Kragujevac Journal of Mathematics
Volume 44(4) (2020), Pages 539–550.

A NEW CLASS OF INTEGRALS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTION AND MULTIVARIABLE ALEPH-FUNCTION

DINESH KUMAR[1], FRÉDÉRIC AYANT[2,3], AND DEVENDRA KUMAR[4]

Abstract. The aim of this paper is to evaluate an interesting integral involving generalized hypergeometric function and the multivariable Aleph-function. The integral is evaluated with the help of an integral involving generalized hypergeometric function obtained recently by Kim et al. [8]. The integral is further used to evaluate an interesting summation formula concerning the multivariable Aleph-function. A few interesting special cases and corollaries have also been discussed.

1. Introduction and Preliminaries

Hypergeometric function is an important and useful tool for special functions that plays an important role in the field of analysis. Transformation theory plays a major role to provide a platform for the development of beautiful transformation. It is important to mention that whenever generalized hypergeometric function reduces to a gamma function, the results are very important from application point of view in mathematics, statistics and mathematical physics [2, 11, 22]. Recently Rohira et al. [17] have evaluated a class of integrals involving generalized hypergeometric function and the H-function defined by Fox [5] (see also, [16]). In this paper, we aim to present a class of integrals involving generalized hypergeometric function and the multivariable Aleph-function.

The multivariable Aleph-function is an extension of the multivariable I-function defined by Sharma and Ahmad [20], which is a generalization of the multivariable H-function defined by Srivastava et al. [24, 25] (see also, [3, 4, 10, 23]). The multivariable

Key words and phrases. Multivariable I-function, multivariable H-function, double finite integrals.
2010 *Mathematics Subject Classification.* Primary: 33C60, 33C99. Secondary: 44A20.
Received: March 13, 2018.
Accepted: June 22, 2018.

Aleph-function is defined by means of the multiple contour integral given by the following manner:

$$\aleph(z_1,\ldots,z_r)$$

$$= \aleph^{0,\mathfrak{n}:m_1,n_1,\ldots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\ldots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|c} z_1 & \left[\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,\mathfrak{n}}\right], \\ \cdot & \ldots\ldots\ldots, \\ \cdot & \\ \cdot & \\ z_r & \end{array}\right.$$

$$\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\ldots,\alpha_{ji}^{(r)}\right)_{\mathfrak{n}+1,p_i}\right]:\left[\left(c_j^{(1)}\right),\left(\gamma_j^{(1)}\right)_{1,n_1}\right],$$
$$\left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\ldots,\beta_{ji}^{(r)}\right)_{m+1,q_i}\right]:\left[\left(d_j^{(1)}\right),\left(\delta_j^{(1)}\right)_{1,m_1}\right],$$

$$\left.\begin{array}{l}\left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right];\ldots;\left[\left(c_j^{(r)}\right),\left(\gamma_j^{(r)}\right)_{1,n_r}\right],\left[\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_i^{(r)}}\right] \\ \left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right];\ldots;\left[\left(d_j^{(r)}\right),\left(\delta_j^{(r)}\right)_{1,m_r}\right],\left[\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_i^{(r)}}\right]\end{array}\right)$$

(1.1)

$$= \frac{1}{(2\pi\omega)^r}\int_{\mathcal{L}_1}\cdots\int_{\mathcal{L}_r}\psi(\xi_1,\ldots,\xi_r)\prod_{k=1}^{r}\theta_k(\xi_k)\,z_k^{\xi_k}\,\mathrm{d}\xi_1\ldots\mathrm{d}\xi_r,$$

with $\omega=\sqrt{-1}$,

$$\psi(\xi_1,\ldots,\xi_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma\left(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}\xi_k\right)}{\sum_{i=1}^{R}\left[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma\left(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}\xi_k\right)\prod_{j=1}^{q_i}\Gamma\left(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}\xi_k\right)\right]}$$

and

$$\theta_k(\xi_k)=\frac{\prod_{j=1}^{m_k}\Gamma\left(d_j^{(k)}-\delta_j^{(k)}\xi_k\right)\prod_{j=1}^{n_k}\Gamma\left(1-c_j^{(k)}+\gamma_j^{(k)}\xi_k\right)}{\sum_{i^{(k)}=1}^{R^{(k)}}\left[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma\left(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}\xi_k\right)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma\left(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}\xi_k\right)\right]}.$$

For more details, reader can refer to recent works [1, 18]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given as $|\arg z_k|<\frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}+\sum_{j=1}^{m_k}\delta_j^{(k)}$$
$$-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}>0,\quad\text{with } k=1,\ldots,r,\ i=1,\ldots,R,\ i^{(k)}=1,\ldots,R^{(k)},$$

where $k=1,\ldots,r$, $i=1,\ldots,R$, $i^{(k)}=1,\ldots,R^{(k)}$.

