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CONNECTEDNESS OF THE CUT SYSTEM COMPLEX ON NONORIENTABLE SURFACES

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Abstract. Let N be a compact, connected, nonorientable surface of genus g with n boundary components. In this note, we show that the cut system complex of N is connected for $g < 4$ and disconnected for $g \geq 4$. We then define a related complex and show that it is connected for $g \geq 4$.

1. Introduction

Complexes of curves and cut system complexes of surfaces are fundamental geometric objects in geometric topology. Let S be a compact, connected, orientable or nonorientable surface of genus $g \geq 1$ with n boundary components. Complexes of curves, denoted by $C(S)$, have been introduced by Harvey in [3, 4]. Other geometric objects on surfaces include the cut system complex introduced by Hatcher and Thurston in [5]. They have played an ever increasing role since then.

In this note, we show that the cut system complexes of nonorientable surfaces are connected for $g < 4$ and disconnected for $g \geq 4$. We then introduce a related complex and show that it is connected for $g \geq 4$.

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2. Preliminaries

Let a be a simple closed curve on S and let S_a denote the surface obtained by cutting S along a. We call a on the surface S nonseparating if S_a is connected, and separating otherwise. We denote a curve or its isotopy class by the same notation throughout this article. Let Σ be a compact, connected, orientable surface of genus g

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with n boundary components. Let us consider collections of g disjoint nonseparating simple closed curves $a_1, a_2, \dots, a_g$ on Σ, whose complement $\Sigma \setminus (a_1 \cup \dots \cup a_g)$ is a sphere with $2g+n$ boundary components. The collection of their isotopy classes is called a cut system on Σ. Let $\langle a_1, a_2, \dots, a_g \rangle$ be a cut system on Σ. Assume that for some k, a'_k is a nonseparating simple closed curve transversely intersecting a_k at exactly one point and disjoint from all a_i for $i \neq k$. Then if we replace a_k by a'_k in the cut system, we obtain another cut system on Σ. This operation of replacing curves is called an elementary move. There are three special types of paths which is described in [5]. These special types of paths play an important role in the construction of the cut system complex. The cut system complex of a surface Σ is a cell complex of dimension 2. Each cut system is a 0-cell (vertex) of this complex. If two cells are related by an elementary move then these two 0-cells are joined by a 1-cell (an unoriented edge) corresponding to this move. Now, we have a graph, in other words; a 1-dimensional cell complex containing the 0-cells and the 1-cells. Finally, we attach 2-cells to this graph along the boundaries resulting from the three special types of paths to get the complex.

Hatcher and Thurston showed that the cut system complex of an orientable surface is connected in [5]. Later, Wajnryb proved the same result by elementary techniques in [6].

An analogous complex for nonorientable surfaces seems to be the following. Let N be a compact, connected nonorientable surface of genus g with n boundary components. If the regular neighborhood of the curve a is a Möbius band or an annulus, then we say that a is one-sided or two-sided, respectively. We note that all one-sided simple closed curves on N are nonseparating. In addition, there are two topological types of one-sided simple closed curves on nonorientable surfaces of odd genus $g \geq 3$. Let a be a one-sided simple closed curve. We call a a one-sided *essential* simple closed curve if either $g = 1$ or $g \geq 2$ and the surface N_a is nonorientable. Otherwise, we say that a is a one-sided *characteristic* simple closed curve. A cut system on the nonorientable surface N is defined by taking a family of pairwise disjoint one-sided essential simple closed curves. Explicitly, let $\{a_1, a_2, \dots, a_g\}$ be a collection of pairwise disjoint one-sided essential simple closed curves on the surface N. Then, the collection of their isotopy classes $\langle a_1, a_2, \dots, a_g \rangle$ is said to be a cut system if the surface obtained from N by cutting along all a_i in the collection is a sphere with $g+n$ boundary components. Let $\langle a_1, a_2, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_g \rangle$ be a cut system on the surface N. Let a'_i be a one-sided essential simple closed curve on the surface N disjoint from a_k for $k \neq i$, $1 \leq i \leq g$ and such that it intersects a_i at one point and does not intersect other one-sided essential simple closed curves in the collection $\{a_1, a_2, \dots, a_g\}$. Similar to the orientable case, if we change a_i by a'_i in the collection, and we get a new cut system $\langle a_1, a_2, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_g \rangle$. This operation, introduced by Ashiba in [1], is called an elementary move. Also, the cut system complex, denoted by $\mathcal{O}(N)$, of a nonorientable surface is described in a similar fashion to the orientable case. As we will show in the next section, unfortunately this complex is not connected for the genus $g \geq 4$.

