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LONG TIME DYNAMICS FOR A COUPLED LAMÉ SYSTEM WITH PAST HISTORY

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Abstract. The focus of the present paper is on the investigation of the long time dynamical behavior for a coupled Lamé system with past history. First, we establish the existence of a global solution under some suitable assumptions. Furthermore, we prove the existence of a global attractor with finite fractal dimension. A result on the existence of exponential attractors for the system is also derived.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. We consider the long-time dynamics of the following coupled Lamé system

$$
\begin{cases}
u_{tt} + \alpha v - \Delta_e u - \int_0^{+\infty} \omega_1(s)\Delta u(t-s)ds + \delta_1 u_t & \\
+ \mu_1 u_t(x, t-\tau_1) + f_1(u) = h_1, & \text{in } \Omega \times (0, +\infty),\\
v_{tt} + \alpha u - \Delta_e v - \int_0^{+\infty} \omega_2(s)\Delta v(t-s)ds + \delta_2 v_t & \\
+ \mu_2 v_t(x, t-\tau_2) + f_2(v) = h_2, & \text{in } \Omega \times (0, +\infty),\\
u(x,t) = v(x,t) = 0, & \text{on } \partial\Omega \times (0, +\infty),\\
(u(x,0), v(x,0)) = (u_0(x), v_0(x)), & \text{in } \Omega,\\
(u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)), & \text{in } \Omega,\\
u_t(x, t-\tau_1) = g_1(x, t-\tau_1), & \text{in } \Omega \times [0, \tau_1],\\
v_t(x, t-\tau_2) = g_2(x, t-\tau_2), & \text{in } \Omega \times [0, \tau_2],
\end{cases}
\tag{1.1}
$$

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where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ represent displacements, f_1 and f_2 represent nonlinear critical source terms, h_1 and h_2 represent external forces, and Δ_e denotes the elasticity operator defined by

$$\Delta_e = \mu\Delta u + (\lambda + \mu)\nabla \mathrm{div}\, u,$$

where λ and μ are Lamé constants which satisfy

$$\mu > 0, \quad \lambda + \mu \geq 0,$$

where $\tau_i > 0$, $i = 1, 2$, is a time delay, $\delta_1, \delta_2, \mu_1, \mu_2$ are positive real numbers and (u_0, u_1, v_0, v_1) are given initial data. The Lamé system is a fundamental mathematical model in the theory of isotropic elasticity. It has been studied by several researchers because of its significant applications. Concerning the Lamé system that includes a memory term we refer to Bchatnia and Guesmia [3], who considered the Lamé system in 3-dimension bounded domain with infinite memories

$$u'' - \Delta_e u + \int_0^{+\infty} g(s)\Delta u(t-s)ds = 0.$$

The authors proved that the system is well-posed and stable. They also found that the solutions converge to zero at infinity in terms of the growth of infinite memory. For the coupled Lamé system, Beniani and Taouaf [5] investigated a coupled Lamé system with viscoelastic damping in the first equation and two strong discrete time delays, proving the existence by using Faedo-Galerkin method and finding an exponential decay. Similar models have been studied by several authors, see [8, 15] and the references therein. In the context of the dynamics of Lamé systems with frictional damping, we first refer to [1, 7, 23, 28], where the authors prove the existence of a global attractor using the quasi-stability results developed by Chueshov and Lasiecka [10, 12].

The paper is organized as follows. In Section 2, we present the preliminaries. In Section 3, we analyze the well-posedness of the system (3.5) using semigroup methods. In Section 4, we give an overview of the abstract results in the theory of infinite dimensional dynamical systems. Moreover, we establish the existence of finite dimensional global and exponential attractors.

2. Assumptions

We consider the following assumptions in this paper.

For the memory terms ω_i, we suppose that $\omega_i : \mathbb{R}^+ \to \mathbb{R}^+$ are differentiable non-increasing function and integrable on $\mathbb{R}^+$ such that

$$\mu - \int_0^{+\infty} \omega_i(s)ds = \sigma_i > 0, \quad \int_0^{+\infty} \omega(s)ds = \omega_i^0, \quad i = 1, 2,$$

and there exist constants k_i satisfying

$$\omega_i'(s) \leq -k_i\omega_i(s), \quad \text{for all } s \in \mathbb{R}^+. \tag{2.1}$$

Let the external force $h \in (L^2(\Omega))^3$ and the nonlinear term $f \in C^2(\Omega)$, we assume that

$$|f''(u)| \leq c_f(1+|u|), \quad c_f > 0, u \in \mathbb{R}, i = 1,2,3, \tag{2.2}$$

which implies that, for some $C > 0$,

$$|f(u) - f(v)| \leq C(1+|u|^2+|v|^2)|u-v|, \quad \text{for all } u, v \in \mathbb{R}. \tag{2.3}$$

Also, we suppose that for some $m \in (0, \lambda_1)$, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$,

$$F(u) = \sum_{i=1}^{3} \int_0^{u_i} f_i(s)ds \geq -\frac{m}{2}|u|^2 - m_f \tag{2.4}$$

and

$$f(u)u \geq m|u|^2 - m_f. \tag{2.5}$$

3. Well Posedness and Energy Estimates

In order to prove the well-posedness result, we introduce as in [28] the following new variables

$$\begin{aligned} z_1(x,\rho,t) &= u_t(x, t-\tau_1\rho), \quad \text{in } \Omega \times (0,1) \times (0,+\infty), \\ z_2(x,\rho,t) &= v_t(x, t-\tau_2\rho), \quad \text{in } \Omega \times (0,1) \times (0,+\infty). \end{aligned} \tag{3.1}$$

Then, we obtain

$$\begin{aligned} \tau_1 z_{1t}(x,\rho,t) + z_{1\rho}(x,\rho,t) &= 0, \quad \text{in } \Omega \times (0,1) \times (0,+\infty), \\ \tau_2 z_{2t}(x,\rho,t) + z_{2\rho}(x,\rho,t) &= 0, \quad \text{in } \Omega \times (0,1) \times (0,+\infty). \end{aligned} \tag{3.2}$$

Additionally, following [19], we define the new variables

$$\begin{aligned} \eta_1(x,t,s) &= u(x,t) - u(x,t-s), \quad \text{in } \Omega \times (0,+\infty) \times (0,+\infty), \\ \eta_2(x,t,s) &= v(x,t) - v(x,t-s), \quad \text{in } \Omega \times (0,+\infty) \times (0,+\infty). \end{aligned} \tag{3.3}$$

These functionals satisfy

$$\begin{aligned} \partial_t\eta_1 + \partial_s\eta_1 - u_t &= 0, \quad \text{in } \Omega \times (0,+\infty) \times (0,+\infty), \\ \partial_t\eta_2 + \partial_s\eta_2 - v_t &= 0, \quad \text{in } \Omega \times (0,+\infty) \times (0,+\infty). \end{aligned} \tag{3.4}$$

To convert our problem to a system of first-order ordinary differential equations, we denote the following

$$\eta_i^0(x,s) = \eta_i(x,0,s), \quad i = 1,2.$$

Therefore, the problem (1.1) takes the form

$$
(3.5)\quad \begin{cases}
u_{tt} + \alpha v - \Delta_e u - \displaystyle\int_0^{+\infty} \omega_1(s)\Delta\eta_1(s)ds + \delta_1 u_t & \\
\quad + \mu_1 z_1(x,1,t) + f_1(u) = h_1, & \text{in } \Omega\times(0,+\infty),\\
v_{tt} + \alpha u - \Delta_e v - \displaystyle\int_0^{+\infty} \omega_2(s)\Delta\eta_2(s)ds + \delta_2 v_t & \\
\quad + \mu_2 z_2(x,1,t) + f_2(v) = h_2, & \text{in } \Omega\times(0,+\infty),\\
\tau_1 z_{1t}(x,\rho,t) + z_{1\rho}(x,\rho,t) = 0, & \text{in } \Omega\times(0,1)\times(0,+\infty),\\
\tau_2 z_{2t}(x,\rho,t) + z_{2\rho}(x,\rho,t) = 0, & \text{in } \Omega\times(0,1)\times(0,+\infty),\\
u(x,t) = v(x,t) = 0, & \text{on } \partial\Omega\times(0,+\infty),\\
\eta_1(x,t,s) = \eta_2(x,t,s) = 0, & \text{on } \partial\Omega\times(0,+\infty)\times(0,+\infty),\\
\eta_1(x,t,0) = \eta_2(x,t,0) = 0, & \text{in } \Omega\times(0,+\infty),\\
(u(x,0),v(x,0)) = (u_0(x),v_0(x)), & \text{in } \Omega,\\
(u_t(x,0),v_t(x,0)) = (u_1(x),v_1(x)), & \text{in } \Omega,\\
u_t(x,t-\tau_1) = g_1(x,t-\tau_1), & \text{in } \Omega\times[0,\tau_1],\\
v_t(x,t-\tau_2) = g_2(x,t-\tau_2), & \text{in } \Omega\times[0,\tau_2].
\end{cases}
$$

Let ξ_1 and ξ_2 be positive constants such that

$$
(3.6)\quad \begin{cases}
\tau_1\mu_1 < \xi_1 < \tau_1(2\delta_1 - \mu_1),\\
\tau_2\mu_2 < \xi_2 < \tau_2(2\delta_2 - \mu_2).
\end{cases}
$$

In order to consider the relative displacement η as a new variable, we introduce the L^2-space

$$
\mathcal{M}^i = L^2_{\omega_i}(\mathbb{R}^+; H_0^1(\Omega)) = \left\{\eta_i : \mathbb{R}^+ \to H_0^1(\Omega) \mid \int_\Omega \omega_i(s)\|\nabla\eta_i(s)\|_2^2 ds\right\},
$$

with the energy space

$$
\mathcal{H} = (H_0^1(\Omega))^2 \times (L^2(\Omega))^2 \times (L^2(\Omega\times(0,1)))^2 \times \mathcal{M}^1 \times \mathcal{M}^2,
$$

which is a Hilbert space with norm

$$
\|\eta_i\|^2_{\mathcal{M}} = \int_0^{+\infty} \omega_i(s)\|\nabla\eta_i(s)\|_2^2 ds, \quad i = 1,2,
$$

and the inner product

$$
(\eta_i, \Upsilon_i)_{\mathcal{M}} = \int_0^{+\infty} \omega_i(s)\nabla\eta_i(s)\nabla\Upsilon_i(s)ds, \quad i = 1,2.
$$

To prove the global well-posedness of (3.15) by using semigroup method, we introduce as in [16], the derivative η_{is}, $i = 1,2$, as an operator form. Define the operator T by

$$
T\eta_i = -\eta_{is}, \quad \eta_i \in D(T), \quad i = 1,2,
$$

with

$$
D(T) = \{\eta_i \in \mathcal{M} \mid \eta_{is} \in \mathcal{M}, \eta_i(0) = 0, i = 1,2\},
$$

which is the infinitesimal generator of translation semigroup. In particular,

$$(T\eta_i, \eta_i)_{\mathcal{M}} = \int_0^{+\infty} \omega_i'(s)\|\nabla \eta_i\|_2^2 ds, \quad \eta_i \in D(T), i = 1, 2,$$

and the solution of

$$\eta_{1t} = T\eta_1 + u, \quad \eta_1(0) = 0 \quad \text{and} \quad \eta_{2t} = T\eta_2 + v, \quad \eta_2(0) = 0,$$

has an explicite formula.

