1)–(1.3).

3. Investigation of Direct Problem

By utilizing the Fourier method, the solution $u(x;t)$ to the problem defined by equations (1.1)–(1.3) can be represented as a uniformly convergent series in terms of its eigenfunctions

$$u(x,t) = \sum_{n=1}^{+\infty} u_n(t) X_n(x), \tag{3.1}$$

where $\langle u(x,t), Y_n(x)\rangle = u_n(t)$. The coefficients $u_n(t)$ for $n \geq 1$ are determined by using the orthogonality of the eigenfunctions $X_n(x)$. Let the expansion coefficients of $\varphi(x)$, $\psi(x)$, and $f(x,t)$ in the eigenfunctions of equation (2.3) for $n \geq 1$ be defined as follows:

$$\langle f(x,t), Y_n(x)\rangle = f_n(t), \quad \langle \varphi(x), Y_n(x)\rangle = \varphi_n, \quad \langle \psi(x), Y_n(x)\rangle = \psi_n, \quad n = 1, 2, \ldots$$

From equations (1.1) and (1.2), we obtain the Cauchy problem for $u_n(t)$

$$\partial_t^\alpha u_n(t) + \lambda_n u_n(t) + q(t) u_n(t) = f_n(t), \tag{3.2}$$

$$u_n(0) = \varphi_n, \quad u_n'(0) = \psi_n. \tag{3.3}$$

We suppose that $f_n(t) \in C[0,T]$. Then, by property 3.1(a) (see [1, p. 232]), Cauchy problem (3.2), (3.3) is equivalent in the space $C^\alpha[0,T]$ to the following Volterra integral equation:

$$\begin{aligned} u_n(t) =& \varphi_n E_{\alpha,1}(-\lambda_n t^\alpha) + \psi_n t E_{\alpha,2}(-\lambda_n t^\alpha) \\ &+ \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^\alpha) \left(f_n(\tau) - q(\tau) u_n(\tau)\right) d\tau. \end{aligned} \tag{3.4}$$

First we prove the following assertions.

Lemma 3.1. *For any $n \in \mathbb{N}$, $f_n(t) \in C[0,T]$ and $q(t) \in C[0,T]$, we have the estimates*

$$|u_n(t)| \leq C_0 \left(|\varphi_n| + |\psi_n| T + \|f_n\| T^\alpha\right) Z_{\alpha,1}\left(\left(\|q\|_{C[0,T]} \Gamma(\alpha)\right)^{\frac{1}{\alpha}} T\right),$$

$$\begin{aligned} t^\gamma |\partial_t^\alpha u_n(t)| \leq& \|f_n\| + C_0 \left(|\lambda_n| + \|q\|_{C[0,T]}\right) \\ &\times \left(|\varphi_n| + |\psi_n| T + \|f_n\| T^\alpha\right) Z_{\alpha,1}\left(\left(\|q\|_{C[0,T]} \Gamma(\alpha)\right)^{\frac{1}{\alpha}} T\right), \end{aligned}$$

where the constant C_0 depends on α, β and C.

Proof. Using the estimates of the Mittag-Leffler function, we estimate the integral equation (3.4)

$$\begin{aligned} |u_n(t)| \leq& |\varphi_n| \cdot |E_{\alpha,1}(-\lambda_n t^\alpha)| \\ &+ |\psi_n| \cdot |t E_{\alpha,2}(-\lambda_n t^\alpha)| + \left| \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^\alpha) f_n(\tau) d\tau \right| \end{aligned}$$

$$+\left|\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)q(\tau)u_n(\tau)d\tau\right|$$
$$\leq C_0\left(|\varphi_n|+|\psi_n|t+\|f_n\|t^\alpha+\|q\|_{C[0,T]}\int_0^t (t-\tau)^{\alpha-1}|u_n(\tau)|d\tau\right).$$

Next, according to Lemma 2.3, we have

$$|u_n(t)| \leq C_0\left(|\varphi_n|+|\psi_n|T+\|f_n\|T^\alpha\right) Z_{\alpha,1}\left(\left(\|q\|_{C[0,T]}\Gamma(\alpha)\right)^{\frac{1}{\alpha}} T\right). \tag{3.5}$$

We get the second part of the Lemma 3.1, from equation (3.3) and the first estimate of Lemma 2.3

$$|\partial_t^\alpha u_n(t)| \leq \|f_n\| + C_0\left(|\lambda_n|+\|q\|_{C[0,T]}\right)$$
$$\times\left(|\varphi_n|+|\psi_n|T+\|f_n\|T^\alpha\right) Z_{\alpha,1}\left(\left(\|q\|_{C[0,T]}\Gamma(\alpha)\right)^{\frac{1}{\alpha}} T\right).$$

