

## PSEUDO GE-ALGEBRAS AS THE EXTENSION OF GE-ALGEBRAS

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**ABSTRACT.** In this paper, the notion of a pseudo GE-algebra as an extension of a GE-algebra is introduced. Basic properties of pseudo GE-algebras are described. The concepts of strong pseudo BE-algebra, good pseudo BE-algebra, good pseudo GE-algebra, and the relationship between them are established. We provide a condition for a good pseudo BE-algebra to be a pseudo GE-algebra and for a strong pseudo BE-algebra to be a pseudo GE-algebra.

### 1. INTRODUCTION

Henkin and Skolem introduced Hilbert algebras in the fifties for investigations in intuitionistic and other non-classical logics. Diego [4] proved that Hilbert algebras form a variety which is locally finite. Bandaru et al. introduced the notion of GE-algebras which is a generalization of Hilbert algebras, and investigated several properties (see [1]). In 1966, Y. Imai and K. Iseki [10, 12] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Pseudo-valuations were introduced and studied by Y. B. Jun [13]. Georgescu and Iorgulescu [9] introduced an extension of BCK-algebra called pseudo BCK-algebra. Di Nola et al. presented pseudo BL-algebras, which are non-commutative BL-algebras [5, 6]. Moreover, they gave the connection of pseudo BCK-algebra with pseudo MV-algebra and with pseudo BL-algebra. Pseudo BCI-algebras were introduced and studied by W. A. Dudek and Y.

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B Jun (see [7]), as generalizations of pseudo BCK-algebras and BCI-algebras, and they form an important tool for an algebraic axiomatization of implicational fragment of non-classical logic (see [8]). A. Walendziak [16] gave a system of axioms defining pseudo BCK-algebras. Pseudo BCK-algebras were intensively studied in [3, 11, 14]. R. A. Borzooei et al. [2] applied pseudo structure to BE-algebras and investigated its properties. They studied the concepts of pseudo-subalgebra, pseudo-filter and pseudo-upper-set and proved that every pseudo-filter is a union of pseudo-upper-sets. Later on, in 2019, Rezaei et al. defined pseudo CI-algebras, which are a generalization of the pseudo BE-algebras, pseudo BCK-algebras and pseudo MV-algebras [15].

In this paper, we introduce the notion of pseudo GE-algebra as a non-commutative generalization of GE-algebra and study its properties. We define the notion of  $(\otimes, \boxplus)$ -pseudo GE-algebra,  $(\boxplus, \otimes)$ -pseudo GE-algebra and investigate its properties. We define the concept of strong pseudo BE-algebra, good pseudo GE-algebra and study relation between them. Finally, we give a condition for a good pseudo BE-algebra to be a pseudo GE-algebra and for a strong pseudo BE-algebra to be a pseudo GE-algebra.

## 2. PRELIMINARIES

**Definition 2.1** ([1]). By a *GE-algebra* we mean a non-empty set  $X$  with a constant 1 and a binary operation  $*$  satisfying the following axioms:

$$(GE1) \quad u * u = 1;$$

$$(GE2) \quad 1 * u = u;$$

$$(GE3) \quad u * (v * w) = u * (v * (u * w)),$$

for all  $u, v, w \in X$ .

In a GE-algebra  $X$ , a binary relation “ $\leq$ ” is defined by

$$(\forall x, y \in X) (x \leq y \Leftrightarrow x * y = 1).$$

**Definition 2.2** ([1]). A GE-algebra  $X$  is said to be *transitive* if it satisfies:

$$(2.1) \quad (\forall x, y, z \in X) (x * y \leq (z * x) * (z * y)).$$

**Proposition 2.1** ([1]). *Every GE-algebra  $X$  satisfies the following items*

$$(\forall u \in X) (u * 1 = 1),$$

$$(\forall u, v \in X) (u * (u * v) = u * v),$$

$$(\forall u, v \in X) (u \leq v * u),$$

$$(\forall u, v, w \in X) (u * (v * w) \leq v * (u * w)),$$

$$(\forall u \in X) (1 \leq u \Rightarrow u = 1),$$

$$(\forall u, v \in X) (u \leq (v * u) * u),$$

$$(\forall u, v \in X) (u \leq (u * v) * v),$$

$$(\forall u, v, w \in X) (u \leq v * w \Leftrightarrow v \leq u * w).$$

If  $X$  is transitive, then

$$(\forall u, v, w \in X) (u \leq v \Rightarrow w * u \leq w * v, v * w \leq u * w),$$