The complex numbers $z_i\neq 0$. Throughout the paper, we assume the existence and absolute convergence conditions of the multivariable Aleph-function. Here and in the following, let $\mathrm{Re}(a)$ be the real part of a complex number a. We establish the asymptotic expansion in the convenient form, below

$$\begin{aligned}\aleph\left(z_1,\ldots,z_r\right) =&0\left(|z_1|^{\alpha_1},\ldots,|z_r|^{\alpha_r}\right), \quad \max\left(|z_1|,\ldots,|z_r|\right)\to 0,\\ \aleph\left(z_1,\ldots,z_r\right) =&0\left(|z_1|^{\beta_1},\ldots,|z_r|^{\beta_r}\right), \quad \min\left(|z_1|,\ldots,|z_r|\right)\to \infty,\end{aligned}$$

where $k = 1,\ldots,r$, $\alpha_k = \min\left[\mathrm{Re}\left(d_j^{(k)}/\delta_j^{(k)}\right)\right]$, $j = 1,\ldots,m_k$ and $\beta_k = \max\left[\mathrm{Re}\left(\left(c_j^{(k)}-1\right)/\gamma_j^{(k)}\right)\right]$, $j = 1,\ldots,n_k$. For convenience, we will also use the following notations in this paper:

$$V = m_1,n_1;\ldots;m_r,n_r,$$

$$W = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\ldots;p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)}, \tag{1.2}$$

(1.3)
$$\begin{aligned}A =&\left\{\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,n}\right\},\ \left\{\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\ldots,\alpha_{ji}^{(r)}\right)_{n+1,p_i}\right\}\left\{\left(c_j^{(1)};\gamma_j^{(1)}\right)_{1,n_1}\right\},\\ &\left\{\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_{i^{(1)}}}\right\};\ldots;\left\{\left(c_j^{(r)};\gamma_j^{(r)}\right)_{1,n_r}\right\},\left\{\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_{i^{(r)}}}\right\}.\end{aligned}$$

(1.4)
$$\begin{aligned}B =&\left\{\tau_i\left(b_{ji};\beta_{ji}^{(1)},\ldots,\beta_{ji}^{(r)}\right)_{m+1,q_i}\right\}:\left\{\left(d_j^{(1)};\delta_j^{(1)}\right)_{1,m_1}\right\},\\ &\left\{\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_{i^{(1)}}}\right\};\ldots;\left\{\left(d_j^{(r)};\delta_j^{(r)}\right)_{1,m_r}\right\},\left\{\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_{i^{(r)}}}\right\}.\end{aligned}$$

2. Required Formula

Recently, Kim et al. [8] have obtained the following integral formula involving generalized hypergeometric function which will be required in our present study. Here and in the following, let $\mathbb{C}$ and $\mathbb{Z}_0^-$ be the sets of complex numbers and non-positive integers, respectively.

Lemma 2.1. *For* $\mathrm{Re}\left(2c-a-b\right) > -1$ *and* $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, *we have the following integral formula, given by*

$$\begin{aligned}&\int_0^1 x^{c-1}\left(1-x\right)^c {}_3F_2\left[\begin{matrix}a,b,d+1;\\ \frac{1}{2}\left(a+b+1\right),d;\end{matrix}\ x\right]\mathrm{d}x\\ =&\frac{\pi\Gamma\left(c\right)4^{-c}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}\\ &+\left(\frac{2c-d}{d}\right)\frac{\pi\Gamma\left(c\right)4^{-c}\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+1\right)}.\end{aligned}$$

3. Main Integrals

In this section, we evaluate the following interesting integral involving generalized hypergeometric function and the multivariable Aleph-function.

Theorem 3.1.
(3.1)

$$\int_0^1 x^{c-1}\,(1-x)^c\;{}_3F_2\left[\begin{array}{c} a,b,d+1; \\ \frac{1}{2}(a+b+1),d; \end{array} x\right] \aleph\begin{pmatrix} z_1\,x^{h_1}\,(1-x)^{h_1} \\ \cdot \\ \cdot \\ z_r\,x^{h_r}\,(1-x)^{h_r}\end{pmatrix}\mathrm{d}x = A_1$$

$$\aleph^{0,\mathfrak{n}+2:V}_{p_i+2,q_i+2,\tau_i;R:W}\left(\begin{array}{c|c} z_1\,4^{-h_1} & (1-c;h_1,\ldots,h_r),\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c;h_1,\ldots,h_r\right),A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\,4^{-h_r} & \left(\frac{1}{2}+\frac{1}{2}a-c;h_1,\ldots,h_r\right),\left(\frac{1}{2}+\frac{1}{2}b-c;h_1,\ldots,h_r\right),B \end{array}\right)$$

$$+\,A_2\,\aleph^{0,\mathfrak{n}+3:V}_{p_i+3,q_i+3,\tau_i;R:W}\left(\begin{array}{c|c} z_1\,4^{-h_1} & (1-c;h_1,\ldots,h_r),(d-2c;2h_1,\ldots,2h_r),\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c;h_1,\ldots,h_r\right),A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\,4^{-h_r} & \left(\frac{1}{2}a-c;h_1,\ldots,h_r\right),\left(\frac{1}{2}b-c;h_1,\ldots,h_r\right),(1+d-2c;2h_1,\ldots,2h_r),B \end{array}\right),$$

where A and B are given by (1.3) *and* (1.4) *respectively. Also,*

$$A_1 = \frac{\pi\,4^{-c}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} \tag{3.2}$$

and

$$A_2 = \frac{\pi\,4^{-c}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{d\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}. \tag{3.3}$$