3. Main Theorem

In this section, firstly, we will explain below why $\mathcal{O}(N)$ is not connected for $g \geq 4$. Let $v_1 = \langle d_1, a_2, \ldots, a_g \rangle$ and $v_2 = \langle d_2, a_2, \ldots, a_g \rangle$ be vertices of the complex which are connected by an edge. Let $\bar{N}$ denote the surface N whose holes are filled with discs. Since d_1 and d_2 intersect transversally once and they are disjoint from all a_i's we see that the homology classes of d_1 and d_2 are the same. This is because in the surface $\bar{N}$ both sums of the homology classes $[d_1] + [a_2] + \cdots + [a_g]$ and $[d_2] + [a_2] + \cdots + [a_g]$ are the Poincaré dual to the first Stiefel Whitney class so that $[d_1] + [a_2] + \cdots + [a_g] = [d_2] + [a_2] + \cdots + [a_g]$ and thus $[d_1] = [d_2]$. As a conclusion, we see that if two vertices of the complex are connected by an edge path then the homology classes of the isotopy classes in these vertices are pairwise identical in $\bar{N}$. In other words, if two vertices have a non-common homology class represented by the isotopy classes contained in them then these two vertices are not connected by an edge path. Such isotopy classes can be found for any $g \geq 4$, hence, the cut system complex cannot be connected in that case.

The above explanation raises the following question.

Question. What if we take $g - 1$ pairwise disjoint one-sided essential simple closed curves $\{a_1, a_2, \ldots, a_{g-1}\}$ on N as vertices, would the corresponding cut system complex made of these vertices be connected?

The answer is still negative. Indeed, if $v_1 = \langle d_1, a_2, \ldots, a_{g-1} \rangle$ and $v_2 = \langle d_2, a_2, \ldots, a_{g-1} \rangle$ are vertices of the complex, which are connected by an edge then cutting N along all a_i's, we see that d_1 and d_2 are two one-sided curves inside a holed Klein Bottle intersecting transversally at one point. Hence again the homology classes $[d_1]$ and $[d_2]$ must coincide in $\bar{N}$. Therefore, the cut system complex still is not connected.

As a result of these observations, we have the following results.

Theorem 3.1. *Let N be a nonorientable surface of genus g with n holes, where $g < 4$. Then the cut system complex $\mathcal{O}(N)$ is connected.*

Let us take $g - 2$ pairwise disjoint one-sided essential simple closed curves $\{a_1, a_2, \ldots, a_{g-2}\}$ on N as vertices and let $X(N)$ denote the corresponding cut system complex made of these vertices. We call it partial cut system complex.

Theorem 3.2. *Let N be a nonorientable surface of genus g with n holes, where $g \geq 4$. Then the partial cut system complex $X(N)$ is connected.*

The idea of the proofs is that any two vertices are connected by an edge path in the complex. We use Wajnryb's technique and follow his proof. The main ingredient used in the proof is the following proposition which is proved by Atalan and Korkmaz in [2].

Proposition 3.1. *Let N be a nonorientable surface of genus g with n boundary components. Let d_1 and d_2 be two one-sided essential simple closed curves on the surface N such that $i(d_1, d_2) = k$, where $k \geq 2$. In this case, there is a one-sided essential simple closed curve d such that $i(d, d_1) < k$ and $i(d, d_2) < k$.*

Proof of Theorem 3.1. Let $g = 1$. In this case, a cut system (on the surface N) contains an isotopy class of a single curve. If two distinct one-sided curves intersect at one point, we connect them by an edge. Since any two essential one-sided curves on a genus one nonorientable surface intersect, using induction it follows from Proposition 3.1 that any two one-sided essential curves can be joined by an edge path in the cut system complex $\mathcal{O}(N)$.