$$(\forall u, v, w \in X) (u * v \leq (v * w) * (u * w)).$$

**Lemma 2.1** ([1]). *In a GE-algebra  $X$ , the following facts are equivalent to each other*

$$(\forall x, y, z \in X) (x * y \leq (z * x) * (z * y)),$$

$$(\forall x, y, z \in X) (x * y \leq (y * z) * (x * z)).$$

### 3. PSEUDO GE-ALGEBRAS

We consider the notion of a pseudo GE-algebra as a generalization of a GE-algebra. Let  $X$  be a set with two binary operations “ $\otimes$ ” and “ $\boxplus$ ”. Then we can consider the following two cases:

$$(3.1) \quad (\forall x, y, z \in X) (x \otimes (y \boxplus z) = x \boxplus (y \otimes (x \boxplus z))$$

$$\text{and } x \boxplus (y \otimes z) = x \otimes (y \boxplus (x \otimes z))),$$

$$(3.2) \quad (\forall x, y, z \in X) (x \otimes (y \boxplus z) = x \otimes (y \boxplus (x \otimes z))$$

$$\text{and } x \boxplus (y \otimes z) = x \boxplus (y \otimes (x \boxplus z))).$$

Hence we can think of two types of pseudo GE-algebra so called type A and type B.

**Definition 3.1.** Let  $X$  be a set with a constant 1 and two binary operations “ $\otimes$ ” and “ $\boxplus$ ”. A structure  $(X, \otimes, \boxplus, 1)$  is called a *pseudo GE-algebra of type A* if it satisfies (3.1) and the following conditions:

$$(3.3) \quad (\forall x \in X) (x \otimes x = 1 \text{ and } x \boxplus x = 1),$$

$$(3.4) \quad (\forall x \in X) (1 \otimes x = x \text{ and } 1 \boxplus x = x).$$

**Definition 3.2.** Let  $X$  be a set with a constant 1 and two binary operations “ $\otimes$ ” and “ $\boxplus$ ”. A structure  $(X, \otimes, \boxplus, 1)$  is called a *pseudo GE-algebra of type B* if it satisfies (3.2), (3.3) and (3.4).

The pseudo GE-algebra  $(X, \otimes, \boxplus, 1)$  of type A or type B is sometimes only shown as  $X$ . It is clear that if a pseudo GE-algebra  $X$  of type A or type B satisfies  $x \otimes y = x \boxplus y$  for all  $x, y \in X$ , then  $X$  is a GE-algebra.

As you can see above, we have defined two types of pseudo GE-algebra. In considering the pseudo theory as a generalization for a given algebraic system, it is not desirable to have multiple types in the development of theory.

The following theorem shows that one of the two types has no meaning.

**Theorem 3.1.** *If  $X$  is a pseudo GE-algebra of type A, then it is just a GE-algebra.*

*Proof.* Let  $X$  be a pseudo GE-algebra of type A and let  $x, y \in X$ . Then  $1 \otimes (x \boxplus y) = 1 \boxplus (x \otimes (1 \boxplus y))$  by (3.1), and so  $x \boxplus y = x \otimes y$  by (3.4). Therefore,  $X$  is a GE-algebra. □

So in thinking about pseudo theory, which is the generalization of GE-algebra, we can see that Definition 3.2 is the only definition expressed. Based on these discussions, we can call pseudo GE-algebra of type B just pseudo GE-algebra.

Now, we give examples of a pseudo GE-algebra.

*Example 3.1.* Let  $X = \{1, a, b, c, d, e\}$  and define binary operations  $\otimes$  and  $\boxplus$  as follows:

$\otimes$	1	a	b	c	d	e	$\boxplus$	1	a	b	c	d	e
1	1	a	b	c	d	e	1	1	a	b	c	d	e
a	1	1	1	d	d	d	a	1	1	b	1	1	1
b	1	a	1	1	d	d	b	1	a	1	c	e	e
c	1	a	1	1	1	1	c	1	a	1	1	1	1
d	1	a	1	1	1	1	d	1	a	1	1	1	1
e	1	a	1	1	1	1	e	1	a	1	1	1	1

It is routine to verify that  $(X, \otimes, \boxplus, 1)$  is a pseudo-GE-algebra.