Next, we introduce the following Cauchy problem of abstract first order evolutionary operator equation

$$\begin{cases} \dfrac{dU}{dt}(t) = \mathcal{A}U + F, \quad t > 0, \\ U(0) = U_0 = (u_0, u_1, v_0, v_1, g_0(, -\tau_1), \tilde{g}_0(, -\tau_2), \eta_1^0, \eta_2^0), \end{cases} \tag{3.7}$$

where $U = (u, u_t, v, v_t, z_1, z_2, \eta_1, \eta_2)^T$ and

$$\mathcal{A}\begin{pmatrix} u \\ u_t \\ v \\ v_t \\ z_1 \\ z_2 \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} u_t \\ -\alpha v + \Delta_e u + \int_0^{+\infty} \omega_1(s)\Delta\eta_1 ds - \delta_1 u_t - \mu_1 z_1(x, 1, t) \\ v_t \\ -\alpha u + \Delta_e v + \int_0^{+\infty} \omega_2(s)\Delta\eta_2 ds - \delta_2 v_t - \mu_2 z_2(x, 1, t) \\ -\dfrac{1}{\tau_1} z_{1\rho}(x, \rho, t) \\ -\dfrac{1}{\tau_2} z_{2\rho}(x, \rho, t) \\ u_t + T\eta_1 \\ v_t + T\eta_2 \end{pmatrix}$$

and

$$F = \begin{pmatrix} 0 \\ -f_1(u) + h_1 \\ 0 \\ -f_2(v) + h_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

with the domain $D(\mathcal{A})$ of $\mathcal{A}$, which is defined by

$$D(\mathcal{A}) = \{U \in \mathcal{H}, \mathcal{A}U \in \mathcal{H}, \eta_i \in D(T), i = 1, 2\}.$$

We define the inner product in $\mathcal{H}$

$$\begin{aligned} \langle U, \tilde{U}\rangle_{\mathcal{H}} = &\int_\Omega u_t\tilde{u}_t dx + \int_\Omega v_t\tilde{v}_t dx + \sigma_1\int_\Omega \nabla u\nabla\tilde{u} dx + \sigma_2\int_\Omega \nabla v\nabla\tilde{v} dx \\ &+ \alpha\int_\Omega (v\tilde{u} + u\tilde{v}) dx + (\mu + \lambda)\int_\Omega \operatorname{div} u \operatorname{div} \tilde{u}\, dx + (\mu + \lambda)\int_\Omega \operatorname{div} u \operatorname{div} \tilde{u}\, dx \end{aligned}$$

$$+\int_0^{+\infty}\omega_1(s)\int_\Omega \nabla\eta_1\nabla\tilde{\eta}_1 dxds+\int_0^{+\infty}\omega_2(s)\int_\Omega \nabla\eta_2\nabla\tilde{\eta}_2 dxds$$
$$+\xi_1\int_\Omega\int_0^1 z_1(x,\rho)\tilde{z}_1(x,\rho)d\rho dx+\xi_2\int_\Omega\int_0^1 z_2(x,\rho)\tilde{z}_2(x,\rho)d\rho dx.$$

The well-posedness of the problem (3.5) is ensured by the following theorem.

Theorem 3.1. *Assume that* (2.1)–(2.4) *and* $\mu_1 \le \gamma_1$, $\mu_2 \le \gamma_2$ *hold. Then, we have the following results.*

(i) *Given* $U_0 \in \mathcal{H}$, *then the problem* (3.5) *has a unique mild solution* $U \in C([0,+\infty),\mathcal{H})$ *with* $U(0)=U_0$.

(ii) *If* U_1 *and* U_2 *are two mild solutions of the problem* (1.1), *then there exists a positive constant* $C_0 = C_0(U_1(0),U_2(0))$, *such that*

$$\|U_1(t)-U_2(t)\|_{\mathcal{H}} \le e^{C_0T}\|U_1(0)-U_2(0)\|_{\mathcal{H}},\quad \text{for all } 0\le t\le T. \tag{3.8}$$

(iii) *If* $U_0 \in D(\mathcal{A})$, *then the above mild solution can be improved as a strong solution.*

Lemma 3.1. *The operator* $\mathcal{A}$ *in* (3.7) *is the infinitesimal generator of a* C_0 *semigroup in* $\mathcal{H}$.

Proof. First, we show that the operator $\mathcal{A}$ is dissipative. For $U \in D(\mathcal{A})$, we have

$$\begin{aligned}\langle \mathcal{A}U,U\rangle_{\mathcal{H}} =& -\delta_1\int_\Omega u_t^2dx-\delta_2\int_\Omega v_t^2dx-\mu_1\int_\Omega z_1(x,1)u_tdx\\ &+\frac{1}{2}\int_0^{+\infty}\omega_1'(s)\|\nabla\eta_1\|_2^2ds+\frac{1}{2}\int_0^{+\infty}\omega_2'(s)\|\nabla\eta_2\|_2^2ds\\ &-\mu_2\int_\Omega z_2(x,1)v_tdx-\frac{\xi_1}{\tau_1}\int_\Omega\int_0^1 z_{1\rho}(x,\rho)z_1(x,\rho)d\rho dx\\ &-\frac{\xi_2}{\tau_2}\int_\Omega\int_0^1 z_{2\rho}(x,\rho)z_2(x,\rho)d\rho dx.\end{aligned}$$

By using the integration by parts and Young inequality

$$\begin{aligned}\langle \mathcal{A}U,U\rangle_{\mathcal{H}} =& \left(-\delta_1+\frac{\mu_1}{2}+\frac{\xi_1}{2\tau_1}\right)\int_\Omega u_t^2dx+\left(-\delta_2+\frac{\mu_2}{2}+\frac{\xi_2}{2\tau_2}\right)\int_\Omega v_t^2dx\\ &+\frac{1}{2}\int_0^{+\infty}\omega_1'(s)\|\nabla\eta_1\|_2^2ds+\frac{1}{2}\int_0^{+\infty}\omega_2'(s)\|\nabla\eta_2\|_2^2ds\\ &+\left(\frac{\mu_1}{2}-\frac{\xi_1}{2\tau_1}\right)\int_\Omega z_1^2(x,1)dx+\left(\frac{\mu_2}{2}-\frac{\xi_2}{2\tau_2}\right)\int_\Omega z_2^2(x,1)dx.\end{aligned}$$

Hence, $\langle \mathcal{A}U,U\rangle \le 0$. Consequently, the operator $\mathcal{A}$ is dissipative.

Now, we will prove that the operator $I-\mathcal{A}$ is surjective. For this purpose let $(f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8)^T \in \mathcal{H}$, we are looking for $U=(u,u_t,v,v_t,z_1,z_2,\eta_1,\eta_2)^T \in$

$D(\mathcal{A})$ solution of the following system of equations

$$
\begin{cases}
u - u_t = f_1, \\
u_t + \alpha v - \left(\mu - \omega_1^0\right) \Delta u - (\lambda+\mu)\nabla(\operatorname{div} u) - \int_0^{+\infty} \omega_1(s)\Delta\eta_1(s)ds \\
+ \delta_1 u_t + \mu_1 z_1(x,1,t) = f_2, \\
v - v_t = f_3, \\
v_t + \alpha u - \left(\mu - \omega_2^0\right) \Delta v - (\lambda+\mu)\nabla(\operatorname{div} v) - \int_0^{+\infty} \omega_2(s)\Delta\eta_2(s)ds \\
+ \delta_2 v_t + \mu_2 z_2(x,1,t) = f_4, \\
z_1 + \frac{1}{\tau_1} z_{1\rho} = f_5, \\
z_2 + \frac{1}{\tau_2} z_{2\rho} = f_6, \\
\eta_1 - u_t - T\eta_1 = f_7, \\
\eta_2 - v_t - T\eta_2 = f_8.
\end{cases} \tag{3.9}
$$

From the first and the third equations in (3.9), we have

$$
\begin{cases}
u_t = u - f_1, \\
v_t = v - f_3.
\end{cases} \tag{3.10}
$$

Then, it is clear that $u_t \in H_0^1(\Omega), v_t \in H_0^1(\Omega)$. Furthermore, by (3.9), we can find $z_i (i = 1, 2)$ as $z_1(x, \rho) = u_t(x)$, $z_2(x, \rho) = v_t(x)$.

Following the same approach as in [24], we obtain easily

$$
\begin{cases}
z_1(x,\rho) = u_t(x)e^{-\rho\tau_1} + \tau_1 e^{-\rho\tau_1} \int_0^1 f_5(x,\sigma)e^{\sigma\tau_1} d\sigma, \\
z_2(x,\rho) = v_t(x)e^{-\rho\tau_2} + \tau_2 e^{-\rho\tau_2} \int_0^1 f_6(x,\sigma)e^{\sigma\tau_2} d\sigma.
\end{cases}
$$

Exploiting (3.10), we get

$$
\begin{aligned}
z_1(x,\rho) &= u(x)e^{-\rho\tau_1} - f_1 e^{-\rho\tau_1} + \tau_1 e^{-\rho\tau_1} \int_0^1 f_5(x,\sigma)e^{\sigma\tau_1} d\sigma, \\
z_2(x,\rho) &= v(x)e^{-\rho\tau_2} - f_3 e^{-\rho\tau_2} + \tau_2 e^{-\rho\tau_2} \int_0^1 f_6(x,\sigma)e^{\sigma\tau_2} d\sigma.
\end{aligned} \tag{3.11}
$$

Using (3.9)–(3.10), the functions u, v satisfy the following system

$$
\begin{cases}
(1+\delta_1)u + \alpha v - \left(\mu - \omega_1^0\right) \Delta u - (\lambda+\mu)\nabla(\operatorname{div} u) - \int_0^{+\infty} \omega_1(s)\Delta\eta_1(s)ds \\
+ \mu_1 z_1(\cdot,1) = f_2 + (1+\delta_1)f_1, \\
(1+\delta_2)v + \alpha u - \left(\mu - \omega_2^0\right) \Delta v - (\lambda+\mu)\nabla(\operatorname{div} v) - \int_0^{+\infty} \omega_2(s)\Delta\eta_2(s)ds \\
+ \mu_2 z_2(\cdot,1) = f_4 + (1+\delta_2)f_3.
\end{cases}
$$

Solving the system is equivalent to finding $(u,v)\in(H^2(\Omega)\cap H_0^1(\Omega))^2$ such that

$$
\begin{cases}
\int_\Omega (1+\delta_1+\mu_1 e^{-\rho\tau_1})u\tilde{u}+\alpha v\tilde{u}+(\mu-\tilde{\omega}_1^0)\nabla u\nabla\tilde{u}+(\lambda+\mu)\operatorname{div} u\operatorname{div}\tilde{u}\\
+\mu_1 z_1(\cdot,1)\tilde{u}dx=\int_\Omega (f_2\tilde{u}+(1+\delta_1)f_1\tilde{u})dx,\\
\int_\Omega (1+\delta_2+\mu_2 e^{-\rho\tau_2})v\tilde{v}+\alpha u\tilde{v}+(\mu-\tilde{\omega}_2^0)\nabla v\nabla\tilde{v}+(\lambda+\mu)\operatorname{div} v\operatorname{div}\tilde{v}\\
+\mu_2 z_2(\cdot,1)\tilde{v}dx=\int_\Omega (f_4\tilde{v}+(1+\delta_2)f_3,\tilde{v})dx,
\end{cases}
\tag{3.12}
$$

with $\tilde{\omega}_i^0=\int_0^{+\infty}e^{-s}\omega_i(s)ds$.