Let $g = 2$. Let v_1 and v_2 be any two vertices of the complex $\mathcal{O}(N)$. We will show that there exists an edge path $P = (v_1 = s_1, s_2, \ldots, s_k = v_2)$ connecting v_1 and v_2. There are two cases.

Case 1. Suppose that the vertices v_1 and v_2 have one isotopy class of one-sided essential simple closed curve in common, say d. Let us cut the surface N along the curve d. The collection of the remaining one-sided essential simple closed curves constitute two vertices of the cut system complex on the obtained surface of genus one. We have showed that the complex $\mathcal{O}(N)$ is connected for $g = 1$. So, they can be connected by a path. Including this common curve d to each of the vertices of this path we obtain a path in $\mathcal{O}(N)$ connecting v_1 to v_2.

Case 2. Suppose that the vertices v_1 and v_2 do not have any common isotopy class of one-sided essential simple closed curves. Let d_1 and d_2 be two different isotopy classes of one-sided essential simple closed curves on N such that v_1 and v_2 contain d_1 and d_2, respectively. Then, we need to show that there exists an edge path connecting v_1 and v_2. To prove this, as in the proof of Lemma 17 in [6], we will use induction on $i(d_1, d_2) = n$.

There are three subcases.

Subcase (i). Let $i(d_1, d_2) = 0$. Then there is a vertex u containing both one-sided essential curves d_1 and d_2. Hence, the vertex u is connected to v_1 and v_2 as in Case 1.

Subcase (ii). Let $i(d_1, d_2) = 1$. The regular neighborhood of $d_1 \cup d_2$ is a two-holed real projective plane. Let us denote $N_{d_1 \cup d_2}$ the surface obtained by cutting N along d_1 and d_2. Since $g = 2$, $N_{d_1 \cup d_2}$has necessarily two components, one of which is a nonorientable surface of genus one, so that we can find a one-sided essential simple closed curve disjoint from d_1 and d_2, say e. Now, we can find two vertices w_1 and w_2 in the complex $\mathcal{O}(N)$ which are joined by an edge and such that w_1 and w_2 contain d_1 and d_2, respectively. In other words, $w_1 = \langle d_1, e \rangle$ and $w_2 = \langle d_2, e \rangle$ are connected by an edge. Finally, we join v_1 to w_1 and v_2 to w_2 as in Case 1. Therefore, we can connect v_1 and v_2.

Subcase (iii). Let $i(d_1, d_2) = n > 1$. By Proposition 3.1, there is a one-sided essential simple closed curve d such that $i(d_1, d) < n$ and $i(d_2, d) < n$. We choose a vertex u containing d. By induction on n, we can connect the vertex u to v_1 and v_2 in the cut system complex $\mathcal{O}(N)$.

For the case $g = 3$ the proof is the same as for the case $g = 2$ except the Subcase (ii), which we include below.

Subcase (ii). Let $i(d_1, d_2) = 1$. Since $g = 3$ and d_1 and d_2 are both essential, $N_{d_1 \cup d_2}$ has necessarily two components. Moreover, either both components are nonorientable of genus one or one of the components is a nonorientable surface of genus two and the other is a sphere with holes. Hence, in each case, we can find two disjoint one-sided essential simple closed curves e_1 and e_2 that they are both disjoint from d_1 and d_2. Now, we can find two vertices w_1 and w_2 in the complex which are joined by an edge and such that w_1 and w_2 contain d_1 and d_2, respectively. In other words, $w_1 = \langle d_1, e_1, e_2\rangle$ and $w_2 = \langle d_2, e_1, e_2\rangle$ are connected by an edge. Finally, we join w_1 to v_1 and v_2 to w_2 as in Case 1. Therefore, we can connect v_1 and v_2.

This completes the proof of the theorem. □

Proof of Theorem 3.2. We use induction on the genus of the surface N for $g \geq 4$.

Let $g = 4$. As in the above proof, there are two cases.

Case 1. Assume that v_1 and v_2 have one isotopy class of one-sided essential curve in common, say d. Let e and f be the other one-sided essential simple closed curves of v_1 and v_2, respectively. In other words, let $v_1 = \langle d, e\rangle$ and $v_2 = \langle d, f\rangle$. We will show that v_1 is connected to v_2 by an edge path. In this case, there are three possibilities.