**Definition 3.3.** A  $(\otimes, \boxplus)$ -pseudo GE-algebra is a structure  $(X, \otimes, \boxplus, 1)$  in which  $X$  is set with a constant 1 and two binary operations “ $\otimes$ ” and “ $\boxplus$ ” satisfying the conditions (3.3), (3.4) and

$$(3.5) \quad (\forall x, y, z \in X) (x \otimes (y \boxplus z) = x \otimes (y \boxplus (x \otimes z))).$$

*Example 3.2.* Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	a	b	c	$\boxplus$	1	a	b	c
1	1	a	b	c	1	1	a	b	c
a	1	1	b	1	a	a	1	c	1
b	1	a	1	a	b	1	a	1	a
c	1	1	1	1	c	a	1	a	1

Then  $X$  is a  $(\otimes, \boxplus)$ -pseudo GE-algebra.

**Definition 3.4.** A  $(\boxplus, \otimes)$ -pseudo GE-algebra is a structure  $(X, \otimes, \boxplus, 1)$  in which  $X$  is set with a constant 1 and two binary operations “ $\otimes$ ” and “ $\boxplus$ ” satisfying the conditions (3.3), (3.4) and

$$(3.6) \quad (\forall x, y, z \in X) (x \boxplus (y \otimes z) = x \boxplus (y \otimes (x \boxplus z))).$$

*Example 3.3.* Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	a	b	c	$\boxplus$	1	a	b	c
1	1	a	b	c	1	1	a	b	c
a	a	1	a	1	a	1	1	c	c
b	a	a	1	a	b	1	1	1	1
c	a	a	a	1	c	1	1	1	1

Then  $X$  is a  $(\boxplus, \otimes)$ -pseudo GE-algebra.

It is clear that if a structure  $(X, \otimes, \boxplus, 1)$  is both a  $(\otimes, \boxplus)$ -pseudo GE-algebra and a  $(\boxplus, \otimes)$ -pseudo GE-algebra, then it is a pseudo GE-algebra.

Every  $(\otimes, \boxplus)$ -pseudo GE-algebra need not be a  $(\boxplus, \otimes)$ -pseudo GE-algebra. In Example 3.2,  $X$  is  $(\otimes, \boxplus)$ -pseudo GE-algebra. But  $X$  is not a  $(\boxplus, \otimes)$ -pseudo GE-algebra, since

$$a \boxplus (a \otimes b) = a \boxplus b = c \neq a = a \boxplus 1 = a \boxplus (a \otimes c) = a \boxplus (a \otimes (a \boxplus b)).$$

Every  $(\boxplus, \otimes)$ -pseudo GE-algebra need not be a  $(\otimes, \boxplus)$ -pseudo GE-algebra. In Example 3.3,  $X$  is  $(\boxplus, \otimes)$ -pseudo GE-algebra. But  $X$  is not a  $(\otimes, \boxplus)$ -pseudo GE-algebra, since

$$a \otimes (a \boxplus b) = a \otimes c = 1 \neq a = a \otimes 1 = a \otimes (a \boxplus a) = a \otimes (a \boxplus (a \otimes b)).$$

In a  $(\otimes, \boxplus)$ -pseudo GE-algebra or a  $(\boxplus, \otimes)$ -pseudo GE-algebra  $(X, \otimes, \boxplus, 1)$ , we define two binary operations “ $\ll_{\otimes}$ ” and “ $\ll_{\boxplus}$ ” as follows:

$$(\forall x, y \in X)(x \ll_{\otimes} y \Leftrightarrow x \otimes y = 1),$$

$$(\forall x, y \in X)(x \ll_{\boxplus} y \Leftrightarrow x \boxplus y = 1),$$

respectively. For every elements  $x$  and  $y$  of a pseudo GE-algebra  $X$ , if  $x \ll_{\otimes} y$  and  $x \ll_{\boxplus} y$  are formed at the same time, it is represented as  $x \ll y$ .

**Proposition 3.1.** *Every  $(\otimes, \boxplus)$ -pseudo GE-algebra  $X$  satisfies:*

$$(3.7) \quad (\forall x \in X)(x \otimes 1 = 1),$$

$$(3.8) \quad (\forall x, y \in X)(x \otimes (x \otimes y) = x \otimes y),$$

$$(3.9) \quad (\forall x, y \in X)(x \ll_{\otimes} (x \otimes y) \boxplus y).$$

*Proof.* Let  $x, y, z \in X$ . Then

$$1 = x \otimes x = x \otimes (1 \boxplus x) = x \otimes ((x \otimes x) \boxplus x) = x \otimes ((x \otimes x) \boxplus (x \otimes x)) = x \otimes 1$$

which proves (3.7). Using (3.4) and (3.5), we have  $x \otimes y = x \otimes (1 \boxplus y) = x \otimes (1 \boxplus (x \otimes y)) = x \otimes (x \otimes y)$  which shows (3.8). Using (3.3), (3.5) and (3.7), we obtain

$$x \otimes ((x \otimes y) \boxplus y) = x \otimes ((x \otimes y) \boxplus (x \otimes y)) = x \otimes 1 = 1. \quad \square$$