From (3.11), we have

$$z_1(x,1)=u_t(x)e^{-\tau_1}+z_0(x),\quad z_2(x,1)=v_t(x)e^{-\tau_2}+\tilde{z}_0(x),$$

where

$$
\begin{aligned}
z_0(x)&=-f_1e^{-\tau_1}+\tau_1e^{-\tau_1}\int_0^1 f_5(x,\sigma)e^{\sigma\tau_1}d\sigma,\\
\tilde{z}_0(x)&=-f_3e^{-\tau_2}+\tau_2e^{-\tau_2}\int_0^1 f_6(x,\sigma)e^{\sigma\tau_2}d\sigma.
\end{aligned}
$$

It is clear from the above formula that $z_0,\tilde{z}_0$ depend only on f_1,f_3,f_5,f_6. Consequently, the problem (3.12) is equivalent to problem

$$a((u,v)(\tilde{u},\tilde{v}))=l(\tilde{u},\tilde{v}),\tag{3.13}$$

where

$$
\begin{aligned}
a((u,v)(\tilde{u},\tilde{v}))=&\int_\Omega((1+\delta_1+\mu_1e^{-\rho\tau_1})u\tilde{u}+(1+\delta_2+\mu_2e^{-\rho\tau_2})v\tilde{v}\\
&+\alpha(v\tilde{u}+u\tilde{v}))dx+\int_\Omega((\mu-\tilde{\omega}_1^0)\nabla u\nabla\tilde{u}+(\mu-\tilde{\omega}_2^0)\nabla v\nabla\tilde{v})dx\\
&+\int_\Omega((\lambda+\mu)\operatorname{div} u\operatorname{div}\tilde{u})+(\lambda+\mu)\operatorname{div} v\operatorname{div}\tilde{v})dx
\end{aligned}
$$

and

$$
\begin{aligned}
l(\tilde{u},\tilde{v})=&\int_\Omega(f_2\tilde{u}+(1+\delta_1)f_1\tilde{u}-\mu_1z_0\tilde{u})dx+\int_\Omega(f_4\tilde{v}+(1+\delta_2)f_3\tilde{v}-\mu_2\tilde{z}_0\tilde{v})dx\\
&+\int_\Omega(\omega_1^0-\tilde{\omega}_1^0)\nabla f_1\nabla\tilde{u}dx+\int_\Omega(\omega_2^0-\tilde{\omega}_2^0)\nabla f_2\nabla\tilde{v}dx\\
&-\int_\Omega\left(\int_0^{+\infty}\omega_1(s)\int_0^s e^{\tau-s}\nabla f_7(\tau)d\tau ds\right)\nabla\tilde{u}dx\\
&-\int_\Omega\left(\int_0^{+\infty}\omega_2(s)\int_0^s e^{\tau-s}\nabla f_8(\tau)d\tau ds\right)\nabla\tilde{v}dx.
\end{aligned}
$$

It is easy to verify that a is continuous and coercive, and l is continuous. So, applying the Lax-Milgram theorem, we deduce that for all $(\tilde{u},\tilde{v})\in(H_0^1(\Omega))^2$, the problem (3.13) admits a unique solution $(u,v)\in(H_0^1(\Omega))^2$. Applying the classical elliptic regularity, it follows from (3.12) that $(u,v)\in(H^2(\Omega)\cap H_0^1(\Omega))^2$. Therefore, the operator $I-\mathcal{A}$ is surjective. Consequently, we can infer that the operator $\mathcal{A}$ is maximal dissipative in $\mathcal{H}$. Since $D(\mathcal{A})$ is dense in $\mathcal{H}$, thus we can conclude that the operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup in $\mathcal{H}$, by the Lummer-Phillips theorem. The proof is now complete. □

Lemma 3.2. *The function F is locally Lipschitz.*

Proof. Let $U^1 = (u^1, u_t^1, v_1^1, v_t^1, z_1^1, z_2^1, \eta_1^1, \eta_2^1)$ and $U^2 = (u^2, u_t^2, v_2^2, v_t^2, z_1^2, z_2^2, \eta_1^2, \eta_2^2)$. Then, we have

$$\begin{aligned}\|F(U) - F(V)\|_{\mathcal{H}} \leq& \|f(u^1) - f(u^2)\|_2^2 + \|f(v^1) - f(v^2)\|_2^2 \\ \leq& C(1 + \|u^1\|_6^2 + \|u^2\|_6^2)\|u^1 - u^2\|_6^2 \\ &+ C(1 + \|v^1\|_6^2 + \|v^2\|_6^2)\|v^1 - v^2\|_6^2 \\ \leq& K_0\|\nabla(u^1 - u^2)\|_2^2 + K_0\|\nabla(v^1 - v^2)\|_2^2 \\ \leq& K_0\|U - V\|_{\mathcal{H}}^2.\end{aligned}$$

So, the operator F is locally Lipschitz in $\mathcal{H}$. The proof is hence complete. □

Proof of the Theorem 3.1. We deduce from Lemma 3.1 and Lemma 3.2, that the Cauchy problem has a unique local mild solution

$$U(t) = e^{\mathcal{A}t}U_0 + \int_0^t e^{\mathcal{A}(t-s)}F(U(s))ds, \tag{3.14}$$

defined in a maximal interval $(0, t_{\max})$. If $t_{\max} < +\infty$, then $\lim_{t\to+\infty} \|U(t)\|_{\mathcal{H}} = +\infty$.

Let $U(t)$ be a mild solution with $U_0 \in D(\mathcal{A})$. By using Theorem 6.1.5 in Pazy [26], we conclude that it is a strong solution. It follows from (3.17) that for all $t \geq 0$

$$\|U(t)\|_{\mathcal{H}}^2 \leq \frac{1}{C_1}(E(0) + C_2),$$

which, by density, holds for mild solutions. Then, it is a contradiction with (3.17) and therefore $t_{\max} = +\infty$, that is, the solution is global. The proof of (i) of Theorem 3.1 is complete. By using (3.14) we obtain the inequality (3.8), the local Lipschitz behavior of F and Gronwall's inequality. Then, we can obtain the continuous dependence on the initial data for mild solutions. This proves the item (ii) of Theorem 3.1. By using Theorem 6.1.5 in Pazy [26], we know that any mild solutions with initial data in $D(\mathcal{A})$ are strong. Then, the proof of Theorem 3.1 is therefore complete. □

In what follows, we present some useful inequalities related to the energy functional. The total energy associated with the problem (3.5) is given by

$$\begin{aligned}E(t) =& \frac{1}{2}\int_\Omega u_t^2 dx + \frac{1}{2}\int_\Omega v_t^2 dx + \frac{\sigma_1}{2}\int_\Omega \nabla u^2 dx + \frac{\sigma_2}{2}\int_\Omega \nabla v^2 dx \\ &+ \frac{\lambda+\mu}{2}\int_\Omega |\operatorname{div} u|^2 dx + \frac{\lambda+\mu}{2}\int_\Omega |\operatorname{div} v|^2 dx + \alpha\int_\Omega uv dx \\ &+ \frac{\xi_1}{2}\int_\Omega\int_0^1 z_1^2(x,\rho,t)d\rho dx + \frac{\xi_2}{2}\int_\Omega\int_0^1 z_2^2(x,\rho,t)d\rho dx\end{aligned} \tag{3.15}$$

$$+\frac{1}{2}\|\eta_1\|^2_{\mathcal{M}}+\frac{1}{2}\|\eta_2\|^2_{\mathcal{M}}+\int_\Omega F_1(u)dx\int_\Omega F_2(v)dx-\int_\Omega h_1 u dx-\int_\Omega h_2 v dx.$$

Then, we can get the following lemma.

Lemma 3.3. *If $(u, v, z_1, z_2, \eta_1, \eta_2)$ is the solution of the problem (3.5), then for any $t \geq 0$ the energy $E(t)$ satisfies*

$$\begin{aligned} E'(t) \leq &-\left(\delta_1-\frac{\xi_1}{2\tau_1}-\frac{\mu_1}{2}\right)\int_\Omega u_t^2 dx-\left(\delta_2-\frac{\xi_2}{2\tau_2}-\frac{\mu_2}{2}\right)\int_\Omega v_t^2 dx\\ &-\left(\frac{\xi_1}{2\tau_1}-\frac{\mu_1}{2}\right)\int_\Omega z_1^2(x,1,t)dx-\left(\frac{\xi_2}{2}-\frac{\mu_2}{2\tau_2}\right)\int_\Omega z_2^2(x,1,t)dx\\ &+\frac{1}{2}\int_0^{+\infty}\omega_1'(s)\|\nabla\eta_1\|_2^2 ds+\frac{1}{2}\int_0^{+\infty}\omega_2'(s)\|\nabla\eta_2\|_2^2 ds, \end{aligned} \tag{3.16}$$

and there exist two positive constants C_1 and C_2 which are independent data in $\mathcal{H}$, such that for any $t \geq 0$, we have

$$\begin{aligned} E(t) \geq &C_1\left(\int_\Omega u_t^2 dx+\int_\Omega v_t^2 dx+\int_\Omega \nabla u^2 dx+\int_\Omega \nabla v^2 dx\right.\\ &+\int_\Omega|\operatorname{div} u|^2 dx+\int_\Omega|\operatorname{div} v|^2 dx+\int_\Omega uv dx+\|\eta_1\|^2_{\mathcal{M}}+\|\eta_2\|^2_{\mathcal{M}}\\ &\left.+\int_\Omega\int_0^1 z_1^2(x,\rho,t)d\rho dx+\int_\Omega\int_0^1 z_2^2(x,\rho,t)d\rho dx\right)\\ &-C_2\left(\int_\Omega h_1^2 dx+\int_\Omega h_2^2 dx\right)-2m_f. \end{aligned} \tag{3.17}$$

Proof. Multiplying the first equation in (3.5) by u_t, the second by v_t and integrating over Ω and using integration by parts, we get

$$\begin{aligned} &\frac{1}{2}\cdot\frac{d}{dt}\left(\int_\Omega u_t^2 dx+\int_\Omega v_t^2 dx+2\alpha\int_\Omega uv dx+\mu\int_\Omega|\nabla u|^2 dx+\mu\int_\Omega|\nabla v|^2 dx\right.\\ &\left.+(\mu+\lambda)\int_\Omega|\operatorname{div} u|^2 dx+(\mu+\lambda)\int_\Omega|v|^2 dx\right)\\ =&-\delta_1\int_\Omega u_t^2 dx-\delta_2\int_\Omega v_t^2 dx-\mu_1\int_\Omega z_1(x,1,t)u_t dx\\ &+\mu_2\int_\Omega z_2(x,1,t)v_t dx-\int_0^{+\infty}\omega_1(s)\int_\Omega\Delta\eta_1(s)u_t dx ds\\ &+\int_0^{+\infty}\omega_2(s)\int_\Omega\Delta\eta_2(s)v_t dx ds. \end{aligned}$$

Since $u_t = \eta_{1t} + \eta_{1s}$, we infer that

$$-\int_0^{+\infty}\omega_1(s)\int_\Omega\nabla\eta_1(s)\nabla u_t dx ds=-\frac{1}{2}\cdot\frac{d}{dt}\|\eta_1\|^2_{\mathcal{M}}+\frac{1}{2}\int_0^{+\infty}\omega_1(s)\frac{d}{ds}\|\nabla\eta_1\|_2^2 ds,$$

we integrate by parts to obtain

$$\int_0^{+\infty}\omega_1(s)\frac{d}{ds}\|\nabla\eta_1\|_2^2 ds=-\int_0^{+\infty}\omega_1'(s)\|\nabla\eta_1\|_2^2 ds. \tag{3.18}$$