Provided that

$$h_i>0,\quad \textit{for } i=1,\ldots,r,\ \mathrm{Re}\,(c)>0,\ d\in\mathbb{C}\backslash\mathbb{Z}_0^-,$$
$$\mathrm{Re}\,(c)+\sum_{i=1}^{r}h_i\min_{1\le j\le m_i}\mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0,\quad \textit{for } i=1,\ldots,r,$$
$$\left|\arg z_k\,x^{h_k}\,(1-x)^{h_k}\right|<\frac{1}{2}A_i^{(k)}\,\pi,$$

where $A_i^{(k)}$ is defined by (1.2) *for $k=1,\ldots,r$.*

Proof. To prove (3.1), first we assume the left side of (3.1) by the notation $\mathcal{F}_1$, and then express the Aleph-function of several variables involved on the left hand side of (3.1) in terms of Mellin-Barnes contour integral with the help of (1.1), and next change the order of integrations which is permissible under the stated conditions, so we obtain

$$\mathcal{F}_1 = \frac{1}{(2\pi\omega)^r}\int_{\mathcal{L}_1}\cdots\int_{\mathcal{L}_r}\psi(s_1,\ldots,s_r)\prod_{k=1}^{r}\theta_k(s_k)\,z_k^{s_k}\left[\int_0^1 x^{c+\sum_{i=1}^r h_i s_i-1}\right.$$
$$\left.\times\ (1-x)^{c+\sum_{i=1}^r h_i s_i}\ {}_3F_2\left[\begin{matrix} a,b,d+1; \\ \frac{1}{2}(a+b+1),d; \end{matrix}\ x\right]\mathrm{d}x\right]\mathrm{d}s_1\ldots\mathrm{d}s_r.$$

Now, we evaluate the inner integral with the help of lemma 2.1, after algebraic manipulations, we have

$$\mathcal{F}_1 = \frac{\pi\,4^{-c}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}\ \frac{1}{(2\pi\omega)^r}\int_{\mathcal{L}_1}\cdots\int_{\mathcal{L}_r}\psi(s_1,\ldots,s_r)\prod_{k=1}^{r}\theta_k(s_k)\ z_k^{s_k}$$
$$\times\ \frac{4^{-\sum_{i=1}^r h_i s_i}\,\Gamma\left(c+\sum_{i=1}^r h_i s_i\right)\Gamma\left(c+\sum_{i=1}^r h_i s_i-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c+\sum_{i=1}^r h_i s_i-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c+\sum_{i=1}^r h_i s_i-\frac{1}{2}b+\frac{1}{2}\right)}\,\mathrm{d}s_1\ldots\mathrm{d}s_r$$
$$+\ \frac{\pi\,4^{-c}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{d\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}\ \frac{1}{(2\pi\omega)^r}\int_{\mathcal{L}_1}\cdots\int_{\mathcal{L}_r}\psi(s_1,\ldots,s_r)\prod_{k=1}^{r}\theta_k(s_k)\ z_k^{s_k}$$
$$\times\ \frac{4^{-\sum_{i=1}^r h_i s_i}\Gamma\left(c+\sum_{i=1}^r h_i s_i\right)\Gamma\left(c+\sum_{i=1}^r h_i s_i-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c+\sum_{i=1}^r h_i s_i-\frac{1}{2}a+1\right)\Gamma\left(c+\sum_{i=1}^r h_i s_i-\frac{1}{2}b+1\right)}$$
$$\times\ \frac{\Gamma\left(2c-d+2\sum_{i=1}^r h_i s_i+1\right)}{\Gamma\left(2c-d+2\sum_{i=1}^r h_i s_i\right)}\,\mathrm{d}s_1\ldots\mathrm{d}s_r,$$

and reinterpreting the multiple Mellin-Barnes contour integral in terms of Aleph-functions of r-variables, we obtain the desired result (3.1). $\square$

Theorem 3.2.