**Proposition 3.2.** *Every  $(\boxplus, \otimes)$ -pseudo GE-algebra  $X$  satisfies:*

$$(3.10) \quad (\forall x \in X)(x \ll_{\boxplus} 1),$$

$$(3.11) \quad (\forall x, y \in X)(x \boxplus (x \boxplus y) = x \boxplus y),$$

$$(3.12) \quad (\forall x, y \in X)(x \ll_{\boxplus} (x \boxplus y) \otimes y).$$

*Proof.* Let  $x, y, z \in X$ . Then

$$1 = x \boxplus x = x \boxplus (1 \otimes x) = x \boxplus (1 \otimes (x \boxplus x)) = x \boxplus (1 \otimes 1) = x \boxplus 1,$$

by (3.3), (3.4) and (3.6), i.e.,  $x \ll_{\boxplus} 1$ . Using (3.4) and (3.6) induces

$$x \boxplus (x \boxplus y) = x \boxplus (1 \otimes (x \boxplus y)) = x \boxplus (1 \otimes y) = x \boxplus y.$$

Using (3.3), (3.6) and (3.10), we obtain

$$x \boxplus ((x \boxplus y) \otimes y) = x \boxplus ((x \boxplus y) \otimes (x \boxplus y)) = x \boxplus 1 = 1,$$

and so (3.12) is valid. □

**Lemma 3.1.** *Every  $(\otimes, \boxplus)$ -pseudo GE-algebra or  $(\boxplus, \otimes)$ -pseudo GE-algebra  $X$  satisfies:*

$$\begin{aligned} (\forall x \in X)(1 \ll_{\otimes} x \Rightarrow x = 1), \\ (\forall x \in X)(1 \ll_{\boxplus} x \Rightarrow x = 1). \end{aligned}$$

*Proof.* Straightforward. □

**Proposition 3.3.** *Every pseudo GE-algebra  $X$  satisfies:*

$$(3.13) \quad (\forall x \in X)(1 \ll x \Rightarrow x = 1).$$

*Proof.* Lemma 3.1 induces (3.13). □

The combination of Propositions 3.1 and 3.2 induces the next proposition.

**Proposition 3.4.** *Every pseudo GE-algebra  $X$  satisfies for all  $x, y, z \in X$*

- (1)  $x \otimes 1 = 1$  and  $x \boxplus 1 = 1$ ;
- (2)  $x \otimes (x \otimes y) = x \otimes y$  and  $x \boxplus (x \boxplus y) = x \boxplus y$ ;
- (3)  $x \ll_{\otimes} (x \otimes y) \boxplus y$  and  $x \ll_{\boxplus} (x \boxplus y) \otimes y$ ;
- (4)  $x \ll_{\boxplus} y \otimes x$  and  $x \ll_{\otimes} y \boxplus x$ ;
- (5)  $x \ll_{\otimes} (y \otimes x) \boxplus x$  and  $x \ll_{\boxplus} (y \boxplus x) \otimes x$ ;
- (6)  $x \ll_{\otimes} (x \otimes y) \boxplus x$  and  $x \ll_{\boxplus} (x \boxplus y) \otimes x$ ;
- (7)  $y \ll_{\otimes} y \boxplus x \Rightarrow x \ll_{\boxplus} y \otimes (y \boxplus x)$ ;
- (8)  $y \ll_{\boxplus} y \otimes x \Rightarrow x \ll_{\otimes} y \boxplus (y \otimes x)$ ;
- (9)  $x \otimes (y \boxplus z) \ll_{\otimes} y \boxplus (x \otimes z)$  and  $x \boxplus (y \otimes z) \ll_{\boxplus} y \otimes (x \boxplus z)$ ;
- (10)  $y \ll_{\boxplus} x \otimes (y \boxplus z) \Rightarrow y \ll_{\otimes} (x \otimes z)$ ;
- (11)  $y \ll_{\otimes} x \boxplus (y \otimes z) \Rightarrow y \ll_{\boxplus} (x \boxplus z)$ ;
- (12)  $x \ll_{\boxplus} y \otimes z \Leftrightarrow y \ll_{\otimes} x \boxplus z$ .