Mulplying the third equation in (3.5) by $\frac{\xi_1}{\tau_1} z_1(x,\rho,t)$ and integrating over $\Omega \times (0,1)$, we obtain

$$\begin{aligned} \xi_1 \int_\Omega \int_0^1 z_{1t}(x,\rho,t) z_1(x,\rho,t) d\rho dx &= -\frac{\xi_1}{2\tau_1} \int_\Omega \int_0^1 \frac{\partial}{\partial \rho} z_1^2(x,\rho,t) d\rho dx \\ &= -\frac{\xi_1}{2\tau_1} \int_\Omega \left(z_1^2(x,1,t) - z_1^2(x,0,t) \right) dx. \end{aligned} \tag{3.19}$$

Due to Young's inequality, we have

$$\begin{aligned} \mu_1 \int_\Omega z_1(x,1,t) u_t dx &\le \frac{\mu_1}{2} \int_\Omega z_1^2(x,1,t) dx + \frac{\mu_1}{2} \int_\Omega u_t^2 dx, \\ \mu_2 \int_\Omega z_2(x,1,t) u_t dx &\le \frac{\mu_2}{2} \int_\Omega z_2^2(x,1,t) dx + \frac{\mu_2}{2} \int_\Omega v_t^2 dx. \end{aligned} \tag{3.20}$$

A combination of (3.18)–(3.20), leads to

$$\begin{aligned} E'(t) \le &- \left(\delta_1 - \frac{\xi_1}{2\tau_1} - \frac{\mu_1}{2} \right) \int_\Omega u_t^2 dx - \left(\delta_2 - \frac{\xi_2}{2\tau_2} - \frac{\mu_2}{2} \right) \int_\Omega v_t^2 dx \\ &- \left(\frac{\xi_1}{2\tau_1} - \frac{\mu_1}{2} \right) \int_\Omega z_1^2(x,1,t) dx - \left(\frac{\xi_2}{2} - \frac{\mu_2}{2\tau_2} \right) \int_\Omega z_2^2(x,1,t) dx \\ &+ \frac{1}{2} \int_0^{+\infty} \omega_1'(s) \|\nabla \eta_1\|_2^2 ds + \frac{1}{2} \int_0^{+\infty} \omega_2'(s) \|\nabla \eta_2\|_2^2 ds. \end{aligned}$$

Invoking the condition (3.6), we have

$$\begin{aligned} \delta_1 - \frac{\xi_1}{2\tau_1} - \frac{\mu_1}{2} > 0, \quad \frac{\xi_1}{2} - \frac{\mu_1}{2\tau_1} > 0, \\ \delta_2 - \frac{\xi_2}{2\tau_2} - \frac{\mu_2}{2} > 0, \quad \frac{\xi_2}{2} - \frac{\mu_2}{2\tau_2} > 0. \end{aligned}$$

Therefore, (3.16) holds.

Let us check the inequality (3.17). Using the assumption (2.5), the Poincaré and Young inequalities, for any κ we infer that

$$\begin{aligned} &\int_\Omega (F(u) + F(v)) dx - \int_\Omega (h_1 u + h_2 v) dx \\ \ge &- \left(\frac{m}{2\lambda_1} + \frac{\kappa}{\lambda_1} \right) \int_\Omega \nabla u^2 dx - \left(\frac{m}{2\lambda_1} + \frac{\kappa}{\lambda_1} \right) \int_\Omega \nabla v^2 dx \\ &- \frac{1}{4\kappa} \left(\int_\Omega h_1^2 dx + \int_\Omega h_2^2 dx \right) - 2m_f, \end{aligned}$$

we put $C_2 = \frac{1}{4\kappa} - 2m_f$, and conclude that (3.17) holds. □

4. Long-time Dynamics

In this section, we establish the existence of finite dimensional global and expontial attractor of the problem (3.5).

4.1. **Generation of dynamical system.** We recall some fundamentals of theory concerning attractors of nonlinear infinite dimensional systems which can be found in [12]. We will focus more specificaly on [10].

• Let X be a Banach space, the one-parameter operator $S(t) : X \to X$, $t \geq 0$, is said to be a semigroup if

$$S(t_1 + t_2) = S(t_1)S(t_2) \quad \text{and} \quad S(0) = \mathrm{Id},$$

hold for all $t_1, t_2 \geq 0$, where Id is the identity operator. The existence of global attractor relies on two properties, namely, dissipativeness and compactness.

• A dynamical system is called dissipative if it possesses an absobing set $B_0 \subset X$ for the semigroup $S(t)$, $t \geq 0$, that attracts any bounded set $B \subset X$ in a finite time $t_1 = t_1(B) > 0$ such that for all $t > t_1$, we have

$$S(t)B \subseteq B_0.$$

• For compactness, a dynamical system $(\mathcal{H}, S(t))$ is called asymptotically compact if for any bounded $B \subset \mathcal{H}$ and sequence $\{x_n\} \subset B$, the sequence $\{S(t_n)x_n\}$ has convergent subsequence whenever $t_n \to +\infty$.

• A compact set $\mathcal{A} \subset X$ is called a global attractor of semigroup $S(t)$ if

(i) $\mathcal{A}$ is strictly invariant with respect to $S(t)$, i.e., for all $t \geq 0$,

$$S(t)\mathcal{A} = \mathcal{A},$$

(ii) $\mathcal{A}$ attracts any bounded set $B \subset X$, i.e., for any $\epsilon > 0$ there exists a time $t_1 = t_1(\epsilon, B) > 0$ such that for all $t \geq t_1(\epsilon, B)$,

$$S(t)B \subseteq \mathcal{O}_\epsilon(\mathcal{A}),$$

where $\mathcal{O}_\epsilon(Y)$ is an ϵ-neighborhood of a set Y in X.

• The fractal dimension of compact set M in a metric space X, is a number defined by

$$\dim_f^X M = \limsup_{\epsilon \to 0} \frac{\ln N(M, \epsilon)}{\ln(1/\epsilon)},$$

where $N_\epsilon(M)$ is the minimal number of closed balls with radius $\epsilon > 0$ which covers M.

Definition 4.1 ([9])**.** The unstable manifold $M_+(\mathcal{N})$ is defined as the family of $y \in X$ such that there exists a full trajectory $u(t)$ satisfying

$$u(0) = y \quad \text{and} \quad \lim_{t \to -\infty} \mathrm{dist}_X(u(t), \mathcal{N}) = 0,$$

where $\mathcal{N}$ is the set of equilibrium for $S(t)$.

Theorem 4.1 ([10])**.** *Assume that the gradient system $(S(t), X)$ with corresponding Lyapunov functional $\mathbf{\Phi}$ is asymptotically compact. Moreover, assume that*

• $\mathbf{\Phi}(S(t)z) \to +\infty$ *if and only if* $\|z\|_X \to +\infty$,

• *the set of equilibrium $\mathcal{N}$ is bounded.*

Then, the gradient system $(S(t), X)$ possesses a compact global attractor $\mathcal{A} \subset X$, which has the structure $\mathcal{A} = M_+(\mathcal{N})$.

Remark 4.1. A semi-norm $n_X(\cdot)$ defined on a Banach space X is compact if there exists a sequence $x_j \to 0$ weakly in X such that $n_X(x_j) \to 0$. Let X, Y, Z be three reflexive Banach spaces with X compactly embedded in Y and put $\mathcal{H} = X \times Y \times Z$. Considering the dynamical system $(\mathcal{H}, S(t))$ given by an evolution operator

$$S(t)U_0 = (u, u_t, v, v_t, z_1, z_2, \eta_1, \eta_1, \eta_2), \quad (u_0, u_1, v_0, v_1, g_0, \tilde{g}_0, \eta_1^0, \eta_2^0) \in \mathcal{H},$$

where u, v, z_1, z_2, and η_1, η_2 have the regularity

$$u, v \in C(\mathbb{R}^+; X) \cap C^1(\mathbb{R}^+; Y), \quad z_1, z_2 \in C(\mathbb{R}^+; Z), \quad \eta_1, \eta_2 \in C(\mathbb{R}^+; Z).$$

The dynamical system $(\mathcal{H}, S(t))$ is quasi-stable on a set $B \subset \mathcal{H}$ if there exists a compact semi-norm n_X on X and nonnegative scalar functions $a(t)$ and $c(t)$, locally bounded in $[0, +\infty)$, and $b(t) \in \mathbb{L}^1(\mathbb{R}^+)$ with $\lim_{t\to+\infty} b(t) = 0$, such that

$$\|S(t)U_1 - S(t)U_2\|_{\mathcal{H}}^2 \leq a(t)\|U_1 - U_2\|_{\mathcal{H}}^2$$

and

$$\begin{aligned}\|S(t)U_1 - S(t)U_2\|_{\mathcal{H}}^2 \leq& b(t)\|U_1 - U_2\|_{\mathcal{H}}^2 \\ &+ c(t) \sup_{0<s<t} \Big[n_X(u^1(s) - u^2(s)) + n_X(v^1(s) - v^2(s))\Big]^2,\end{aligned}$$

for any $U_1, U_2 \in B$.

Theorem 4.2 ([10])**.** *Let $(X, S(t))$ be a dynamical system and suppose that the system is quasi-stable on every bounded positively invariant set $B \subset X$. Then, $(X, S(t))$ is a asymptotically compact.*

Theorem 4.3 ([10])**.** *Let $(X, S(t))$ be a gradient system and suppose that the system is quasi-stable on every bounded positively invariant set $B \subset X$. Then, $(X, S(t))$ has a global attractor $\mathcal{A} = M_+(\mathcal{N})$ with finite fractal dimension, where $\mathcal{N}$ is the set of equilibrium for $S(t)$, $M_+(\mathcal{N})$ is the unstable manifold for $\mathcal{N}$. Moreover, the generalized finite fractal dimensional exponential attractor also exists under suitable condition for $S(t)$.*

Our main result is the following.

Theorem 4.4 ([10])**.** *Suppose that assumptions of Theorem* 3.1 *and the given initial data $(u_0, u_1, v_0, v_1, g_0, \tilde{g}_0, \eta_1^0, \eta_2^0) \in \mathcal{H}$ for the problem* (3.5) *hold. Then, we have the following results.*

The gradient system $(\mathcal{H}, S(t))$ for the problem (3.5) *possesses a compact finite fractal dimensional global attractor $\mathcal{A} \subset \mathcal{H}$,which has the structure as*

$$\mathcal{A} = M_+(\mathcal{N}),$$

where $\mathcal{N} = \{y \in \mathcal{H} \mid S(t)y = y\}$ for all $t > 0$ is the set of stationary points and $M_+(\mathcal{N})$ be the unstable manifold from the set emanating from the set $\mathcal{N}$.

Moreover, the gradient system has a generalized exponential $\mathcal{A}^{exp} \subset \mathcal{H}$ with finite fractal dimension.

To prove Theorem 4.4, we will show that the gradient system $(\mathcal{H}, S(t))$ is dissipative and asymptotically compact.