$$\int_0^1 x^{\alpha-1}\,(1-x)^{\beta-1}\,\aleph\begin{pmatrix} z_1\,x^{h_1}\ (1-x)^{l_1} \\ \cdot \\ \cdot \\ z_r\,x^{h_r}\ (1-x)^{l_r}\end{pmatrix}\mathrm{d}x$$

$$= \aleph^{0,\mathfrak{n}+2:V}_{p_i+2,q_i+1,\tau_i;R:W}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_r\end{matrix}\ \middle|\ \begin{matrix}(1-\alpha;h_1,\ldots,h_r),\,(1-\beta;l_1,\ldots,l_r),\ A \\ \cdot \\ \cdot \\ (1-\alpha-\beta;h_1+l_1,\ldots,h_r+l_r),\ B\end{matrix}\right), \tag{3.4}$$

here provided that

$$h_i>0,\ l_i>0,\quad \textit{for } i=1,\ldots,r,$$

$$\begin{aligned}
&\mathrm{Re}\,(\alpha)+\sum_{i=1}^{r}h_i\min_{1\le j\le m_i}\mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0,\\
&\mathrm{Re}\,(\beta)+\sum_{i=1}^{r}l_i\min_{1\le j\le m_i}\mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0,\quad i=1,\ldots,r,\\
&\left|\arg z_k\,x^{h_k}\,(1-x)^{l_k}\right|<\frac{1}{2}A_i^{(k)}\pi,
\end{aligned}$$

where $A_i^{(k)}$ *is given by* (1.2) *for* $k=1,\ldots,r$.

Proof. To prove (3.4), we express the Aleph-function of several variables involved on the left hand side of (3.4) in the terms of Mellin-Barnes contour integral with the help of (1.1), and change the order of integrations which is permissible under the stated conditions and use the formula concerning beta-integral to evaluate the inner integral. Now reinterpreting the multiple Mellin-Barnes contour integrals in terms of Aleph-functions of r-variables, we obtain the desired result (3.4). □

4. Application in Obtaining a New Summation Formula

We have the following summation formula concerning the multivariable Aleph-function, defined as

Theorem 4.1.

(4.1)

$$\sum_{s=0}^{\infty}\frac{(a)_s\,(b)_s\,(d+1)_s}{\left(\frac{1}{2}(a+b+1)\right)_s(d)_s\;s!}$$

$$\times\;\aleph^{0,\mathfrak{n}+2:V}_{p_i+2,q_i+1,\tau_i;R:W}\left(\begin{array}{c|c} z_1 & (1-c-s;h_1,\ldots,h_r),(-c;h_1,\ldots,h_r),\,A\\ \cdot & \cdot\\ \cdot & \cdot\\ z_r & (-2c-s;2h_1,\ldots,2h_r),\,B\end{array}\right)=A_1$$

$$\aleph^{0,\mathfrak{n}+2:V}_{p_i+2,q_i+2,\tau_i;R:W}\left(\begin{array}{c|c} z_1\,4^{-h_1} & (1-c;h_1,\ldots,h_r),\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c;h_1,\ldots,h_r\right),\,A\\ \cdot & \cdot\\ \cdot & \cdot\\ z_r\,4^{-h_r} & \left(\frac{1}{2}+\frac{1}{2}a-c;h_1,\ldots,h_r\right),\left(\frac{1}{2}+\frac{1}{2}b-c;h_1,\ldots,h_r\right),\,B\end{array}\right)$$

$$+\,A_2\,\aleph^{0,\mathfrak{n}+3:V}_{p_i+3,q_i+3,\tau_i;R:W}\left(\begin{array}{c|c} z_1\,4^{-h_1} & (1-c;h_1,\ldots,h_r),(d-2c;2h_1,\ldots,2h_r),\\ \cdot & \cdot\\ \cdot & \cdot\\ z_r\,4^{-h_r} & \left(\frac{1}{2}a-c;h_1,\ldots,h_r\right),\left(\frac{1}{2}b-c;h_1,\ldots,h_r\right),\end{array}\right.$$

$$\left.\begin{matrix} \left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c;h_1,\ldots,h_r\right),\ A \\ \cdot \\ \cdot \\ \left(1+d-2c;2h_1,\ldots,2h_r\right),\ B \end{matrix}\right),$$

where A_1 and A_2 are defined in (3.2) *and* (3.3) *respectively, also the validity conditions can easily be obtained from* (3.1).

Proof. We have the following integral denoted by $\mathcal{I}$ (say), given as

$$\mathcal{I}=\int_0^1 x^{c-1}\,(1-x)^c\,{}_3F_2\left[\begin{matrix} a,b,d+1; \\ \frac{1}{2}\,(a+b+1)\,,d; \end{matrix}\ x\right]\aleph\begin{pmatrix} z_1\,x^{h_1}\,(1-x)^{l_1} \\ \cdot \\ \cdot \\ z_r\,x^{h_r}\,(1-x)^{l_r} \end{pmatrix}\mathrm{d}x.$$

Expressing the generalized hypergeometric function ${}_3F_2$ as a series, and after algebraic manipulations we have

$$\mathcal{I}=\sum_{s=0}^{\infty}\frac{(a)_s\,(b)_s\,(d+1)_s}{\left(\frac{1}{2}(a+b+1)\right)_s\,(d)_s\,s!}\int_0^1 x^{c+s-1}\,(1-x)^c\,\aleph\begin{pmatrix} z_1\,x^{h_1}\,(1-x)^{l_1} \\ \cdot \\ \cdot \\ z_r\,x^{h_r}\,(1-x)^{l_r} \end{pmatrix}\mathrm{d}x.$$