*Proof.* Propositions 3.1 and 3.2 prove (1), (2), (3). Now,

$$x \boxplus (y \otimes x) = x \boxplus (y \otimes (x \boxplus x)) = x \boxplus (y \otimes 1) = x \boxplus 1 = 1$$

and

$$x \otimes (y \boxplus x) = x \otimes (y \boxplus (x \otimes x)) = x \otimes (y \boxplus 1) = x \otimes 1 = 1.$$

Hence  $x \ll_{\boxplus} y \otimes x$  and  $x \ll_{\otimes} y \boxplus x$ . Hence, (4) follows. (5) and (6) follow from (4). Assume that  $y \ll_{\otimes} y \boxplus x$ . Then  $y \otimes (y \boxplus x) = 1$ , which implies that

$$\begin{aligned} x \boxplus (y \otimes (y \boxplus x)) &= x \boxplus (y \otimes (x \boxplus (y \boxplus x))) \\ &= x \boxplus (y \otimes (x \boxplus (y \otimes (y \boxplus x)))) \\ &= x \boxplus (y \otimes (x \boxplus 1)) \\ &= x \boxplus (y \otimes 1) = x \boxplus 1 = 1. \end{aligned}$$

Thus  $x \ll_{\boxplus} y \circledast (y \boxplus x)$ . If  $y \ll_{\boxplus} y \circledast x$ , then  $y \boxplus (y \circledast x) = 1$ . Hence,

$$\begin{aligned} x \circledast (y \boxplus (y \circledast x)) &= x \circledast (y \boxplus (x \circledast (y \circledast x))) \\ &= x \circledast (y \boxplus (x \circledast (y \boxplus (y \circledast x)))) \\ &= x \circledast (y \boxplus (x \circledast 1)) \\ &= x \circledast (y \boxplus 1) = x \circledast 1 = 1, \end{aligned}$$

and so,  $x \ll_{\circledast} y \boxplus (y \circledast x)$ . The combination of (3.2), (3.3) and (1) induces

$$(x \circledast (y \boxplus z)) \circledast (y \boxplus (x \circledast z)) = (x \circledast (y \boxplus z)) \circledast (y \boxplus (x \circledast (y \boxplus z))) = 1, \quad \text{by (4)}$$

and

$$(x \boxplus (y \circledast z)) \boxplus (y \circledast (x \boxplus z)) = (x \boxplus (y \circledast z)) \boxplus (y \circledast (x \boxplus (y \circledast z))) = 1, \quad \text{by (4)}.$$

Hence,  $x \circledast (y \boxplus z) \ll_{\circledast} y \boxplus (x \circledast z)$  and  $x \boxplus (y \circledast z) \ll_{\boxplus} y \circledast (x \boxplus z)$ , that is, (9) is true. If  $y \ll_{\boxplus} x \circledast (y \boxplus z)$ , then  $y \boxplus (x \circledast (y \boxplus z)) = 1$  and hence  $y \boxplus (x \circledast z) = 1$ . Therefore  $y \ll_{\boxplus} (x \circledast z)$ . Thus (10) follows. (11) is similar to (10). If  $x \ll_{\boxplus} y \circledast z$ , then  $1 = x \boxplus (y \circledast z) \ll_{\boxplus} y \circledast (x \boxplus z)$  by (9) and so  $y \circledast (x \boxplus z) = 1$  by Lemma 3.1, i.e.,  $y \ll_{\circledast} x \boxplus z$ . Similarly, if  $y \ll_{\circledast} x \boxplus z$ , then  $x \ll_{\boxplus} y \circledast z$ . Therefore, (12) holds.  $\square$

We consider the following four items

$$(3.14) \quad (\forall x, y, z \in X) (x \circledast y \ll_{\circledast} (z \circledast x) \boxplus (z \circledast y)),$$

$$(3.15) \quad (\forall x, y, z \in X) (x \circledast y \ll_{\boxplus} (y \circledast z) \circledast (x \circledast z)),$$

$$(3.16) \quad (\forall x, y, z \in X) (x \boxplus y \ll_{\boxplus} (z \boxplus x) \circledast (z \boxplus y)),$$

$$(3.17) \quad (\forall x, y, z \in X) (x \boxplus y \ll_{\circledast} (y \boxplus z) \boxplus (x \boxplus z)).$$