4.2. **Gradient systems.**

Lemma 4.1. *Assume that assumptions of Theorem* 3.1 *and the given initial data* $(u_0, u_1, v_0, v_1, g_0, \tilde{g}_0, \eta_1^0, \eta_2^0) \in \mathcal{H}$ *for the problem* (3.5) *hold. Then, the dynamical system* $(\mathcal{H}, S(t))$ *is gradient.*

Proof. For the given initial data $U_0 = (u_0, u_1, v_0, v_1, g_0, \tilde{g}_0, \eta_1^0, \eta_2^0) \in \mathcal{H}$, it yields that

$$U(t) = S(t)U_0 = (u, u_t, v, v_t, z_1, z_2, \eta_1, \eta_2)$$

is a solution trajectory for the dynamical system $(\mathcal{H}, S(t))$ of the problem (3.5). Let $\mathcal{K}(S(t)U)$ be the energy along the solution $S(t)U$. Then, from (3.16), we see that $t \mapsto \mathcal{K}(S(t)U)$ is non-increasing. Let us suppose $\mathcal{K}(S(t)U_0) = S(t)U_0$, for all $t \geq 0$. Then, we discover that

$$\begin{aligned}
&-\left(\delta_1 - \frac{\xi_1}{2\tau_1} - \frac{\mu_1}{2}\right)\|u_t\|_2^2 - \left(\delta_1 - \frac{\xi_2}{2\tau_2} - \frac{\mu_2}{2}\right)\|v_t\|_2^2\\
&-\left(\frac{\xi_1}{2\tau_1} - \frac{\mu_1}{2}\right)\int_\Omega z_1^2(x,1,t)dx + \frac{1}{2}\int_0^{+\infty}\omega_1'\|\nabla\eta_1\|_2^2 ds\\
&-\left(\frac{\xi_2}{2\tau_2} - \frac{\mu_2}{2}\right)\int_\Omega z_2^2(x,1,t)dx + \frac{1}{2}\int_0^{+\infty}\omega_2'\|\nabla\eta_1\|_2^2 ds = 0.
\end{aligned}$$

Since all terms have the same sign, we discover

$$\int_0^{+\infty}\omega_i'(s)\|\nabla\eta_i\|_2^2 ds \geq k_i\|\eta_i\|_{\mathcal{M}}^2 = 0.$$

Then, $\eta_1(x,s) = \eta_2(x,s) = 0$. Using (3.4) we conclude that

$$u_t(x,t) = v_t(x,t) = 0, \quad \text{a.e. in } \Omega \times \mathbb{R}^+. \tag{4.1}$$

From (4.1) and (3.1), we conclude $U_0 = (u_0, v_0, 0, 0, 0, 0, 0, 0)$ is a stationary point of dynamical system $(\mathcal{H}, S(t))$, which implies $(\mathcal{H}, S(t))$ is a gradient system. ☐

4.3. **Existence of absorbing set.**

Theorem 4.5. *Suppose that assumptions of Theorem* 3.1 *hold and the given initial data*
$(u_0, u_1, v_0, v_1, g_0, \tilde{g}_0, \eta_1^0, \eta_2^0) \in \mathcal{H}$. *Then, the gradient system* $(\mathcal{H}, S(t))$ *has a bounded set* $B \subset \mathcal{H}$.

Lemma 4.2. *Let* $(u, u_t, v, v_t, z_1, z_2, \eta_1, \eta_2)$ *be the solution of the problem* (3.5). *We define the functional* $F(t)$ *by*

$$F(t) = \int_\Omega uu_t dx + \int_\Omega vv_t dx + \frac{\delta_1}{2}\int_\Omega u^2 dx + \frac{\delta_2}{2}\int_\Omega v^2 dx.$$

Then, it holds for any $\vartheta > 0$

$$
\begin{aligned}
F'(t) \leq & \int_\Omega u_t^2 dx + \int_\Omega v_t^2 dx - 2\alpha \int_\Omega uv dx - \int_\Omega (\lambda+\mu)|\operatorname{div} u|^2 dx \\
& - \int_\Omega (\lambda+\mu)|\operatorname{div} v|^2 dx + \frac{1}{4c}(\|\eta_1\|^2_{\mathcal{M}} + \|\eta_2\|^2_{\mathcal{M}}) \\
& - \left(\sigma_1 - \frac{1}{2\lambda_1} - \frac{m}{\lambda_1} - \frac{\vartheta}{\lambda_1} - \omega_0^1 c\right) \int_\Omega \nabla u^2 dx \\
& - \left(\sigma_2 - \frac{1}{2\lambda_1} - \frac{m}{\lambda_1} - \frac{\vartheta}{\lambda_1} - \omega_0^2 c\right) \int_\Omega \nabla v^2 dx \\
& + \frac{\mu_1^2}{2} \int_\Omega z_1^2(x,1,t) dx + \frac{\mu_2^2}{2} \int_\Omega z_2^2(x,1,t) dx \\
& + \frac{1}{4\vartheta} \int_\Omega h_1^2 dx + \frac{1}{4\vartheta} \int_\Omega h_2^2 dx + 2m_f,
\end{aligned} \tag{4.2}
$$

where $\lambda_1 > 0$ denotes the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

Proof. After differentiation, we easily obtain

$$
\begin{aligned}
F'(t) = & \int_\Omega u_t^2 \, dx + \int_\Omega u u_{tt} \, dx + \int_\Omega v_t^2 \, dx + \int_\Omega v v_{tt} \, dx \\
= & \int_\Omega u_t^2 dx - 2\alpha \int_\Omega u.v dx - \sigma_1 \int_\Omega \nabla u^2 dx - \sigma_2 \int_\Omega \nabla v^2 dx \\
& - (\lambda+\mu) \int_\Omega |\operatorname{div} u|^2 dx - (\lambda+\mu) \int_\Omega |\operatorname{div} v|^2 dx \\
& - \mu_1 \int_\Omega z_1(x,1,t) u \, dx - \mu_2 \int_\Omega z_1(x,1,t) v \, dx \\
& - \int_\Omega f_1(u) u dx - \int_\Omega f_2(v) v dx + \int_\Omega h_1 u dx + \int_\Omega h_2 v dx \\
& - \int_0^{+\infty} \omega_1(s) \int_\Omega \nabla u(t) \nabla \eta_1(s) dx ds \\
& - \int_0^{+\infty} \omega_2(s) \int_\Omega \nabla v(t) \nabla \eta_2(s) dx ds.
\end{aligned} \tag{4.3}
$$

Young's inequality and Poincaré's inequality imply

$$
\int_\Omega z_1(x,1,t) u dx \leq \frac{\mu_1^2}{2} \int_\Omega z_1^2(x,1,t) dx + \frac{1}{2\lambda_1} \int_\Omega \nabla u^2 dx, \tag{4.4}
$$

similarly, we have

$$
\int_\Omega z_2(x,1,t) v dx \leq \frac{\mu_2^2}{2} \int_\Omega z_2^2(x,1,t) dx + \frac{1}{2\lambda_1} \int_\Omega \nabla v^2 dx \tag{4.5}
$$

and

$$
\left| - \int_0^{+\infty} \omega_1(s) \int_\Omega \nabla u(t) \nabla \eta_1(s) dx ds \right| \leq \omega_0^1 c \int_\Omega \nabla u^2 dx + \frac{1}{4c} \|\eta_1\|^2_{\mathcal{M}}. \tag{4.6}
$$

Using (2.5), we obtain

$$-\int_\Omega f(u)udx \le \frac{m}{\lambda_1}\int_\Omega \nabla u^2 dx + m_f. \tag{4.7}$$

Inserting (4.4)–(4.7) into (4.3), (4.2) is proven. □

Lemma 4.3. *Let $(u, u_t, v, v_t, z_1, z_2, \eta_1, \eta_2)$ be the solution of the problem (3.5). We define the functional $J_1(t)$ by*

$$J_1(t) = -\int_0^{+\infty} \omega_1(s)\left(\int_\Omega u_t(t)\eta_1(s)dx\right)ds,$$

which satisfies

$$\begin{aligned} J_1'(t) \le& \left(\omega_0^1\delta_1 + \frac{3\omega_0^1}{2}\right)\int_\Omega u_t^2 dx + \left(\sigma_1\omega_0^1 + \frac{\omega_0^1 m}{\lambda_1} + \frac{1}{4\lambda_1}\right)\int_\Omega \nabla u^2 dx \\ &+ \frac{\alpha\omega_0^1}{\lambda_1}\int_\Omega \nabla v^2 dx + (\lambda+\mu)\,\omega_0^1\int_\Omega |\operatorname{div} u|^2 dx + \mu_1\omega_0^1\int_\Omega z_1^2(x,1,t)dx \\ &+ \left(\frac{\alpha}{4\lambda_1} + \frac{\sigma_1}{4} + \frac{\lambda+\mu}{4} + \omega_0^1 + \frac{\delta_1}{4\lambda_1} + \frac{\mu_1}{4\lambda_1} + \frac{1}{2\lambda_1}\right)\|\eta_1\|_{\mathcal{M}}^2 \\ &- \frac{l_1}{2\omega_0^1\lambda_1}\int_0^{+\infty}\omega_1'(s)\|\nabla\eta_1\|_2^2 ds. \end{aligned} \tag{4.8}$$

Proof. It is clear that

$$\begin{aligned} J_1'(t) =& -\int_0^{+\infty}\omega_1(s)\left(\int_\Omega u_{tt}(t)\eta_1(s)dx\right)ds - \int_0^{+\infty}\omega_1(s)\left(\int_\Omega u_t(t)u(t)dx\right)ds \\ &+ \int_0^{+\infty}\omega_1(s)\left(\int_\Omega u_t(t)\eta_{1s}(s)dx\right)ds. \end{aligned} \tag{4.9}$$

Exploiting Young inequality and Poincaré inequality, we get

$$\int_0^{+\infty}\omega_1(s)\left(\alpha v(t)\eta_1(s)dx\right)ds \le \frac{\alpha\omega_0^1}{\lambda_1}\int_\Omega \nabla v^2 dx + \frac{\alpha}{4\lambda_1}\|\eta_1\|_{\mathcal{M}}^2, \tag{4.10}$$

$$\int_\Omega\left(\int_0^{+\infty}\omega_1(s)\Delta\eta_1(s)ds\right)\left(\int_0^{+\infty}\omega_1(s)\eta_1(s)ds\right) \le \omega_0^1\|\eta_1\|_{\mathcal{M}}^2$$

and

$$\begin{aligned} \int_0^{+\infty}\omega_1(s)\left(\int_\Omega u_t(t)\eta_{1s}(s)dx\right)ds =& \int_0^{+\infty}(-\omega_1'(s))\left(\int_\Omega u_t(t)\eta_1(s)dx\right)ds \\ \le& \frac{\omega_0^1}{2}\int_\Omega u_t^2 dx - \frac{l_1}{2\omega_0^1\lambda_1}\int_0^{+\infty}\omega_1'(s)\|\nabla\eta_1\|_2^2 ds, \end{aligned} \tag{4.11}$$

where $l_1 = -\int_0^{+\infty}\omega_1'(s)ds$.

Noticing the estimtes (4.10)–(4.11), we easily deduce the inequality (4.8). The proof is hence complete. □

Lemma 4.4. *Let $(u, u_t, v, v_t, z_1, z_2, \eta_1, \eta_2)$ be the solution of the problem* (3.5). *We define the functional $J_2(t)$ by*

$$J_2(t) = -\int_0^{+\infty} \omega_1(s) \left(\int_\Omega u_t(t)\eta_1(s)dx \right) ds,$$

which satisfies

$$\begin{aligned} J_2'(t) \leq & \left(\omega_0^2\delta_2 + \frac{3\omega_0^2}{2} \right) \int_\Omega v_t^2 dx + \left(\sigma_2\omega_0^2 + \frac{\omega_0^2 m}{\lambda_1} + \frac{1}{4\lambda_1} \right) \int_\Omega \nabla v^2 dx \\ & + \frac{\alpha\omega_0^2}{\lambda_1} \int_\Omega \nabla u^2 dx + (\lambda+\mu)\,\omega_0^2 \int_\Omega |\operatorname{div} v|^2 dx + \mu_1\omega_0^2 \int_\Omega z_2^2(x,1,t)dx \\ & + \left(\frac{\alpha}{4\lambda_1} + \frac{\sigma_2}{4} + \frac{\lambda+\mu}{4} + \omega_0^2 + \frac{\delta_2}{4\lambda_1} + \frac{\mu_2}{4\lambda_1} + \frac{1}{2\lambda_1} \right) \|\eta_2\|_{\mathcal{M}}^2 \\ & - \frac{l_2}{2\omega_0^2\lambda_1} \int_0^{+\infty} \omega_2'(s) \|\nabla\eta_2\|_2^2 ds, \end{aligned}$$

where $l_2 = -\int_0^{+\infty} \omega_2'(s)ds$.