From the last two inequalities we immediately obtain the estimates of Lemma 3.1 for any $t\in[0,T]$. Lemma 3.1 is proven. □

Formally differentiate series (3.1) and construct the following series

$$\partial_t^\alpha u(x,t) = \sum_{n=1}^{+\infty} \partial_t^\alpha u_n(t)X_n(x), \tag{3.6}$$

$$\partial_x^\beta u(x,t) = \sum_{n=1}^{+\infty} \lambda_n u_n(t)X_n(x). \tag{3.7}$$

Let us prove the uniform convergence of series (3.1), (3.6) and (3.7) in the domain $\overline{\Omega}_T := \{(x,t) : 0 \leq x \leq l,\ 0 \leq t \leq T\}$. These series, for any $(x,t)\in\overline{\Omega}_T$, are respectively majorized by the following numerical series

$$C_0\sum_{n=1}^{+\infty}\left(|\varphi_n|+|\psi_n|T+\|f_n\|T^\alpha\right) Z_{\alpha,1}\left(\left(\|q\|_{C[0,T]}\Gamma(\alpha)\right)^{\frac{1}{\alpha}} T\right), \tag{3.8}$$

$$\sum_{n=1}^{+\infty}\left(\|f_n\|+C_0\left(|\lambda_n|+\|q\|_{C[0,T]}\right)\left(|\varphi_n|+|\psi_n|T+\|f_n\|T^\alpha\right)\right.$$
$$\left.\times Z_{\alpha,1}\left(\left(\|q\|_{C[0,T]}\Gamma(\alpha)\right)^{\frac{1}{\alpha}} T\right)\right), \tag{3.9}$$

$$C_0\sum_{n=1}^{+\infty}|\lambda_n|\left(|\varphi_n|+|\psi_n|T+\|f_n\|T^\alpha\right) Z_{\alpha,1}\left(\left(\|q\|_{C[0,T]}\Gamma(\alpha)\right)^{\frac{1}{\alpha}} T\right). \tag{3.10}$$

It is not difficult to notice that the sums of the series (3.8)-(3.10) do not exceed the following expression:

$$\sum_{n=1}^{+\infty}|\lambda_n|\left(|\varphi_n|+|\psi_n|T+\|f_n\|T^\alpha\right). \tag{3.11}$$

We state the following auxiliary lemma.

Lemma 3.2. *If the conditions A1)-A3) are satisfied, then we have the following estimates*

$$|\varphi_n| \le \frac{C_0}{|\lambda_n|^2}\|\varphi^{(3)}\|_{L_2(0,1)}, \quad |\psi_n| \le \frac{C_0}{|\lambda_n|^2}\|\psi^{(3)}\|_{L_2(0,1)},$$

$$|f_n(t)| \le \frac{C_0}{|\lambda_n|^2}\|f_{xxx}(t)\|_{L_2(0,1)}. \tag{3.12}$$

Proof. It is easy to see that

$$\frac{d}{dx}E_{\beta,1}\left(\lambda x^{\beta}\right) = \lambda x^{\beta-1}E_{\beta,\beta}\left(\lambda x^{\beta}\right). \tag{3.13}$$

Consider

$$\varphi_n = \langle \varphi(x), Y_n(x)\rangle = \int_0^1 \varphi(x)(1-x)^{\beta-1}E_{\beta,\beta}(\lambda_n(1-x)^{\beta})dx.$$

Based on the equality (3.13)

$$\varphi_n = \frac{1}{\lambda_n}\int_0^1 \varphi(x)\frac{d}{dx}E_{\beta,1}(\lambda_n(1-x)^{\beta})dx.$$

The following inequality is the result of integration by parts

$$\varphi_n = \frac{1}{\lambda_n}\int_0^1 \varphi'(x)E_{\beta,1}(\lambda_n(1-x)^{\beta})dx.$$

By Proposition 2.1, we get

$$|\varphi_n| \le \frac{C_0}{|\lambda_n|^2}\left|\int_0^1 \varphi'(x)(1-x)^{-\beta}dx\right|.$$

Integrating by parts twice and using the condition of Lemma 3.2, we obtain

$$\begin{aligned}|\varphi_n| \le &\frac{C_0}{|\lambda_n|^2}\left(\frac{|\varphi'(0)|}{\beta-1} + \frac{|\varphi''(0)|}{(\beta-1)(2-\beta)}\right)\\ &+ \frac{C_0}{|\lambda_n|^2(\beta-1)(2-\beta)}\left|\int_0^1 \varphi^{(3)}(x)(1-x)^{2-\beta}dx\right|.\end{aligned}$$