*Example 3.4.* 1. Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\circledast$ ” and “ $\boxplus$ ” given in the following tables:

$\circledast$	1	a	b	c	,	$\boxplus$	1	a	b	c
1	1	a	b	c		1	1	a	b	c
a	a	1	a	a		a	1	1	1	1
b	a	a	1	a		b	1	1	1	1
c	a	a	a	1		c	1	1	1	1

Then  $X$  is a  $(\boxplus, \circledast)$ -pseudo GE-algebra. But it does not satisfy (3.14), since

$$(a \circledast 1) \circledast ((1 \circledast a) \boxplus (1 \circledast 1)) = a \circledast (a \boxplus 1) = a \circledast 1 = a \neq 1.$$

2. Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\circledast$ ” and “ $\boxplus$ ” given in the following tables:

$\circledast$	1	a	b	c	,	$\boxplus$	1	a	b	c
1	1	a	b	c		1	1	a	b	c
a	1	1	c	c		a	a	1	a	a
b	1	1	1	c		b	a	a	1	a
c	1	1	1	1		c	a	a	a	1

Then  $X$  is a  $(\otimes, \boxplus)$ -pseudo GE-algebra. But it does not satisfy (3.15), since

$$(a \otimes 1) \boxplus ((1 \otimes b) \otimes (a \otimes b)) = 1 \boxplus (b \otimes c) = 1 \boxplus c = c \neq 1.$$

3. Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	$a$	$b$	$c$	,	$\boxplus$	1	$a$	$b$	$c$
1	1	$a$	$b$	$c$	,	1	1	$a$	$b$	$c$
$a$	1	1	1	1	,	$a$	$a$	1	$a$	$a$
$b$	1	$a$	1	1	,	$b$	1	$a$	1	1
$c$	1	$a$	1	1	,	$c$	1	$a$	1	1

Then  $X$  is a  $(\otimes, \boxplus)$ -pseudo GE-algebra. But it does not satisfy (3.16), since

$$(a \boxplus 1) \boxplus ((1 \boxplus a) \otimes (1 \boxplus 1)) = a \boxplus (a \otimes 1) = a \boxplus 1 = a \neq 1.$$

4. Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	$a$	$b$	$c$	,	$\boxplus$	1	$a$	$b$	$c$
1	1	$a$	$b$	$c$	,	1	1	$a$	$b$	$c$
$a$	$a$	1	$a$	$a$	,	$a$	1	1	$c$	$c$
$b$	$a$	$a$	1	1	,	$b$	1	1	1	$c$
$c$	$a$	1	1	1	,	$c$	1	1	$b$	1

Then  $X$  is a  $(\boxplus, \otimes)$ -pseudo GE-algebra. But it does not satisfy (3.17), since

$$(a \boxplus 1) \otimes ((1 \boxplus b) \boxplus (a \boxplus b)) = 1 \otimes (b \boxplus c) = 1 \otimes c = c \neq 1.$$

**Proposition 3.5.** *Let  $X$  be a  $(\boxplus, \otimes)$ -pseudo GE-algebra. If  $X$  meets condition (3.14), it also meets condition (3.15).*

*Proof.* Assume that a  $(\boxplus, \otimes)$ -pseudo GE-algebra  $X$  satisfies (3.14). Then  $(x \otimes y) \otimes ((z \otimes x) \boxplus (z \otimes y)) = 1$  for all  $x, y, z \in X$ , and so

$$\begin{aligned} (x \otimes y) \boxplus ((y \otimes z) \otimes (x \otimes z)) &= (x \otimes y) \boxplus ((y \otimes z) \otimes ((x \otimes y) \boxplus (x \otimes z))) \\ &= (x \otimes y) \boxplus 1 = 1, \end{aligned}$$

by (3.6) and (3.10). Therefore  $x \otimes y \ll_{\boxplus} (y \otimes z) \otimes (x \otimes z)$  for all  $x, y, z \in X$ . □

**Proposition 3.6.** *Let  $X$  be a  $(\otimes, \boxplus)$ -pseudo GE-algebra. If  $X$  meets condition (3.15), it also meets condition (3.14).*

*Proof.* Assume that a  $(\otimes, \boxplus)$ -pseudo GE-algebra  $X$  satisfies (3.15). Then  $(x \otimes y) \boxplus ((y \otimes z) \otimes (x \otimes z)) = 1$  for all  $x, y, z \in X$ , which implies from (3.5) and (3.7) that

$$\begin{aligned} (x \otimes y) \otimes ((z \otimes x) \boxplus (z \otimes y)) &= (x \otimes y) \otimes ((z \otimes x) \boxplus ((x \otimes y) \otimes (z \otimes y))) \\ &= (x \otimes y) \otimes 1 = 1. \end{aligned}$$

Thus,  $x \otimes y \ll (z \otimes x) \boxplus (z \otimes y)$  for all  $x, y, z \in X$ . □



**Corollary 3.1.** *In a pseudo GE-algebra  $X$ , (3.14) and (3.15) are equivalent with each other.*

The following example shows that the converse of Propositions 3.5 and 3.6 is not true in general.