Lemma 4.5. *Let $(u, u_t, v, v_t, z_1, z_2, \eta_1, \eta_2)$ be the solution of the problem* (3.5). *Then, the functional $I_1(t)$ defined by*

$$I_1(t) = \tau_1 \int_\Omega \int_0^1 e^{-\tau_1\rho} z_1^2(x,\rho,t)d\rho dx, \tag{4.12}$$

satisfies the following estimate

$$I_1'(t) \leq -e^{-\tau_1} \left(\tau_1 \int_\Omega \int_0^1 z_1^2(x,\rho,t)d\rho dx + \int_\Omega z_1^2(x,\rho,t)dx \right) + \int_\Omega u_t^2 dx. \tag{4.13}$$

Proof. Differentiating (4.12) with respect to t and using the third equation of the system (3.5), we have

$$\begin{aligned} I_1'(t) &= 2\tau_1 \int_\Omega \int_0^1 e^{-\tau_1\rho} z_{1t}(x,\rho,t) z_1(x,\rho,t) d\rho dx \\ &= -2 \int_\Omega \int_0^1 e^{-\tau_1\rho} z_{1\rho}(x,\rho,t) z_1(x,\rho,t) d\rho dx \\ &= -\int_\Omega \int_0^1 e^{-\tau_1\rho} \frac{\partial}{\partial\rho} (z_1^2(x,\rho,t)) d\rho dx \\ &= -\tau_1 \int_\Omega \int_0^1 z_1^2(x,\rho,t)d\rho dx + \int_\Omega u_t^2 dx - e^{-\tau_1\rho} \int_\Omega z_1^2(x,\rho,t)dx \\ &\leq -e^{-\tau_1} \left(\tau_1 \int_\Omega \int_0^1 z_1^2(x,\rho,t)d\rho dx + \int_\Omega z_1^2(x,\rho,t)dx \right) + \int_\Omega u_t^2 dx. \end{aligned}$$

Therefore, (4.13) holds. □

Lemma 4.6. *Let $(u, u_t, v, v_t, z_1, z_2, \eta_1, \eta_2)$ be the solution of the problem* (3.5). *Then, for the functional $I_2(t)$ defined by*

$$I_2(t) = \tau_2 \int_\Omega \int_0^1 e^{-\tau_2\rho} z_2^2(x,\rho,t)d\rho dx,$$

the following inequality holds

$$I_2'(t) \leq -e^{-\tau_2}\left(\tau_2\int_\Omega\int_0^1 z_2^2(x,\rho,t)d\rho dx + \int_\Omega z_2^2(x,1,t)dx\right) + \int_\Omega v_t^2 dx.$$

Now, we define the Lyapunov functional

$$L(t) = NE(t) + \varepsilon F(t) + J_1(t) + J_2(t) + I_1(t) + I_2(t). \tag{4.14}$$

Then, we obtain the following lemma using the multiplier method.

Lemma 4.7. *Let* $(u, u_t, v, v_t, z_1, z_2, \eta_1, \eta_2)$ *be the solution of the problem* (3.5) *for* N *large enough there exist two constants* ν_1 *and* ν_2 *depending on* N, ε *such that for any* $t \geq 0$

$$\nu_1 E(t) - C_2(\|h_1\|_2^2 + \|h_2\|_2^2) \leq L(t) \leq \nu_2 E(t) + C_2(\|h_1\|_2^2 + \|h_2\|_2^2). \tag{4.15}$$

Proof. From (4.14), we obtain

$$\begin{aligned}
|L(t) - NE(t)| \leq & \left(\frac{\varepsilon}{2} + \omega_0^1\right)\int_\Omega u_t^2 dx + \left(\frac{\varepsilon}{2\lambda_1} + \frac{\varepsilon\delta_1}{2\lambda_1}\right)\int_\Omega \nabla u^2 dx \\
& + \left(\frac{\varepsilon}{2} + \omega_0^2\right)\int_\Omega v_t^2 dx + \left(\frac{\varepsilon}{2\lambda_1} + \frac{\varepsilon\delta_2}{2\lambda_1}\right)\int_\Omega \nabla v^2 dx \\
& + \tau_1\int_\Omega\int_0^1 z_1^2(x,\rho,t)d\rho dx + \tau_2\int_\Omega\int_0^1 z_2^2(x,\rho,t)d\rho dx \\
& + \frac{1}{4\lambda_1}\left(\|\eta_1\|_{\mathcal{M}}^2 + \|\eta_1\|_{\mathcal{M}}^2\right) \\
\leq & CE(t),
\end{aligned} \tag{4.16}$$

with $C = \max\{\frac{\varepsilon}{2} + \omega_0^i, \frac{\varepsilon}{2\lambda_1} + \frac{\varepsilon\delta_i}{2\lambda_1}, \tau_i, \frac{1}{4\lambda_1}\}$, $i = 1, 2$.

Combining (3.17) and (4.16), we choose N large enough that $\nu_1 = N - \tilde{C} > 0$ and $\nu_2 = N + \tilde{C} > 0$. This complete the proof. □

Proof of Theorem 4.5. From the previous lemmas we get

$$\begin{aligned}
L'(t) \leq & -\theta_1\int_\Omega u_t^2 dx - \theta_2\int_\Omega v_t^2 dx - \theta_3\int_\Omega \nabla u^2 dx - \theta_4\int_\Omega \nabla v^2 dx \\
& - \theta_5\int_\Omega |\operatorname{div} u|^2 dx - \theta_6\int_\Omega |\operatorname{div} v|^2 dx - 2\alpha\int_\Omega uv dx \\
& + \varsigma_1\|\eta_1\|_{\mathcal{M}}^2 + \varsigma_2\|\eta_2\|_{\mathcal{M}}^2 + \frac{\varepsilon\omega_0^1}{4\vartheta}\int_\Omega h_1^2 dx + \frac{\varepsilon\omega_0^2}{4\vartheta}\int_\Omega h_2^2 dx \\
& - \tau_1 e^{-\tau_1}\int_\Omega\int_0^1 z_1^2(x,\rho,t)d\rho dx - \tau_2 e^{-\tau_2}\int_\Omega\int_0^1 z_2^2(x,\rho,t)d\rho dx \\
& - \theta_7\int_\Omega z_1^2(x,1,t)dx - \theta_8\int_\Omega z_2^2(x,1,t)dx \\
& + \left(\frac{N}{2} - \frac{l_1}{2\omega_0^1\lambda_1}\right)\int_0^{+\infty}\omega_1'(s)\|\nabla\eta_1\|_2^2 ds
\end{aligned}$$

$$+\left(\frac{N}{2}-\frac{l_2}{2\omega_0^2\lambda_1}\right)\int_0^{+\infty}\omega_2'(s)\|\nabla\eta_2\|_2^2ds.$$

We choose N large enough so that

$$\begin{aligned}
\theta_1 =&N\left(\delta_1-\frac{\xi_1}{2\tau_1}-\frac{\mu_1}{2}\right)-\varepsilon-\omega_0^1\delta_1-\frac{3\omega_0^1}{2}-1>0,\\
\theta_2 =&N\left(\delta_2-\frac{\xi_2}{2\tau_2}-\frac{\mu_2}{2}\right)-\varepsilon-\omega_0^2\delta_2-\frac{3\omega_0^2}{2}-1>0,\\
\theta_3 =&\varepsilon\left(\sigma_1-\frac{1}{2\lambda_1}-\frac{m}{\lambda_1}-\frac{\vartheta}{\lambda_1}-\omega_0^1\right)-\left(\sigma_1\omega_0^1c+\frac{\omega_0^1m}{\lambda_1}+\frac{1}{4\lambda_1}+\frac{\alpha\omega_0^2}{\lambda_1}\right)>0,\\
\theta_4 =&\varepsilon\left(\sigma_2-\frac{1}{2\lambda_1}-\frac{m}{\lambda_1}-\frac{\vartheta}{\lambda_1}-\omega_0^2c\right)-\left(\sigma_2\omega_0^2+\frac{\omega_0^2m}{\lambda_1}+\frac{1}{4\lambda_1}+\frac{\alpha\omega_0^1}{\lambda_1}\right)>0,\\
\theta_5 =&(\varepsilon-\omega_0^1)(\lambda+\mu)>0,\\
\theta_6 =&(\varepsilon-\omega_0^2)(\lambda+\mu)>0,\\
\theta_7 =&N\left(\frac{\xi_1}{2\tau_1}-\frac{\mu_1}{2}\right)+e^{-\tau_1}-\frac{\varepsilon\mu_1^2}{2}-\mu_1\omega_0^1>0,\\
\theta_8 =&N\left(\frac{\xi_2}{2\tau_2}-\frac{\mu_2}{2}\right)+e^{-\tau_2}-\frac{\varepsilon\mu_2^2}{2}-\mu_2\omega_0^2>0,\\
\varsigma_1 =&\frac{\varepsilon}{4c}+\frac{\alpha}{4\lambda_1}+\frac{\sigma_1}{4}+\frac{\lambda+\mu}{4}+\omega_0^1+\frac{\delta_1}{4\lambda_1}+\frac{\mu_1}{4\lambda_1}+\frac{1}{2\lambda_1}>0,\\
\varsigma_2 =&\frac{\varepsilon}{4c}+\frac{\alpha}{4\lambda_1}+\frac{\sigma_2}{4}+\frac{\lambda+\mu}{4}+\omega_0^2+\frac{\delta_2}{4\lambda_1}+\frac{\mu_2}{4\lambda_1}+\frac{1}{2\lambda_1}>0.
\end{aligned}$$

We choose our constants very carefully and properly so there exists a constant $\theta=\min\{\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6,\theta_7,\theta_8\}$. Hence,

$$\begin{aligned}
L'(t)\leq &-\theta E(t)+\left(\frac{N}{2}-\frac{l_1}{2\omega_0^1\lambda_1}\right)\int_0^{+\infty}\omega_1'(s)\|\nabla\eta_1\|_2^2ds\\
&+\left(\frac{N}{2}-\frac{l_2}{2\omega_0^2\lambda_1}\right)\int_0^{+\infty}\omega_2'(s)\|\nabla\eta_2\|_2^2ds+\varsigma_1\|\eta_1\|_{\mathcal{M}}^2\\
&+\varsigma_2\|\eta_2\|_{\mathcal{M}}^2+C_3\|h_1\|_2^2+C_4\|h_2\|_2^2+C_5,
\end{aligned}$$

and we can choose N such that $\frac{N}{2}-\frac{l_1}{2\omega_0^1\lambda_1}\geq 0$, $\frac{N}{2}-\frac{l_2}{2\omega_0^2\lambda_1}\geq 0$, so it follows that

$$L'(t)\leq -\theta E(t)+\varsigma_1\|\eta_1\|_{\mathcal{M}}^2+\varsigma_2\|\eta_2\|_{\mathcal{M}}^2+C_3\|h_1\|_2^2+C_4\|h_2\|_2^2+C_5. \tag{4.17}$$