According to Hölder's inequality, we get the following inequality

$$|\varphi_n| \le \frac{C_0}{|\lambda_n|^2}\left(\frac{|\varphi'(0)|}{\beta-1} + \frac{|\varphi''(0)|}{(\beta-1)(2-\beta)} + \frac{\|\varphi^{(3)}\|_{L_2(0,1)}}{(\beta-1)(2-\beta)\sqrt{5-2\beta}}\right).$$

From the above inequality according to condition A1), we obtain the following inequality:

$$|\varphi_n| \leq \frac{C_0}{|\lambda_n|^2}\|\varphi^{(3)}\|_{L_2(0,1)}.$$

The next inequalities in Lemma 3.2 are easily obtained using the same process. □

According to inequalities (3.12), the series in (3.11) converge. From this, it follows that the series (3.8), (3.9) and (3.10) are convergent.

Theorem 3.1. *Let* $q(t) \in C[0,T]$ *and the conditions* $A1)$, $A2)$ *are satisfied, then there exists a unique solution of the direct problem* (1.1)–(1.3), $u(x,t) \in C^{\alpha,\beta}(\overline{\Omega})$.

From the conditions $A1)$, $A2)$ and the inequality (3.12), the series (3.1), (3.6), and (3.7) are absolutely and uniformly convergent. From this it follows the proof of Theorem 3.1.

Let us derive an estimate for the norm of the difference between the solution of the original integral equation (3.4) and the solution of this equation with perturbed functions $\widetilde{q}$, $\widetilde{\varphi}_n$, $\widetilde{\psi}_n$ and $\widetilde{f}_n$. Let $\widetilde{u}_n(t)$ be solution of the integral equation (3.4) corresponding to the functions $\widetilde{q}$, $\widetilde{\varphi}_n$, $\widetilde{\psi}_n$, $\widetilde{f}_n$, i.e.,

$$\begin{aligned}\widetilde{u}_n(t) =&\widetilde{\varphi}_n E_{\alpha,1}(-\lambda_n t^\alpha) + \widetilde{\psi}_n t E_{\alpha,2}(-\lambda_n t^\alpha)\\ &+\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^\alpha)\left(\widetilde{f}_n(\tau) - \widetilde{q}(\tau)\widetilde{u}_n(\tau)\right) d\tau.\end{aligned} \tag{3.14}$$

Composing the difference $u_n(t) - \widetilde{u}_n(t)$ with the help of the equations (3.4), (3.14) and introducing the notations $\widehat{u}_n(t) = u_n(t) - \widetilde{u}_n(t)$, $\widehat{q} = q(t) - \widetilde{q}(t)$, $\widehat{\varphi}_n(t) = \varphi_n(t) - \widetilde{\varphi}_n(t)$, $\widehat{\psi}_n(t) = \psi_n(t) - \widetilde{\psi}_n(t)$, $\widehat{f}_n(t) = f_n(t) - \widetilde{f}_n(t)$ we obtain the integral equation

$$\begin{aligned}\widehat{u}_n(t) =&\widehat{\varphi}_n E_{\alpha,1}(-\lambda_n t^\alpha) + \widehat{\psi}_n t E_{\alpha,2}(-\lambda_n t^\alpha)\\ &+\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^\alpha)\left(\widehat{f}_n(\tau) - \widehat{q}(\tau)u_n(\tau)\right) d\tau\\ &-\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^\alpha)\widetilde{q}(\tau)\widehat{u}_n(\tau) d\tau,\end{aligned} \tag{3.15}$$

from which, is derived the following linear integral inequality for $|\widehat{u}_n(t)|$

$$\begin{aligned}|\widehat{u}_n(t)| \leq& C_0\left(|\widehat{\varphi}_n| + |\widehat{\psi}_n|T + \|\widehat{f}_n\|T^\alpha + \|\widehat{q}\|_{C[0,T]}\left(|\varphi_n| + |\psi_n|T + \|f_n\|T^\alpha\right)\right)\\ &\times Z_{\alpha,1}\left(\left(\|q\|_{C[0,T]}\Gamma(\alpha)\right)^{\frac{1}{\alpha}} t\right) T^\alpha B(\alpha,1) + \|\widetilde{q}\|_{C[0,T]}\int_0^t (t-\tau)^{\alpha-1}|\widehat{u}_n(\tau)| d\tau.\end{aligned}$$

Using the Lemma 2.3 from last inequality, we arrive at the estimate:

$$|\widehat{u}_n(t)| \leq C_0\Big[|\widehat{\varphi}_n| + |\widehat{\psi}_n|T + \|\widehat{f}_n\|T^\alpha + \|\widehat{q}\|_{C[0,T]}\left(|\varphi_n| + |\psi_n|T + \|f_n\|T^\alpha\right)$$

$$\times Z_{\alpha,1}\Big(\left(\|q\|_{C[0,T]}\Gamma(\alpha)\right)^{\frac{1}{\alpha}} t\Big)T^{\alpha}B(\alpha,1)\Big]Z_{\alpha,1}\left(\left(\|\tilde{q}\|_{C[0,T]}\Gamma(\alpha)\right)^{\frac{1}{\alpha}} t\right). \tag{3.16}$$

The estimate will be used in the next section.