*Example 3.5.* 1. Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	$a$	$b$	$c$	,	$\boxplus$	1	$a$	$b$	$c$
1	1	$a$	$b$	$c$		1	1	$a$	$b$	$c$
$a$	$a$	1	$a$	$a$		$a$	1	1	1	1
$b$	$a$	$a$	1	$a$		$b$	1	1	1	1
$c$	$a$	$a$	$a$	1		$c$	1	1	1	1

Then  $X$  is a  $(\boxplus, \otimes)$ -pseudo GE-algebra satisfying (3.15). But it does not satisfy (3.14), since

$$(a \otimes 1) \otimes ((1 \otimes a) \boxplus (1 \otimes 1)) = a \otimes (a \boxplus 1) = a \otimes 1 = a \neq 1.$$

2. Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	$a$	$b$	$c$	,	$\boxplus$	1	$a$	$b$	$c$
1	1	$a$	$b$	$c$		1	1	$a$	$b$	$c$
$a$	1	1	$b$	$c$		$a$	1	1	1	1
$b$	1	$a$	1	$a$		$b$	1	$a$	1	$a$
$c$	1	1	1	1		$c$	1	1	1	1

Then  $X$  is a  $(\otimes, \boxplus)$ -pseudo GE-algebra satisfying (3.14). But it does not satisfy (3.15), since

$$(a \otimes b) \boxplus ((b \otimes c) \otimes (a \otimes c)) = b \boxplus (a \otimes c) = b \boxplus c = a \neq 1.$$

**Proposition 3.7.** *Let  $X$  be a  $(\otimes, \boxplus)$ -pseudo GE-algebra. If  $X$  meets condition (3.16), it also meets condition (3.17).*

*Proof.* If a  $(\otimes, \boxplus)$ -pseudo GE-algebra  $X$  meets condition (3.16), then  $(x \boxplus y) \boxplus ((z \boxplus x) \otimes (z \boxplus y)) = 1$  for all  $x, y, z \in X$ . Hence,

$$\begin{aligned} (x \boxplus y) \otimes ((y \boxplus z) \boxplus (x \boxplus z)) &= (x \boxplus y) \otimes ((y \boxplus z) \boxplus ((x \boxplus y) \otimes (x \boxplus z))) \\ &= (x \boxplus y) \otimes 1 = 1, \end{aligned}$$

by (3.5) and (3.7), that is,  $x \boxplus y \ll_{\otimes} (y \boxplus z) \boxplus (x \boxplus z)$  for all  $x, y, z \in X$ . □

**Proposition 3.8.** *Let  $X$  be a  $(\boxplus, \otimes)$ -pseudo GE-algebra. If  $X$  meets condition (3.17), it also meets condition (3.16).*

*Proof.* Suppose that the condition (3.17) holds in a  $(\boxplus, \otimes)$ -pseudo GE-algebra  $X$ . Then

$$(x \boxplus y) \otimes ((y \boxplus z) \boxplus (x \boxplus z)) = 1,$$

for all  $x, y, z \in X$ . Using (3.6) and (3.10) induces

$$\begin{aligned} (x \boxplus y) \boxplus ((z \boxplus x) \otimes (z \boxplus y)) &= (x \boxplus y) \boxplus ((z \boxplus x) \otimes ((x \boxplus y) \boxplus (z \boxplus y))) \\ &= (x \boxplus y) \boxplus 1 = 1, \end{aligned}$$

that is,  $x \boxplus y \ll_{\boxplus} (z \boxplus x) \otimes (z \boxplus y)$  for all  $x, y, z \in X$ . □

**Corollary 3.2.** *In a pseudo GE-algebra  $X$ , (3.16) and (3.17) are equivalent with each other.*

The following example shows that the converse of Propositions 3.7 and 3.8 is not true in general.