From the assumption (2.1), we have that

$$\|\eta_i\|_{\mathcal{M}}^2\leq -\frac{1}{k_i}\int_0^{+\infty}\omega_i'(s)\|\nabla\eta_i(s)\|_2^2ds.$$

Together with (4.17), we conclude that there exists a positive constant θ such that

$$
\begin{aligned}
L'(t) \leq & -\theta E(t) - \varsigma_3 \int_0^{+\infty} \omega_1'(s)\|\nabla \eta_1\|_2^2 ds \\
& - \varsigma_4 \int_0^{+\infty} \omega_2'(s)\|\nabla \eta_2\|_2^2 ds + C_3\|h_1\|_2^2 + C_4\|h_2\|_2^2 + C_5.
\end{aligned}
$$

Let $\tilde{L}(t) = L(t) + \varsigma_5 E(t)$. Using (3.15), we get

$$\tilde{L}'(t) \leq -\theta E(t) + C_3\|h_1\|_2^2 + C_4\|h_2\|_2^2 + C_5.$$

From (4.17) we have

$$\tilde{\nu_1} E(t) - C_2(\|h_1\|_2^2 + \|h_2\|_2^2) \leq \tilde{L}(t) \leq \tilde{\nu_2} E(t) + C_2(\|h_1\|_2^2 + \|h_2\|_2^2), \tag{4.18}$$

where $\tilde{\nu_1} = \nu_1 + \varsigma_5$, $\tilde{\nu_2} = \nu_2 + \varsigma_5$. Along with (4.18), we obtain

$$\tilde{L}'(t) \leq \frac{-\theta}{\tilde{\nu_2}} \tilde{L}(t) + C_3'\|h_1\|_2^2 + C_4'\|h_2\|_2^2 + C_5'$$

and

$$\tilde{L}'(t) \leq \tilde{L}(0) e^{\frac{-\theta}{\tilde{\nu_2} t}} + C_3'\|h_1\|_2^2 + C_4'\|h_2\|_2^2 + C_5'.$$

Using (4.18) again, we get

$$E(t) \leq \frac{1}{\tilde{\nu_1}} \left(\tilde{\nu_1} E(0) + C_2\|h_1\|_2^2 + C_2\|h_2\|_2^2 \right) e^{\frac{-\theta}{\tilde{\nu_2} t}} + C_3''\|h_1\|_2^2 + C_4''\|h_2\|_2^2 + C_5''.$$

In view of (3.15), we infer

$$\|(u, u_t, v, v_t, z_1, z_2, \eta_1, \eta_2)\|_{\mathcal{H}}^2 \leq C_0 e^{\frac{-\theta}{\tilde{\nu_2}} t} + C_2'\|h_1\|_2^2 + C_2''\|h_2\|_2^2 + C_2'''.$$

Then, there exists an absorbing ball $B(0, R)$ with radius

$$R > \sqrt{C_2'\|h_1\|_2^2 + C_2''\|h_2\|_2^2 + C_2'''},$$

for the dynamical system $(\mathcal{H}, S(t))$. Hence, we complete the proof of Theorem 4.5. □

4.4. **Quasi-stability.**

Lemma 4.8. *Suppose that assumptions of Theorem* 3.1 *and the given initial data* $(u_0, u_1, v_0, v_1, g_0, \tilde{g_0}, \eta_1^0, \eta_2^0) \in \mathcal{H}$. *Let us consider a bounded subset* $B \subset \mathcal{H}$ *and two weak solutions* $U^1 = (u^1, u_t^1, v^1, v_t^1, z_1^1, z_2^1, \eta_1^1, \eta_2^1)$ *and* $U^2 = (u^2, u_t^2, v^2, v_t^2, z_1^2, z_2^2, \eta_1^2, \eta_2^2)$ *of the problem* (3.5) *and initial data* $U_0^1, U_0^2 \in B$. *Then,*

$$
\begin{aligned}
& \|S(t)U^1 - S(t)U^2\|_{\mathcal{H}}^2 \\
\leq & \|Z_0^1 - Z_0^2\|_{\mathcal{H}}^2 + C_B \sup_{0<s<t} \left(\|u^1(t) - u^2(t)\|_4^2 + \|v^1(t) - v^2(t)\|_4^2 \right).
\end{aligned}
$$

Proof. For any $U_0^1, U_0^2 \in B$, let U^1, U^2 be the corresponding solutions. Let $W(t) = (\Phi, \Psi, Z_1, Z_2, \zeta_1, \zeta_2)^T = U_1(t) - U_2(t) = (u^1 - u^2, v^1 - v^2, z_1^1 - z_1^2, z_2^1 - z_2^2, \eta_1^1 - \eta_1^2, \eta_2^1 - \eta_2^2)$. Then, $W(t)$ verifies

$$\begin{cases} \Phi_{tt} + \alpha\Psi - \Delta_e\Phi + \delta_1\Phi_t - \int_0^{+\infty} \omega_1(s)\Delta\zeta_1(s)ds & \\ + \mu_1 Z_1(x,1,t) + f(u^1) - f(u^2) = 0, & \text{in } \Omega \times (0,+\infty), \\ \Psi_{tt} + \alpha\Phi - \Delta_e\Psi + \delta_2\Psi_t - \int_0^{+\infty} \omega_2(s)\Delta\zeta_2(s)ds & \\ + \mu_2 Z_2(x,1,t) + f(v^1) - f(v^2) = 0, & \text{in } \Omega \times (0,+\infty), \\ \tau_1 Z_{1t}(x,\rho,t) + Z_{1\rho}(x,\rho,t) = 0, & \text{in } \Omega \times (0,1) \times (0,+\infty), \\ \tau_2 Z_{2t}(x,\rho,t) + Z_{2\rho}(x,\rho,t) = 0, & \text{in } \Omega \times (0,1) \times (0,+\infty), \\ \zeta_{1t} + \zeta_{1s} = \Phi_t, & \text{in } \Omega \times (0,+\infty) \times (0,+\infty), \\ \zeta_{2t} + \zeta_{2s} = \Psi_t, & \text{in } \Omega \times (0,+\infty) \times (0,+\infty), \end{cases}$$

with the initial and boundary conditions

$$U_1(0) - U_2(0) = (\Phi_0, \Phi_1, \Psi_0, \Psi_1, J_0, \tilde{J}_0, \zeta_1^0, \zeta_2^0).$$

We denote the associated energy functional by

$$\begin{aligned} \tilde{E}(t) =& \frac{1}{2}\int_\Omega \Phi_t^2 dx + \frac{1}{2}\int_\Omega \Psi_t^2 dx + \frac{\sigma_1}{2}\int_\Omega \nabla\Phi^2 dx + \frac{\sigma_2}{2}\int_\Omega \nabla\Psi^2 dx \\ &+ \alpha\int_\Omega \Phi\Psi dx + \frac{\lambda+\mu}{2}\int_\Omega |\mathrm{div}\,\Phi|^2 dx + \frac{\lambda+\mu}{2}\int_\Omega |\mathrm{div}\,\Psi|^2 dx \\ &+ \frac{\xi_1}{2}\int_\Omega\int_0^1 Z_1^2(x,\rho,t)d\rho dx + \frac{\xi_2}{2}\int_\Omega\int_0^1 Z_2^2(x,\rho,t)d\rho dx \\ &+ \frac{1}{2}\|\zeta_1\|_{\mathcal{M}}^2 + \frac{1}{2}\|\zeta_2\|_{\mathcal{M}}^2. \end{aligned} \tag{4.19}$$

To adress the difference between the nonlinear terms $f(u^1) - f(u^2)$ and $f(v^1) - f(v^2)$, we recall the following result. The proof can be found in [9].

Proposition 4.1. *There exists a constant K_0 such that*

$$\begin{aligned} &\int_0^t \int_\Omega e^{\gamma s}(f(u^1(s)) - f(u^2(s)))\Phi_t(s)dxds \\ \le& K_0 e^{\gamma s}\sup_{0<s<t}\|\Phi\|_4^2 + K_0\int_0^t e^{\gamma s}(\|u_t^1(s)\|_2^2 + \|u_t^2(s)\|_2^2)\tilde{E}(s)ds, \end{aligned} \tag{4.20}$$

where γ is any positive constant. Moreover,

$$\int_\Omega (f(u^1) - f(u^2))\Phi dx \le K_0\|\Phi\|_4^2.$$

Now, we define the following multipliers by

$$B(t) = \int_\Omega \Phi_t\Phi dx + \int_\Omega \Psi_t\Psi dx + \frac{\delta_1}{2}\int_\Omega \Phi^2 dx + \frac{\delta_2}{2}\int_\Omega \Psi^2 dx,$$

$$\begin{aligned}
D_1(t) =& -\int_0^{+\infty} \omega_1(s) \left(\int_\Omega \Phi_t(t)\zeta_1(s)dx \right) ds,\\
D_2(t) =& -\int_0^{+\infty} \omega_2(s) \left(\int_\Omega \Psi_t(t)\zeta_2(s)dx \right) ds,\\
G_1(t) =&\tau_1 \int_\Omega \int_0^1 e^{-\tau_1\rho} Z_1^2(x,\rho,t)d\rho dx,\\
G_2(t) =&\tau_1 \int_\Omega \int_0^1 e^{-\tau_1\rho} Z_2^2(x,\rho,t)d\rho dx.
\end{aligned}$$

Using the similar technique as in the proof of Theorem 4.5, we have

$$\begin{aligned}
B'(t) \le& \int_\Omega \Phi_t^2 dx + \int_\Omega \Psi_t^2 dx - \left(\sigma_1 - \frac{1}{2\lambda_1} - \omega_0^1 c\right) \int_\Omega \nabla\Phi^2 dx\\
&+ \left(\sigma_2 - \frac{1}{2\lambda_1} - \omega_0^2 c\right) \nabla\Psi^2 dx - 2\alpha \int_\Omega \Phi\Psi dx\\
&- (\lambda+\mu) \int_\Omega |\operatorname{div}\Phi|^2 dx - (\lambda+\mu) \int_\Omega |\operatorname{div}\Psi|^2 dx\\
&+ \frac{\mu_1^2}{2} \int_\Omega Z_1^2(x,1,t)dx + \frac{\mu_2^2}{2} \int_\Omega Z_2^2(x,1,t)dx\\
&+ \frac{1}{4c} \left(\|\zeta_1\|_{\mathcal{M}}^2 + \|\zeta_2\|_{\mathcal{M}}^2 \right) - \int_\Omega (f(u^1) - f(u^2))\Phi dx\\
&- \int_\Omega (f(v^1) - f(v^2))\Psi dx
\end{aligned}$$