4. Investigation of the Inverse Problem

Let us now we investigate to inverse problem (1.1)–(1.4). By integrating equation (1.1) with respect to x from 0 to 1, and applying the condition given in equation (1.4), we obtain

$$\int_0^1 \left\{\left(\partial_t^{\alpha} u\right)(x,t) - \left(\partial_x^{\beta} u\right)(x,t) + q(t)u(x,t)\right\} dx = \int_0^1 f(x,t)dx,$$

we form

$$\left(\partial_t^{\alpha} g\right)(t) + q(t)g(t) - \int_0^1 \left(\partial_x^{\beta} u\right)(x,t)dx = \int_0^1 f(x,t)dx,$$

which yields

$$q(t) = \frac{1}{g(t)}\left(\int_0^1 f(x,t)dx - \left(\partial_{0+}^{\alpha} g\right)(t) + \int_0^1 \left(\partial_x^{\beta} u\right)(x,t)dx\right). \tag{4.1}$$

Let's consider the integral $\int_0^1 \left(\partial_x^{\beta} u\right)(x,t)dx$ on the right side of the above integral equation. We write this integral according to (2.4) as follows

$$\int_0^1 \left(\partial_x^{\beta} u\right)(x,t)dx = \int_0^1 \sum_{n=1}^{+\infty} u_n(t)\left(\partial_x^{\beta} X_n\right)(x)dx.$$

Considering that the series under this integral is uniformly convergent, we obtain

$$\int_0^1 \left(\partial_x^{\beta} u\right)(x,t)dx = \sum_{n=1}^{+\infty} u_n(t)\int_0^1 \left(\partial_x^{\beta} X_n\right)(x)dx.$$

Using $\int_0^1 \left(\partial_x^{\beta} X_n\right)(x)dx = E_{\alpha,1}(\lambda_n) - 1$ in this equality, we get

$$\int_0^1 \left(\partial_x^{\beta} u\right)(x,t)dx = \sum_{n=1}^{+\infty} u_n(t)\left(E_{\alpha,1}(\lambda_n) - 1\right). \tag{4.2}$$

Using equality (4.2) in the integral equation (4.1) and taking into account the dependence of the functions $u_n(t)$ on $q(t)$, i.e., $u_n(t;q)$, we obtain the following integral equation with respect to $q(t)$:

$$q(t) = q_0(t) + \frac{1}{g(t)}\sum_{n=1}^{+\infty} u_n(t;q)\left(E_{\alpha,1}(\lambda_n) - 1\right), \tag{4.3}$$

where

$$q_0(t) = \frac{1}{g(t)} \left(\int_0^1 f(x,t)dx - \left(\partial_{0+}^{\alpha} g\right)(t) \right).$$

We introduce an operator $\mathcal{F}$ defining it by the right hand side of (4.3):

$$\mathcal{F}[q](t) = q_0(t) + \frac{1}{g(t)} \sum_{n=1}^{+\infty} u_n(t;q) \left(E_{\alpha,1}(\lambda_n) - 1\right). \tag{4.4}$$

Then, the equation (4.4) is written in more convenient form as

$$\mathcal{F}[q](t) = q(t). \tag{4.5}$$

Let

$$q_{00} := \max_{t\in[0;T]} |q_0(t)| = \left\| \frac{1}{g(t)} \left(\int_0^1 f(x,t)dx - \left(\partial_{0+}^{\alpha} g\right)(t) \right) \right\|_{C[0,T]} \leq \frac{1}{g_0} \left(\|f\| + \|g\|\right),$$

where

$$\|f\| := \max_{(x,t)\in\overline{\Omega}} |f(x,t)|, \quad \|g\| := \max_{t\in[0,T]} |g(t)|.$$

Fix a number $r > 0$ and consider the ball

$$B(q_0, r) := \{q(t) : q(t) \in C[0,T], \|q - q_0\| \leq r\}.$$

Theorem 4.1. *Let A1)-A3) be satisfied. Then, there exists a number $T^* \in (0,T)$ and a radius r, such that there exists a unique solution $q(t) \in C[0,T^*]$ of the inverse problem (1.1)–(1.4), $q(t) \in B(q_0, r)$.*