Finally, evaluating the above integral with the help of (3.4), we arrive at

$$\begin{aligned} \mathcal{I}=&\sum_{s=0}^{\infty}\frac{(a)_s\,(b)_s\,(d+1)_s}{\left(\frac{1}{2}\,(a+b+1)\right)_s\,(d)_s\,s!} \\ &\times\aleph^{0,\mathfrak{n}+2:V}_{p_i+2,q_i+1,\tau_i;R:W}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_r \end{matrix}\ \middle|\ \begin{matrix} (1-c-s;h_1,\ldots,h_r)\,,(-c;h_1,\ldots,h_r)\,,\ A \\ \cdot \\ \cdot \\ (-2c-s;2h_1,\ldots,2h_r)\,,\ B \end{matrix}\right). \end{aligned}\tag{4.2}$$

Hence, the summation formula (4.1) follows from equating the two integrals (3.1) and (4.2). ☐

When $d=2c$, then above result reduces to the following interesting relation:

$$\begin{aligned} &\sum_{s=0}^{\infty}\frac{(a)_s\,(b)_s\,(2c+1)_s}{\left(\frac{1}{2}\,(a+b+1)\right)_s\,(2c)_s\,s!} \\ &\times\aleph^{0,\mathfrak{n}+2:V}_{p_i+2,q_i+1,\tau_i;R:W}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_r \end{matrix}\ \middle|\ \begin{matrix} (1-c-s;h_1,\ldots,h_r)\,,(-c;h_1,\ldots,h_r)\,,A \\ \cdot \\ \cdot \\ (-2c-s;2h_1,\ldots,2h_r)\,,B \end{matrix}\right)=A_1, \end{aligned}$$

$$\aleph^{0,\mathfrak{n}+2:V}_{p_i+2,q_i+2,\tau_i;R:W}\left(\begin{array}{c|c} z_1\,4^{-h_1} & (1-c;h_1,\dots,h_r)\,,\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c;h_1,\dots,h_r\right),\,A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\,4^{-h_r} & \left(\frac{1}{2}+\frac{1}{2}a-c;h_1,\dots,h_r\right),\left(\frac{1}{2}+\frac{1}{2}b-c;h_1,\dots,h_r\right),\,B \end{array}\right)$$

$$+\,A_2\,\aleph^{0,\mathfrak{n}+3:V}_{p_i+3,q_i+2,\tau_i;R:W}\left(\begin{array}{c|c} z_1\,4^{-h_1} & (1-c;h_1,\dots,h_r)\,,(0;2h_1,\dots,2h_r)\,, \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\,4^{-h_r} & \left(\frac{1}{2}a-c;h_1,\dots,h_r\right),\left(\frac{1}{2}b-c;h_1,\dots,h_r\right), \end{array}\right.$$

$$\left.\begin{array}{l} \left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c;h_1,\dots,h_r\right),\,A \\ \cdot \\ \cdot \\ (1;2h_1,\dots,2h_r)\,,\,B \end{array}\right).$$

5. Special Cases

In this section, we will see the interesting special cases of integral formula (3.1) and summation formula (4.1).

Let $b=-2s$ and replace a by $a+2s$, where s is zero or a positive integer. In such case, one of the two terms on the right hand side of (3.1) will be vanished and we get the following interesting result, as concerning by the following corollary.

Corollary 5.1.

$$\int_0^1 x^{c-1}\,(1-x)^c\;{}_3F_2\left[\begin{array}{c} a+2s,-2s,d+1; \\ \frac{1}{2}(a+1)\,,d; \end{array}\; x\right]\aleph\left(\begin{array}{c} z_1\,x^{h_1}\,(1-x)^{h_1} \\ \cdot \\ \cdot \\ z_r\,x^{h_r}\,(1-x)^{h_r} \end{array}\right)\mathrm{d}x$$

$$=\frac{(-)^s\,\sqrt{\pi}\,\left(\frac{1}{2}\right)_s}{4^c\,\left(\frac{1}{2}a+\frac{1}{2}\right)_s}\;\aleph^{0,\mathfrak{n}+2:V}_{p_i+2,q_i+2,\tau_i;R:W}\left(\begin{array}{c|c} z_1\,4^{-h_1} & (1-c;h_1,\dots,h_r)\,, \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\,4^{-h_r} & \left(\frac{1}{2}+\frac{1}{2}a+s-c;h_1,\dots,h_r\right), \end{array}\right.$$

$$\left.\begin{array}{c} \left(\frac{1}{2}+\frac{1}{2}a-c;h_1,\dots,h_r\right),\,A \\ \cdot \\ \cdot \\ \left(\frac{1}{2}-s-c;h_1,\dots,h_r\right),\,B \end{array}\right),$$

provided that the condition easily obtainable from (3.1) *is satisfied.*

Let $b=-2s-1$ and replace a by $a+2s+1$, where s is zero or a positive integer. Then, one of the two terms on the right hand side of (3.1) will vanish and we get the following corollary.