- The one-sided essential curves e and f are disjoint. Then, since d is disjoint from both e and f, and these all curves are one-sided essential, there is another essential simple closed curve, say g, in the complement of $d \cup e \cup f$. Hence, we can find an essential one-sided simple closed curve c representing $\mathbb{Z}_2$-homology class $[e] + [f] + [g]$ such that c intersects each of e and f at only one point. Thus, we obtain the required path.
- The one-sided essential curves e and f intersect at one point. Then there is nothing to prove.
- The one-sided essential curves e and f intersect at least two points. Let us cut the surface N along the curve d. We get a nonorientable surface of genus three, say N_d, in which $i(e, f) = k \geq 2$. First assume that the curves e and f are still essential in N_d. Then, by Proposition 3.1, there is a one-sided essential simple closed curve c such that $i(e, c) < k$ and $i(f, c) < k$.

Now assume without loss of generality that the curve e is not essential in N_d. Hence e is characteristic in N_d. Since f is one-sided and e is characteristic the integer k must be odd. Take representatives for e and f and a push-off of f that intersects f transversally at one point. Also a take a slight perturbation f' of the push-off. By inspection we see that our curves must be as in the diagram at Figure 1.

This new curve f' has the same homology class as f. It has at most two components. If it is connected then it, call it c, intersects f in one point and e in $k-2$ points. Since $k-2$ is odd, c is still one-sided. If the f' is not connected, then by homology arguments

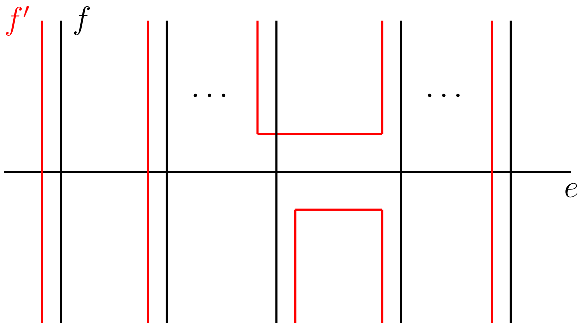

FIGURE 1.

one of the components must be one-sided and the other one must be two-sided. The one-sided component, call it c again, will intersect f in at most one point and e in at most $k-2$ points. The curve c may not be essential in N_d but certainly is essential in N.

In particular, in all cases we have obtained a one-sided essential simple closed curve c such that $i(e,c) < k$ and $i(f,c) < k$. Finally, using an induction argument we obtain two sequences of essential one-sided simple closed curves $e = e_0, e_1, \dots, e_j = c$ and $c = c_0, c_1, \dots, c_s = f$ such that any two adjacent essential one-sided simple closed curves in the sequences intersect once. Then the sequence $e = e_0, e_1, \dots, e_j = c = c_0, c_1, \dots, c_s = f$ gives the required path in $X(N)$.

Case 2. Assume that the vertices v_1 and v_2 do not have any common isotopy class of one-sided essential simple closed curves. Let d_1 and d_2 be two different isotopy classes of one-sided essential simple closed curves on N such that v_1 and v_2 contain d_1 and d_2, respectively. Then, we show that there exists an edge path connecting v_1 and v_2. To prove this, as in the proof of Lemma 17 in [6], we will use induction on $i(d_1, d_2) = n$.

There are three subcases.

Subcase (i). Let $i(d_1, d_2) = 0$. Then, we can construct a vertex $u = \langle d_1, d_2 \rangle$. Thus, u is connected to v_1 and v_2 as in Case 1.

Subcase (ii). Let $i(d_1, d_2) = 1$. The regular neighborhood of $d_1 \cup d_2$ is a two-holed real projective plane. Then, there are two possibilities. One possibility is that $N_{d_1 \cup d_2}$ is a connected nonorientable surface of genus 1 (note that $N_{d_1 \cup d_2}$ cannot be connected and orientable since both d_1 and d_2 are essential). Hence, we can find a one-sided essential simple closed curve disjoint from d_1 and d_2. Again we can find two vertices w_1 and w_2 containing d_1 and d_2, respectively, such that they are connected by an edge. Hence, we join w_1 to v_1 and v_2 to w_2 as in Case 1. The other possibility is that $N_{d_1 \cup d_2}$ is disconnected. We can notice that at least one component of $N_{d_1 \cup d_2}$ must be nonorientable, so in this case we can also find a one-sided essential simple closed curve disjoint from d_1 and d_2, which allows to treat the disconnected case similarly. Therefore, we can connect v_1 and v_2 by a path.