Proof. Let us first prove that for sufficiently small $T > 0$, the operator $\mathcal{F}$ maps the ball $B(q_0, r)$ into itself, i.e., $\mathcal{F}[q](t) \in B(q_0, r)$. Specifically, for any continuous function $q(t)$, the function $\mathcal{F}[q](t)$, calculated using formula (4.5), will also be continuous. Furthermore, by estimating the norm of the difference, we can establish the desired result

$$\|\mathcal{F}[q](t) - q_0(t)\| \leq \frac{C_0}{g_0} \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} \left(|\varphi_n| + |\psi_n|T + \|f_n\|T^{\alpha}\right) Z_{\alpha,1}\left(\left(\|q\|_{C[0,T]}\Gamma(\alpha)\right)^{\frac{1}{\alpha}} T \right).$$

Here, we have utilized the estimate (3.5). According to Lemma 2.3, the series in this inequality is convergent. It is important to note that the function on the right-hand side of this inequality is monotone increasing with respect to T, and since the function $q(t)$ belongs to the ball $B(q_0, r)$, we have the inequality $\|q\| \leq \|q_0\| + r$. Thus, the inequality is strengthened if we replace $\|q\|$ with $\|q_0\| + r$. After making this substitution, we obtain the improved estimate.

$$\begin{aligned} \|\mathcal{F}[q](t) - q_0(t)\| \leq & \frac{C_0}{g_0} \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} \left(|\varphi_n| + |\psi_n|T + \|f_n\|T^{\alpha}\right) \\ & \times Z_{\alpha,1}\left(\left(\left(\|q_0\| + r\right)\Gamma(\alpha)\right)^{\frac{1}{\alpha}} T \right) =: m_1(T). \end{aligned} \tag{4.6}$$

Let us consider two functions $q(t), \widetilde{q}(t) \in B(q_0, r)$ and estimate the distance between their images $\mathcal{F}[q](t)$ and $\mathcal{F}[\widetilde{q}](t)$ in the space $C[0,T]$. The function $\widetilde{u}_n(t)$, corresponding to $\widetilde{q}(t)$, satisfies the integral equation (20) with $\varphi_n = \widetilde{\varphi}_n$, $\psi = \widetilde{\psi}_n$, and $f_n = \widetilde{f}_n$. By expressing the difference $\mathcal{F}[q](t) - \mathcal{F}[\widetilde{q}](t)$ using equations (3.4), (3.15) and inequalities (3.5), (3.16) and then estimating its norm, we derive

$$\begin{aligned} \|\mathcal{F}[q](t) - \mathcal{F}[\widetilde{q}](t)\| \leq & \frac{C_0}{g_0} \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} \| \left(u_n(t,q) - \widetilde{u}_n(t,\widetilde{q})\right) \| \\ \leq & \frac{T^\alpha C_0}{g_0} \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} \left(|\varphi_n| + |\psi_n| T + \|f_n\| T^\alpha\right) Z_{\alpha,1}\left(\left(\|q\|_{C[0,T]}\Gamma(\alpha)\right)^{\frac{1}{\alpha}} T\right) \\ & \times Z_{\alpha,1}\left(\left(\|\widetilde{q}\|_{C[0,T]}\Gamma(\alpha)\right)^{\frac{1}{\alpha}} T\right) \|\widehat{q}\|_{C[0,T]}. \end{aligned} \tag{4.7}$$

The functions $q(t)$ and $\widetilde{q}(t)$ belong to the ball $B(q_0, r)$, meaning that for each of these functions, the inequality $\|q\| \leq \|q_0\| + r$ holds. Note that the function on the right-hand side of inequality (4.7), at the factor $\|\widehat{q}\|_{C[0,T]}$, is monotone increasing with respect to $\|q\|$, $\|\widetilde{q}\|$ and T. Therefore, replacing $\|q\|$ and $\|\widetilde{q}\|$ in inequality (4.7) with $\|q_0\| + r$ will only strengthen the inequality. Hence, we obtain

$$\begin{aligned} \|\mathcal{F}[q](t) - \mathcal{F}[\widetilde{q}](t)\| \leq & \frac{C_0}{g_0} \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} \left(|\varphi_n| + |\psi_n| T + \|f_n\| T^\alpha\right) Z_{\alpha,1}\left(\left(\left(\|q_0\| + r\right)\Gamma(\alpha)\right)^{\frac{1}{\alpha}} T\right) \\ & \times T^\alpha Z_{\alpha,1}\left(\left(\left(\|q_0\| + r\right)\Gamma(\alpha)\right)^{\frac{1}{\alpha}} T\right) \|\widehat{q}\|_{C[0,T]} =: m_2(T) \|\widehat{q}\|_{C[0,T]}. \end{aligned} \tag{4.8}$$

Note that the expressions on the right-hand sides of these inequalities are monotonically increasing functions of T, $\|q\|$, and $\|\widetilde{q}\|$. Given that $q, \widetilde{q} \in B(q_0, r)$, it follows from the definition of the ball that the inequality

$$\|(q,\, \widetilde{q})\| \leq \|q_0\| + r$$

holds.