Corollary 5.2. *By assuming that the validity condition easily obtainable from* (3.1) *is satisfied, then we have*

$$\int_0^1 x^{c-1}\,(1-x)^c\,{}_3F_2\left[\begin{array}{c} a+2s+1,-2s-1,d+1;\\ \frac{1}{2}(a+1),d; \end{array} x\right]\aleph\left(\begin{array}{c} z_1\,x^{h_1}\,(1-x)^{h_1}\\ \cdot\\ \cdot\\ z_r\,x^{h_r}\,(1-x)^{h_r}\end{array}\right)\mathrm{d}x$$

$$=\frac{(-)^{s-1}\,\sqrt{\pi}\,\left(\frac{3}{2}\right)_s}{d\,2^{2c+1}\,\left(\frac{1}{2}a+\frac{1}{2}\right)_s}\,\aleph^{0,\mathfrak{n}+3:V}_{p_i+3,q_i+3,\tau_i;R:W}\left(\begin{array}{c|c} z_1\,4^{-h_1} & (1-c;h_1,\ldots,h_r),\\ \cdot & \cdot\\ \cdot & \cdot\\ z_r\,4^{-h_r} & \left(\frac{1}{2}+\frac{1}{2}a+s-c;h_1,\ldots,h_r\right),\end{array}\right.$$

$$\left.\begin{array}{l}\left(\frac{1}{2}+\frac{1}{2}a-c;h_1,\ldots,h_r\right),(d-2c;2h_1,\ldots,2h_r),\ A\\ \cdot\\ \cdot\\ \left(-\frac{1}{2}-s-c;h_1,\ldots,h_r\right),(1+d-2c;2h_1,\ldots,2h_r),\ B\end{array}\right).$$

Next, we will provide the special cases of the summation formula (4.1). Concerning the following corollary, we consider the Aleph-function of one variable defined by Südland et al. [26, 27] (see also, Saxena et al. [18]).

Corollary 5.3.

$$\sum_{s=0}^{\infty}\frac{(a)_s\,(b)_s\,(d+1)_s}{\left(\frac{1}{2}(a+b+1)\right)_s\,(d)_s\,s!}\,\aleph^{m_1,n_1+2}_{p_1+2,q_1+1,\tau_{i^{(1)}},R^{(1)}}\left(z_1\left|\begin{array}{c}(1-c-s;h_1),(-c;h_1),\ \mathbf{A}\\ \cdot\\ \cdot\\ (-2c-s;2h_1),\ \mathbf{B}\end{array}\right.\right)$$

$$=A_1\,\aleph^{m_1,n_1+2}_{p_1+2,q_1+2,\tau_{i^{(1)}},R^{(1)}}\left(z_1\,4^{-h_1}\left|\begin{array}{c}(1-c;h_1),\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c;h_1\right),\ \mathbf{A}\\ \cdot\\ \cdot\\ \left(\frac{1}{2}+\frac{1}{2}a-c;h_1\right),\left(\frac{1}{2}+\frac{1}{2}b-c;h_1\right),\ \mathbf{B}\end{array}\right.\right)$$

$$+A_2\,\aleph^{m_1,n_1+3}_{p_1+3,q_1+3,\tau_{i^{(1)}},R^{(1)}}\left(z_1\,4^{-h_1}\left|\begin{array}{c}(1-c;h_1),(d-2c;2h_1),\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c;h_1\right),\mathbf{A}\\ \cdot\\ \cdot\\ \left(\frac{1}{2}a-c;h_1\right),\left(\frac{1}{2}b-c;h_1\right),(1+d-2c;2h_1),\mathbf{B}\end{array}\right.\right),$$

where

$$\mathbf{A}=\left\{\left(c_j^{(1)};\gamma_j^{(1)}\right)_{1,n_1}\right\},\ \left\{\tau_{i^{(1)}}\left(c^{(1)}_{ji^{(1)}};\gamma^{(1)}_{ji^{(1)}}\right)_{n_1+1,p_{i^{(1)}}}\right\}$$

and

$$\mathbf{B}=\left\{\left(d_j^{(1)};\delta_j^{(1)}\right)_{1,m_1}\right\},\ \left\{\tau_{i^{(1)}}\left(d^{(1)}_{ji^{(1)}};\delta^{(1)}_{ji^{(1)}}\right)_{m_1+1,q_{i^{(1)}}}\right\}.$$

Provided that:

$$h_1 > 0,\ \mathrm{Re}\,(c) > 0, \quad d \neq 0, -1, -2, \ldots,$$

$$\mathrm{Re}\,(c) + h_1 \min_{1 \le l \le m_1} \mathrm{Re}\left(\frac{d_l^{(1)}}{\delta_l^{(1)}}\right) > 0, \quad \left|\arg z_1 x^{h_1}\ (1-x)^{h_1}\right| < \frac{1}{2}\pi,$$

$$\left(\sum_{j=1}^{n_1} \gamma_j^{(1)} - \tau_{i^{(1)}} \sum_{j=n_1+1}^{p_{i^{(1)}}} \gamma_{ji^{(1)}}^{(1)} + \sum_{j=1}^{m_1} \delta_j^{(1)} - \tau_{i^{(1)}} \sum_{j=m_1+1}^{q_{i^{(1)}}} \delta_{ji^{(1)}}^{(1)}\right) > 0.$$

Now, we consider the I-function defined by Saxena [19]. We have the following result.