and

$$\begin{aligned}
D_1'(t) \le& \left(\omega_0^1\delta_1 + \frac{3\omega_0^1}{2}\right) \int_\Omega \Phi_t^2 dx + \left(\sigma_1\omega_0^1 + \frac{1}{4\lambda_1} + \frac{K_0\tilde{\kappa}}{2}\right) \int_\Omega \nabla\Phi^2 dx\\
&+ \left(\frac{\alpha}{4\lambda_1} + \frac{\sigma_1}{4} + \frac{\lambda+\mu}{4} + \omega_0^1 + \frac{\delta_1}{4\lambda_1} + \frac{\mu_1}{4\lambda_1} + \frac{\omega_0^1}{2\tilde{\kappa}\lambda_1}\right) \|\eta_1\|_{\mathcal{M}}^2\\
&+ \frac{\alpha\omega_0^1}{\lambda_1} \int_\Omega \nabla\Psi^2 dx + (\lambda+\mu)\,\omega_0^1 \int_\Omega |\operatorname{div}\Phi|^2 dx\\
&- \frac{l_1}{2\omega_0^1\lambda_1} \int_0^{+\infty} \omega_1'(s)\|\nabla\zeta_1\|_2^2 ds + \mu_1\omega_0^1 \int_\Omega Z_1^2(x,1,t)dx,\\
D_2'(t) \le& \left(\omega_0^2\delta_2 + \frac{3\omega_0^2}{2}\right) \int_\Omega \Psi_t^2 dx + \left(\sigma_2\omega_0^2 + \frac{1}{4\lambda_1} + \frac{K_0\tilde{\kappa}}{2}\right) \int_\Omega \nabla\Psi^2 dx\\
&+ \left(\frac{\alpha}{4\lambda_1} + \frac{\sigma_2}{4} + \frac{\lambda+\mu}{4} + \omega_0^2 + \frac{\delta_2}{4\lambda_1} + \frac{\mu_2}{4\lambda_1} + \frac{\omega_2^0}{2\tilde{\kappa}\lambda_1}\right) \|\eta_2\|_{\mathcal{M}}^2\\
&+ \frac{\alpha\omega_0^2}{\lambda_1} \int_\Omega \nabla\Phi^2 dx + (\lambda+\mu)\,\omega_0^2 \int_\Omega |\operatorname{div}\Phi|^2 dx\\
&- \frac{l_2}{2\omega_0^2\lambda_1} \int_0^{+\infty} \omega_2'(s)\|\nabla\zeta_2\|_2^2 ds + \mu_2\omega_0^2 \int_\Omega Z_2^2(x,1,t)dx.
\end{aligned}$$

In addition, we define the Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) = N\tilde{E}(t) + \varepsilon B(t) + D_1(t) + D_2(t) + G_1(t) + G_2(t).$$

Following the same approach as in the previous argument, we get

$$\begin{aligned}
\mathcal{L}'(t) \leq & - \chi_1 \int_\Omega \Phi_t^2 dx - \chi_2 \int_\Omega \Psi_t^2 dx - \chi_3 \int_\Omega \nabla\Phi^2 dx - \chi_4 \int_\Omega \nabla\Psi^2 dx \\
& - \chi_5 \int_\Omega |\operatorname{div}\Phi|^2 dx - \chi_6 \int_\Omega |\operatorname{div}\Psi|^2 dx - 2\alpha \int_\Omega \Phi\Psi dx \\
& + \chi_7 \|\zeta_1\|^2_{\mathcal{M}} + \chi_8 \|\zeta_2\|^2_{\mathcal{M}} - \tau_1 e^{-\tau_1} \int_\Omega \int_0^1 Z_1^2(x,\rho,t) d\rho dx \\
& - \tau_2 e^{-\tau_2} \int_\Omega \int_0^1 Z_2^2(x,\rho,t) d\rho dx - \chi_9 \int_\Omega Z_1^2(x,1,t) dx \\
& - \chi_{10} \int_\Omega Z_2^2(x,1,t) dx + \left(\frac{N}{2} - \frac{l_1}{2\omega_0^1\lambda_1}\right) \int_0^{+\infty} \omega_1'(s) \|\nabla\zeta_1\|_2^2 ds \\
& + \left(\frac{N}{2} - \frac{l_2}{2\omega_0^2\lambda_1}\right) \int_0^{+\infty} \omega_2'(s) \|\nabla\zeta_2\|_2^2 ds \\
& + N \int_\Omega (f(u^1) - f(u^2)) \Phi_t dx + N \int_\Omega (f(v^1) - f(v^2)) \Psi_t dx \\
& - \varepsilon \int_\Omega (f(u^1) - f(u^2)) \Phi dx - \varepsilon \int_\Omega (f(u^1) - f(u^2)) \Phi dx.
\end{aligned}$$

Then, setting N large enough, we have

$$\begin{aligned}
\chi_1 =& N\left(\delta_1 - \frac{\xi_1}{2} - \frac{\mu_1}{2}\right) - \varepsilon - \left(\omega_0^1\delta_1 + \frac{3\omega_0^1}{2}\right) - 1 > 0, \\
\chi_2 =& (N\left(\delta_2 - \frac{\xi_2}{2} - \frac{\mu_2}{2}\right) - \varepsilon - \left(\omega_0^2\delta_2 + \frac{3\omega_0^2}{2}\right) - 1 > 0, \\
\chi_3 =& \varepsilon\left(\sigma_1 - \frac{1}{2\lambda_1} - \omega_0^1 c\right) - \frac{\alpha\omega_0^2}{\lambda_1} > 0, \\
\chi_4 =& \varepsilon\left(\sigma_2 - \frac{1}{2\lambda_1} - \omega_0^2 c\right) - \frac{\alpha\omega_0^1}{\lambda_1} > 0, \\
\chi_5 =& (\varepsilon - \omega_0^1)(\lambda + \mu) > 0, \\
\chi_6 =& (\varepsilon - \omega_0^2)(\lambda + \mu) > 0, \\
\chi_7 =& \frac{\varepsilon}{4c} + \frac{\alpha}{4\lambda_1} + \frac{\sigma_1}{4} + \frac{\lambda+\mu}{4} + \omega_0^1 + \frac{\delta_1}{4\lambda_1} + \frac{\mu_1}{4\lambda_1} + \frac{\omega_0^1}{2\tilde{\kappa}\lambda_1} > 0, \\
\chi_8 =& \frac{\varepsilon}{4c} + \frac{\alpha}{4\lambda_1} + \frac{\sigma_2}{4} + \frac{\lambda+\mu}{4} + \omega_0^2 + \frac{\delta_2}{4\lambda_1} + \frac{\mu_2}{4\lambda_1} + \frac{\omega_0^2}{2\tilde{\kappa}\lambda_1} > 0, \\
\chi_9 =& N\left(\frac{\xi_1}{2} - \frac{\mu_1}{2}\right) + e^{-\tau_1} - \frac{\varepsilon\mu_1^2}{2} - \mu_1\omega_0^1 > 0,
\end{aligned}$$

$$\chi_{10} = N\left(\frac{\xi_2}{2} - \frac{\mu_2}{2}\right) + e^{-\tau_2} - \frac{\varepsilon\mu_2^2}{2} - \mu_2\omega_0^2 > 0.$$

Consequently, there exists a positive constant $\chi = \min\{\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_9, \chi_{10}\}$ such that

$$\mathcal{L}'(t) \leq -\chi\tilde{E}(t) - \varpi_1 \int_0^{+\infty} \omega_1'(s)\|\nabla\zeta_1\|_2^2 ds - \varpi_2 \int_0^{+\infty} \omega_2'(s)\|\nabla\zeta_2\|_2^2 ds + \mathcal{Q}(t),$$

with

$$\mathcal{Q}(t) = N\int_\Omega (f(u^1) - f(u^2))\Phi_t dx + N\int_\Omega (f(v^1) - f(v^2))\Psi_t dx + K_0\left(\|\Phi\|_4^2 + |\Psi\|_4^2\right).$$

Let $\tilde{\mathcal{L}}(t) = \mathcal{L}(t) + \varpi_3\tilde{E}(t)$. It is easy to verify that there exist two positive constants β_1, β_2 such that

$$\beta_1\tilde{E}(t) \leq \tilde{\mathcal{L}}(t) \leq \beta_2\tilde{E}(t). \tag{4.21}$$

Taking into account (4.19), we conclude that

$$\frac{d\tilde{\mathcal{L}}(t)}{dt} \leq -\chi\tilde{E}(t) + \mathcal{Q}(t).$$

We set $\gamma = \frac{\chi}{\beta_2}$, which leads to

$$\tilde{\mathcal{L}}(t) \leq \tilde{\mathcal{L}}(0)e^{-\gamma t} + \int_\Omega e^{-\gamma(t-s)}\mathcal{Q}(s)dx.$$

Combining this with (4.21), we obtain

$$\begin{aligned}\beta_1\tilde{E}(t) \leq& \beta_2 e^{-\gamma t}\tilde{E}(0) + K_0 \sup_{0<s<t}\left(\|\Phi\|_4^2 + \|\Psi\|_4^2\right)\\ &+ Ne^{-\gamma t}\int_0^t e^{\gamma s}\int_\Omega (f(u^1) - f(u^2))\Phi_t dx\\ &+ Ne^{-\gamma t}\int_0^t e^{\gamma s}\int_\Omega (f(v^1) - f(v^2))\Psi_t dx.\end{aligned}$$

Using the estimate (4.20), we get, for some $K_0' > 0$,

$$\begin{aligned}\tilde{E}(t) \leq& K_0'\tilde{E}(0)e^{-\gamma t} + K_0' \sup_{0<s<t}\left(\|\Phi\|_4^2 + \|\Psi\|_4^2\right)\\ &+ K_0' e^{-\gamma t}\int_0^t e^{\gamma s}\left(\|u_t^1(s)\|_2^2 + \|u_t^2(s)\|_2^2\right)\tilde{E}(t)ds\\ &+ K_0' e^{-\gamma t}\int_0^t e^{\gamma s}\left(\|v_t^1(s)\|_2^2 + \|v_t^2(s)\|_2^2\right)\tilde{E}(t)ds.\end{aligned}$$

Then, applying the Gronwall's lemma for $e^{\frac{\sigma}{\beta_2}}\tilde{E}(t)$, we obtain

$$\begin{aligned}e^{\gamma t}\tilde{E}(t) \leq& \left(K_0'\tilde{E}(0)e^{-\gamma t} + K_0' \sup_{0<s<t}\left(\|\Phi\|_4^2 + \|\Psi\|_4^2\right)\right)\\ &\times \exp\left(K_0' \int_0^t \left(\|u_t^1(s)\|_2^2 + \|u_t^2(s)\|_2^2 + \|v_t^1(s)\|_2^2 + \|v_t^2(s)\|_2^2\right) ds\right).\end{aligned} \tag{4.22}$$

By the uniform bounded of Z in $\mathcal{H}$, we can denote

$$\int_0^1 \left(\|u_t^1(s)\|_2^2 + \|u_t^2(s)\|_2^2 + \|v_t^1(s)\|_2^2 + \|v_t^2(s)\|_2^2 \right) ds = \tilde{C}.$$

Therefore, the inequality (4.22) implies

$$\begin{aligned}\tilde{E}(t) \leq & K_0' e^{K_0' \tilde{C}} \|Z_0^1 - Z_0^2\|_{\mathcal{H}} e^{-\gamma t} \\ & + K_0' e^{K_0' \tilde{C}} \sup_{0<s<t} \left(\|u^1(t) - u^2(t)\|_4^2 + \|v^1(t) - v^2(t)\|_4^2 \right).\end{aligned}$$

Hence, the proof of Lemma 4.8 is complete. □

Proof of Theorem 4.4. Lemma 4.1, Lemma 4.5 and Lemma 4.8 imply that the gradient system $(\mathcal{H}, S(t))$ is dissipative and assymptotical smothness. Along with Theorem 4.3, we can obtain that $(\mathcal{H}, S(t))$ has a finite dimensional global and exponential attractors $\mathcal{A}$ and $\mathcal{A}^{\exp}$, respectively. Moreover, $\mathcal{A}$ has the structure $\mathcal{A} = M_+(\mathcal{N})$. □

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