Considering these facts, for $(x,t) \in Q_T$, we obtain the following inequalities:

$$\|\mathcal{F}[q](t) - q_0(t)\| \leq m_1(T), \quad \|\mathcal{F}[q](t) - \mathcal{F}[\widetilde{q}](t)\| \leq m_2(T)\|\widehat{q}\|_{C[0,T]},$$

where $m_1(T)$ denotes the right-hand side of inequality (4.6), and $m_2(T)$ represents the multiplier $\|\widehat{q}\|_{C[0,T]}$, on the right-hand side of inequality (4.8). The only difference is that in both cases, q and $\widetilde{q}$ are replaced by $\|q_0\| + r$.

It is important to note that $m_i(T)$ for $i = 1,2$ are positive, monotonically increasing functions of T, with initial conditions $m_1(0) = \frac{C}{g_0}\|\varphi\|_{L_2[0,1]}$ and $m_2(0) = 0$. From this, we conclude that the equations $m_1(T) = r$, where $\frac{C}{g_0}\|\varphi\|_{L_2[0,1]} < r$, and $m_2(T) = 1$ have unique positive roots, denoted by T_1 and T_2, respectively. Therefore, if we choose $T^* < \min\{T_1, T_2\}$, the operator $\mathcal{F}$ becomes a contraction on the ball $B(q_0, r)$. By the Banach fixed-point theorem (see [44, pp. 87–97]), the operator $\mathcal{F}$ has a unique fixed point in the ball $B(q_0, r)$, which corresponds to a unique solution of equation (4.5). Thus, Theorem 4.1 is proven. $\square$

5. Conclusion

In this paper, we investigate the inverse coefficient problem for a space-time fractional differential equation. The associated direct problem is formulated as an initial-boundary value problem, incorporating Cauchy-type initial conditions and Dirichlet boundary conditions. We first analyze the direct problem, employing estimates of the Mittag-Leffler function and a generalized form of the Gronwall inequality to derive an a priori estimate for the solution in terms of the unknown coefficient. This estimate forms the basis for studying the inverse problem. To facilitate the identification of the unknown coefficient, a nonlocal integral condition involving the solution of the direct problem is imposed as an overdetermination condition. An inverse problem was addressed using a set of biorthogonal functions constructed from Mittag-Leffler-type functions, derived from a fractional-order spectral analysis and an associated auxiliary problem. The inverse problem was reduced to an equivalent Volterra-type integral equation. The contraction mapping principle was employed to solve this equation. Results on local existence and uniqueness of the solution were established.

In our view, the problem under consideration can be generalized to a more comprehensive form of equation (1.1) by replacing the Caputo fractional operator with respect to the spatial variable with a fractional Laplacian operator, subject to appropriately modified conditions analogous to those in (1.2)–(1.4).

This paper establishes local existence and uniqueness of solutions. Global existence is not considered. While global existence may be attainable via a suitable continuation principle, its rigorous justification requires additional analysis and therefore remains an open problem.

References

[1] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Application of Fractional Differential Equations*, North-Holland Mathematical Studies, Elsevier, Amsterdam, 2006.

[2] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach Science Publishers, Amsterdam, 1993.

[3] G. Sorrentino, *Fractional Derivative Linear Models for Describing the Viscoelastic Dynamic Behavior of Polymeric Beams*, Proceedings of IMAS, Saint Louis, Mo, USA, 2006.

[4] K. Diethelm, *The Analysis of Fractional Differential Equations, an Application Oriented Exposition Using Differential Operators of Caputo Type*, Lecture Notes in Mathematics, Vol. 2004, Springer, Heidelberg, 2010.

[5] K. S. Miller and B. Ross, *An Introduction to Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[6] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 2009.

[7] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publishing Co, Singapore, 2003.