Corollary 5.4.

$$\sum_{s=0}^{\infty} \frac{(a)_s\ (b)_s\ (d+1)_s}{\left(\frac{1}{2}(a+b+1)\right)_s\ (d)_s\ s!}\ I^{m_1,n_1+2}_{p_1+2,q_1+1;R^{(1)}}\left(z_1 \middle| \begin{matrix} (1-c-s;h_1), (-c;h_1),\ \mathbf{A}' \\ \cdot \\ \cdot \\ (-2c-s;2h_1),\ \mathbf{B}' \end{matrix}\right)$$

$$= A_1\ I^{m_1,n_1+2}_{p_1+2,q_1+2;R^{(1)}}\left(z_1\, 4^{-h_1} \middle| \begin{matrix} (1-c;h_1), \left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c;h_1\right),\ \mathbf{A}' \\ \cdot \\ \cdot \\ \left(\frac{1}{2}+\frac{1}{2}a-c;h_1\right), \left(\frac{1}{2}+\frac{1}{2}b-c;h_1\right),\ \mathbf{B}' \end{matrix}\right)$$

$$+ A_2\ I^{m_1,n_1+3}_{p_1+3,q_1+3;R^{(1)}}\left(z_1\, 4^{-h_1} \middle| \begin{matrix} (1-c;h_1), (d-2c;2h_1), \left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c;h_1\right),\ \mathbf{A}' \\ \cdot \\ \cdot \\ \left(\frac{1}{2}a-c;h_1\right), \left(\frac{1}{2}b-c;h_1\right), (1+d-2c;2h_1),\ \mathbf{B}' \end{matrix}\right),$$

where

$$\mathbf{A}' = \left\{\left(c_j^{(1)};\gamma_j^{(1)}\right)_{1,n_1}\right\}, \left\{\left(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_{i^{(1)}}}\right\}$$

and

$$\mathbf{B}' = \left\{\left(d_j^{(1)};\delta_j^{(1)}\right)_{1,m_1}\right\}, \left\{\left(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_{i^{(1)}}}\right\}.$$

Provided that

$$h_1 > 0,\ \mathrm{Re}\,(c) > 0, \quad d \neq 0, -1, -2, \ldots,$$

$$\mathrm{Re}\,(c) + h_1 \min_{1 \le l \le m_1} \mathrm{Re}\left(\frac{d_l^{(1)}}{\delta_l^{(1)}}\right) > 0, \quad \left|\arg z_1 x^{h_1}\ (1-x)^{h_1}\right| < \frac{1}{2}\pi,$$

$$\left(\sum_{j=1}^{n_1} \gamma_j^{(1)} - \sum_{j=n_1+1}^{p_{i^{(1)}}} \gamma_{ji^{(1)}}^{(1)} + \sum_{j=1}^{m_1} \delta_j^{(1)} - \sum_{j=m_1+1}^{q_{i^{(1)}}} \delta_{ji^{(1)}}^{(1)}\right) > 0.$$

Remark 5.1. By the similar methods, we can obtain the similar summation formula with the Aleph-function of two variables (see [9]), the I-function of two variables (see [12, 21]), the multivariable I-function (see [13, 15]), the multivariable A-function (see [7]), the A-function [6], the modified multivariable H-function (see [14]) and the multivariable H-function (see [3, 4, 10, 24, 25]).

6. Concluding Remarks

In this paper, we have established two integrals formulas and one summation formula involving the generalized hypergeometric function and Aleph-function of r-variables. On account of the most general character of the multivariable Aleph-function in Theorems 3.1, 3.2 and 4.1, numerous other special cases associated with potentially useful higher transcendental functions, orthogonal polynomials of one and several variables can be deduced.

References

[1] F. Ayant, *An integral associated with the aleph-functions of several variables*, International Journal of Mathematics Trends and Technology **31** (2016), 142–154.

[2] F. Ayant and D. Kumar, *Certain finite double integrals involving the hypergeometric function and aleph-function*, International Journal of Mathematics Trends and Technology **35** (2016), 49–55.

[3] J. Choi, J. Daiya, D. Kumar and R. Saxena, *Fractional differentiation of the product of appell function F_3 and multivariable H-function*, Commun. Korean Math. Soc. **31** (2016), 115–129.

[4] J. Daiya, J. Ram and D. Kumar, *The multivariable H-function and the general class of srivastava polynomials involving the generalized mellin-barnes contour integrals*, Filomat **30** (2016), 1457–1464.