Subcase (iii). Let $i(d_1, d_2) = n > 1$. By Proposition 3.1, there is a one-sided essential simple closed curve d such that $i(d_1, d) < n$ and $i(d_2, d) < n$. Let us pick a vertex u containing d. By induction on n, we can join u to v_1 and v_2.

Let $g \geq 5$. By the induction hypothesis, we assume that the theorem holds for a nonorientable surface of genus less than g. We will prove that the complex $X(N)$ is connected for a nonorientable surface of genus g. Let v_1 and v_2 be any two vertices of the complex $X(N)$. We will prove that these two vertices are connected by an edge path.

Case 1. Suppose that v_1 and v_2 have one isotopy class of one-sided essential curve in common, say d. Let us cut the surface N along the curve d. The collection of the remaining one-sided essential simple closed curves constitute two vertices of the cut system complex on the obtained surface of smaller genus. By the induction hypothesis, they can be connected by a path. Including this common curve d to each of the vertices of this path we obtain a path in $X(N)$ connecting v_1 to v_2.

Case 2. Suppose that v_1 and v_2 do not have any common isotopy classes of one-sided essential simple closed curves. Let d_1 and d_2 be two different isotopy classes of one-sided essential simple closed curves on N such that v_1 and v_2 contain d_1 and d_2, respectively. To prove the existence of an edge path joining v_1 and v_2, as in the proof of Lemma 17 in [6], we will use induction on $i(d_1, d_2) = n$.

There are three subcases.

Subcase (i). Let $i(d_1, d_2) = 1$. The regular neighborhood of $d_1 \cup d_2$ is a two-holed real projective plane. Again, we have two possibilities. One possibility is that $N_{d_1 \cup d_2}$ is a connected nonorientable surface of genus $g - 3$. So, one can choose pairwise disjoint $g - 3$ one-sided essential curves disjoint from d_1 and d_2. Then, there are vertices w_1 and w_2 containing d_1 and d_2, respectively, such that they are joined by an edge. Thus, we connect w_1 to v_1 and v_2 to w_2 as in Case 1. The other possibility is that $N_{d_1 \cup d_2}$ is disconnected. However, by Theorem 3.10 in [2], we can find a sequence of essential one-sided simple closed curves $d_1 = a_1, a_2, \ldots, a_k = d_2$ such that any two adjacent curves a_i and a_{i+1} in the sequence intersect once and $N_{a_i \cup a_{i+1}}$ is connected, where $N_{a_i \cup a_{i+1}}$ is the surface obtained by cutting N along a_i and a_{i+1}. Therefore, using the idea of the previous possibility, we can connect v_1 and v_2 by a path.

Subcase (ii). Let $i(d_1, d_2) = 0$. If $[d_1] + [d_2]$ is not characteristic, then there is a vertex u containing both curves d_1 and d_2. Hence, the vertex u is connected to v_1 and v_2 as in Case 1. Now, assume that $[d_1] + [d_2]$ is characteristic, in this case $g \geq 6$. Then, $[d_2]$ is characteristic on N_{d_1} which is a connected nonorientable surface of genus $g \geq 5$. Without lost of generality, we can choose a one-sided essential curve c on N_{d_1}, which is not characteristic such that $i(c, d_2) = 1$. This implies that $[c] + [d_1]$ is not characteristic on N. So there is a vertex u containing both curves d_1 and c. Now, we can connect the vertex u and v_1 as in Case 1. Moreover, u can be connected to v_2 because $i(c, d_2) = 1$ by Subcase (i) above.

Subcase (iii). Let $i(d_1, d_2) = n > 1$. By Proposition 3.1, there is a one-sided essential simple closed curve d such that $i(d_1, d) < n$ and $i(d_2, d) < n$. We pick a vertex u containing d. By induction on n, we can connect the vertex u to v_1 and v_2 in the partial complex $X(N)$.

This finishes the proof of the theorem. □

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