[8] J. A. Tenreiro Machado, M. F. Silva, R. S. Barbosa, I. S. Jesus, C. M. Reis, M. G. Marcos and A. F. Galhano, *Some applications of fractional calculus in engineering*, Math. Probl. Eng. (2010). https://doi.org/10.1155/2010/639801

[9] T. S. Aleroev, M. Kirane and Y. F. Tang, *Boundary-value problems for differential equations of fractional order*, J. Math. Sci. **194**(5) (2013), 499–512. https://doi.org/10.1007/s10958-013-1543-y

[10] A. Y. Popov, *On the number of real eigenvalues of the boundary-value problem for a second-order equation with fractional derivative*, Fundam. Prikl. Mat. **151** (2008), 2726–2740. https://doi.org/10.1007/s10948-008-0169-7

[11] M. V. Khasambiev and T. S. Aleroev, *Boundary-value problem for one-dimensional fractional differential advection-diffusion equation*, Vestn. Mos. Gos. Stroit. Univ. **6** (2014), 71–76.

[12] R. Ashurov and S. Umarov, *Determination of the order of fractional derivative for subdiffusion equations*, Fract. Calc. Appl. Anal. **23**(6) (2020), 1647–1662. https://doi.org/10.1515/fca-2020-0081

[13] Sh. Alimov and R. Ashurov, *Inverse problem of determining an order of the Caputo time-fractional derivative for a subdiffusion equation*, J. Inverse Ill-Posed Probl. **28**(5) (2020), 651–658. https://doi.org/10.1515/jiip-2020-0072

[14] Zh. Li and M. Yamamoto, *Uniqueness for inverse problems of determining orders of multi-term time-fractional derivatives of diffusion equation*, Appl. Anal. **4**(3) (2014), 570–579. https://doi.org/10.1080/00036811.2014.926335

[15] K. Sakamoto and M. Yamamoto, *Inverse source problem with a final overdetermination for a fractional diffusion equation*, Math. Control Relat. Fields **1** (2011), 509–518. https://doi.org/10.3934/mcrf.2011.1.509

[16] X. Gong and T. Wei, *Reconstruction of a time-dependent source term in a time-fractional diffusion-wave equation*, Inverse Probl. Sci. Eng. **27**(11) (2019), 1577–1594. https://doi.org/10.1080/17415977.2018.1539481

[17] B. Wu and S. Wu, *Existence and uniqueness of an inverse source problem for a fractional integrodifferential equation*, Comput. Math. Appl. **68** (2014), 1123–1136. https://doi.org/10.1016/j.camwa.2014.08.014

[18] Y. Zhang and X. Xu, *Inverse source problem for a fractional diffusion equation*, Inverse Problems **27**(3) (2011), 035010. https://doi.org/10.1088/0266-5611/27/3/035010

[19] M. Kirane, S. A. Malik and M. A. Al-Gwaiz, *An inverse source problem for a two-dimensional time fractional diffusion equation with nonlocal boundary conditions*, Math. Methods Appl. Sci. **36** (2013), 1056–1069. https://doi.org/10.1002/mma.2661

[20] E. Karimov, N. Al-Salti and S. Kerbal, *An inverse source non-local problem for a mixed type equation with a Caputo fractional differential operator*, East Asian J. Appl. Math. **7**(2) (2017), 417–438. https://doi.org/10.4208/eajam.051216.280217a

[21] D. K. Durdiev, A. A. Rahmonov and Z. R. Bozorov, *A two-dimensional diffusion coefficient determination problem for the time-fractional equation*, Math. Methods Appl. Sci. **44**(13) (2021), 10753–10761. https://doi.org/10.22541/au.160916886.64925851/v1

[22] L. Miller and M. Yamamoto, *Coefficient inverse problem for a fractional diffusion equation*, Inverse Problems **29**(7) (2013), Article ID 075013. https://doi.org/10.1088/0266-5611/29/7/075013

[23] H. H. Turdiev, *Inverse coefficient problems for a time-fractional wave equation with the generalized Riemann-Liouville time derivative*, Russian Math. (Iz. VUZ) **10** (2023), 46–59. https://doi.org/10.3103/S1066369X23100092

[24] D. K. Durdiev and H. H. Turdiev, *Inverse coefficient problem for fractional wave equation with the generalized Riemann-Liouville time derivative*, Indian J. Pure Appl. Math. (2023). https://doi.org/10.1007/s13226-023-00517-9

[25] D. K. Durdiev and H. H. Turdiev, *Inverse coefficient problem for a time-fractional wave equation with initial-boundary and integral type overdetermination conditions*, Math. Methods Appl. Sci. **47**(6) (2024), 5329–5340. https://doi.org/10.1002/mma.9867

[26] D. K. Durdiev and H. H. Turdiev, *Determining of a space dependent coefficient of fractional diffusion equation with the generalized Riemann-Liouville time derivative*, Lobachevskii J. Math. **45**(2) (2024), 648–662. https://doi.org/10.1134/S1995080224600316