[5] C. Fox, *The G and H-functions as symmetrical fourier kernels*, Trans. Amer. Math. Soc. **98** (1961), 395–429.

[6] B. Gautam and A. Asga, *The A-Function*, Revista Mathematica, Tucuman, 1980.

[7] B. Gautam and A. Asga, *On the multivariable A-function*, Vijnana Parishas Anusandhan Patrika **29** (1986), 67–81.

[8] Y. Kim, M. Rakha and A. Rathie, *Extension of certain classical summations theorems for the series ${}_2F_1$, ${}_3F_2$ and ${}_4F_3$ with applications in ramanujan's summations*, Int. J. Math. Sci. **2010** (2010), Article ID 309503, 26 pages.

[9] D. Kumar, *Generalized fractional differintegral operators of the aleph-function of two variables*, Journal of Chemical, Biological and Physical Sciences - Section C **6** (2016), 1116–1131.

[10] D. Kumar, S. Purohit and J. Choi, *Generalized fractional integrals involving product of multivariable H-function and a general class of polynomials*, J. Nonlinear Sci. Appl. **9** (2016), 8–21.

[11] D. Kumar and J. Singh, *Application of generalized M-series and $\bar{H}$-function in electric circuit theory*, MESA **7** (2016), 503–512.

[12] K. Kumari, T. V. Nambisan and A. Rathie, *A study of I-function of two variables*, Le Matematiche **69** (2014), 285–305.

[13] Y. Prasad, *Multivariable I-function*, Vijnana Parishad Anusandhan Patrika **29** (1986), 231–237.

[14] Y. Prasad and A. Singh, *Basic properties of the transform involving and H-function of r-variables as kernel*, Indian Acad Math. **2** (1982), 109–115.

[15] J. Prathima, V. Nambisan and S. Kurumujji, *A study of I-function of several complex variables*, International Journal of Engineering Mathematics **2014** (2014), 1–12.

[16] J. Ram and D. Kumar, *Generalized fractional integration involving appell hypergeometric of the product of two H-functions*, Vijanana Parishad Anusandhan Patrika **54** (2011), 33–43.

[17] V. Rohira, K. Kumari and A. Rathie, *A new class of integral involving generalized hypergeometric function and the H-function*, International Journal of Latest Engineering Research and Applications **2** (2017), 5–9.

[18] R. Saxena, J. Ram and D. Kumar, *Generalized fractional integral of the product of two aleph-functions*, Appl. Appl. Math. **8** (2013), 631–646.

[19] V. Saxena, *Formal solution of certain new pair of dual integral equations involving H-function*, Proc. Nat. Acad. Sci. India Sect. A **51** (2001), 366–375.

[20] C. Sharma and S. Ahmad, *On the multivariable I-function*, Acta Ciencia Indica: Mathematics **20** (1994), 113–116.

[21] C. Sharma and P. Mishra, *On the I-function of two variables and its properties*, Acta Ciencia Indica: Mathematics **17** (1991), 667–672.

[22] J. Singh and D. Kumar, *On the distribution of mixed sum of independent random variables one of them associated with srivastava's polynomials and $\bar{H}$-function*, J. Appl. Math. Stat. Inform. **10** (2014), 53–62.

[23] H. Srivastava, K. Gupta and S. Goyal, *The H-Function of One and Two Variables with Applications*, South Asian Publications, New Delhi, Madras, 1982.

[24] H. Srivastava and R. Panda, *Some expansion theorems and generating relations for the H-function of several complex variables*, Comment. Math. Univ. St. Paul. **25** (1976), 119–137.

[25] H. Srivastava and R. Panda, *Some expansion theorems and generating relations for the H-function of several complex variables II*, Comment. Math. Univ. St. Paul. **25** (1976), 167–197.

[26] N. Südland, B. Baumann and T. Nonnenmacher, *Open problem: who knows about the aleph-function?* Fract. Calc. Appl. Anal. **1** (1998), 401–402.

[27] N. Südland, B. Baumann and T. Nonnenmacher, *Fractional drift-less fokker-planck equation with power law diffusion coefficients*, in: V. G. Gangha, E. W. Mayr and E. V. Vorozhtsov (Eds.), *Computer Algebra in Scientific Computing* (*CASC Konstanz 2001*), Springer-Verlag, Berlin, 2001, 513–525.

[1]Department of Applied Sciences,
College of Agriculture, Sumerpur- Pali,
Agriculture University of Jodhpur, Jodhpur 342304, India
Email address: dinesh_dino03@yahoo.com

[2]Collége Jean L'herminier,
Allée des Nymphéas, 83500 La Seyne-sur-Mer, France

[3]Six-Fours-les-Plages-83140, Department of Var, France
Email address: fredericayant@gmail.com

[4]Department of Mathematics,
University of Rajasthan,
Jaipur-302004, Rajasthan, India
Email address: devendra.maths@gmail.com