[27] K. Sakamoto and M. Yamamoto, *Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems*, J. Math. Anal. Appl. **382**(1) (2011), 426–447. https://doi.org/10.1016/j.jmaa.2011.04.058

[28] U. D. Durdiev, *Problem of determining the reaction coefficient in a fractional diffusion equation*, Differ. Equ. **57**(9) (2021), 1195–1204. https://doi.org/10.1134/S0012266121090081

[29] M. Kirane and S. A. Malik, *Determination of an unknown source term and the temperature distribution for the linear heat equation involving fractional derivative in time*, Appl. Math. Comput. **218**(1) (2011), 163–170. https://doi.org/10.1016/j.amc.2011.05.084

[30] T. S. Aleroev, M. Kirane and S. A. Malik, *Determination of a source term for a time fractional diffusion equation with an integral type-over determining condition*, Electron. J. Differential Equations **270** (2013), 1–16. https://api.semanticscholar.org/CorpusID:55899337

[31] S. Tatar and S. Ulusoy, *A uniqueness result for an inverse problem in space-time fractional diffusion equation*, Electron. J. Differential Equations **258** (2013), 1–9. https://www.researchgate.net/publication/268028028

[32] M. Ali, S. Aziz and S. A. Malik, *Inverse problem for a space-time fractional diffusion equation: application of fractional Sturm-Liouville operator*, Math. Methods Appl. Sci. **41** (2018), 2733–2744. https://api.semanticscholar.org/CorpusID:206258062

[33] M. Ali, S. Aziz and S. A. Malik, *Inverse source problem for a space-time fractional diffusion equation*, Fract. Calc. Appl. Anal. **21**, (2018), 844–863. https://doi.org/10.1515/fca-2018-0045

[34] M. Ali, S. Aziz and S. A. Malik, *Inverse source problems for a space-time fractional diffusion equation*, Inverse Probl. Sci. Eng. **28**(1) (2019), 1–22. https://doi.org/10.1080/17415977.2019.1597079

[35] J. Jia, J. Peng and J. Yang, *Harnack's inequality for a space-time fractional diffusion equation and application to an inverse source problem*, J. Differential Equations **262**(8) (2017), 4415–4450. https://doi.org/10.1016/j.jde.2017.01.002

[36] D. K. Durdiev and Z. D. Totieva, *The problem of determining the one-dimensional matrix kernel of the system of viscoelasticity equations*, Math. Methods Appl. Sci. **41**(17) (2018), 8019–8032. https://doi.org/10.1002/mma.5267

[37] D. K. Durdiev and Z. Z. Zhumaev, *Memory kernel reconstruction problems in the integro-differential equation of rigid heat conductor*, Math. Methods Appl. Sci. **45**(14) (2022), 8374–8388. https://doi.org/10.1002/mma.7133

[38] D. K. Durdiev and H. H. Turdiev, *Inverse problem for a first-order hyperbolic system with memory*, Differ. Equ. **56** (2020), 1634–1643. https://doi.org/10.1134/S00122661200120125

[39] U. D. Durdiev and Z. D. Totieva, *A problem of determining a special spatial part of 3D memory kernel in an integrodifferential hyperbolic equation*, Math. Methods Appl. Sci. **42** (2019), 7440–7451. https://doi.org/10.1002/mma.5863

[40] D. K. Durdiev, *Inverse coefficient problem for the time-fractional diffusion equation*, Eurasian J. Math. Comput. Appl. **9**(1) (2021), 44–54. https://doi.org/10.32523/2306-6172-2021-9-1-44-54

[41] M. M. Dzhrbashyan, *The boundary-value problem for a differential fractional-order operator of the Sturm-Liouville type*, Izv. Akad. Nauk Arm. SSR, Ser. Mat. **5**(2) (1970), 71–96.

[42] R. Gorenflo, A. A. Kilbas, F. Mainardi and S. V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*, Springer-Verlag, Berlin, Germany, 2014.

[43] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer, Berlin, 1981.

[44] A. N. Kolmogorov and S. V. Fomin, *Elements of the Theory of Functions and Functional Analysis*, Dover Books on Mathematics, Mineola, New York, 1976.

[1]V.I.Romanovskiy Institute of Mathematic,
Uzbekistan Academy of Sciences,
Email address: durdievdd@gmail.com
ORCID iD: https://orcid.org/0000-0002-6054-2827

[2]Differential equation,
Physical and Mathematical Faculty,
Bukhara State University
Email address: hturdiev@mail.ru, h.h.turdiyev@gmail.com
ORCID iD: https://orcid.org/0000-0002-1152-9159