*Example 3.6.* 1. Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	$a$	$b$	$c$	,	$\boxplus$	1	$a$	$b$	$c$
1	1	$a$	$b$	$c$		1	1	$a$	$b$	$c$
$a$	1	1	1	1		$a$	$a$	1	$a$	$a$
$b$	1	$a$	1	1		$b$	1	$a$	1	1
$c$	1	$a$	1	1		$c$	1	$a$	1	1

Then  $X$  is a  $(\otimes, \boxplus)$ -pseudo GE-algebra satisfying (3.17). But it does not satisfy (3.16), since

$$(a \boxplus 1) \boxplus ((1 \boxplus a) \otimes (1 \boxplus 1)) = a \boxplus (a \otimes 1) = a \boxplus 1 = a \neq 1.$$

2. Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	$a$	$b$	$c$	,	$\boxplus$	1	$a$	$b$	$c$
1	1	$a$	$b$	$c$		1	1	$a$	$b$	$c$
$a$	1	1	1	1		$a$	1	1	$b$	$c$
$b$	1	$a$	1	$a$		$b$	1	$a$	1	$a$
$c$	1	1	1	1		$c$	1	1	1	1

Then  $X$  is a  $(\boxplus, \otimes)$ -pseudo GE-algebra satisfying (3.16). But it does not satisfy (3.17), since

$$(a \boxplus b) \otimes ((b \boxplus c) \boxplus (a \boxplus c)) = b \otimes (a \boxplus c) = b \otimes c = a \neq 1.$$

**Definition 3.5.** A  $(\otimes, \boxplus)$ -pseudo GE-algebra  $X$  is said to be

- $\otimes$ -transitive if it satisfies (3.15);
- $\boxplus$ -transitive if it satisfies (3.16).

If a  $(\otimes, \boxplus)$ -pseudo GE-algebra  $X$  is both  $\otimes$ -transitive and  $\boxplus$ -transitive, we say  $X$  is a *transitive  $(\otimes, \boxplus)$ -pseudo GE-algebra*.

*Example 3.7.* 1. Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	$a$	$b$	$c$	,	$\boxplus$	1	$a$	$b$	$c$
1	1	$a$	$b$	$c$		1	1	$a$	$b$	$c$
$a$	1	1	$b$	1		$a$	1	1	$a$	1
$b$	1	1	1	1		$b$	1	1	1	1
$c$	1	$a$	$b$	1		$c$	1	$a$	$a$	1

Then  $X$  is a  $\otimes$ -transitive  $(\otimes, \boxplus)$ -pseudo GE-algebra. But it is not  $\boxplus$ -transitive since

$$(a \boxplus b) \boxplus ((1 \boxplus a) \otimes (1 \boxplus b)) = a \boxplus (a \otimes b) = a \boxplus b = a \neq 1.$$

2. Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	$a$	$b$	$c$	,	$\boxplus$	1	$a$	$b$	$c$
1	1	$a$	$b$	$c$		1	1	$a$	$b$	$c$
$a$	1	1	1	$c$		$a$	1	1	1	$c$
$b$	1	$a$	1	1		$b$	1	$a$	1	$c$
$c$	1	$a$	1	1		$c$	1	$a$	1	1

Then  $X$  is a  $(\otimes, \boxplus)$ -pseudo GE-algebra which is  $\boxplus$ -transitive. But it is not  $\otimes$ -transitive since

$$(a \otimes b) \boxplus ((b \otimes c) \otimes (a \otimes c)) = 1 \boxplus (1 \otimes c) = 1 \boxplus c = c \neq 1.$$

3. Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	$a$	$b$	$c$	,	$\boxplus$	1	$a$	$b$	$c$
1	1	$a$	$b$	$c$		1	1	$a$	$b$	$c$
$a$	1	1	$b$	$b$		$a$	1	1	$c$	$b$
$b$	1	$a$	1	1		$b$	1	$a$	1	1
$c$	1	$a$	1	1		$c$	1	$a$	1	1

Then  $X$  is a transitive  $(\otimes, \boxplus)$ -pseudo GE-algebra.

**Proposition 3.9.** *Every  $\otimes$ -transitive  $(\otimes, \boxplus)$ -pseudo GE-algebra  $X$  satisfies*

$$(3.18) \quad (\forall x, y, z \in X)(x \ll_{\otimes} y \Rightarrow y \otimes z \ll_{\otimes} x \otimes z, z \otimes x \ll_{\boxplus} z \otimes y),$$

$$(3.19) \quad (\forall x, y, z \in X)((x \otimes y) \boxplus y) \otimes z \ll_{\otimes} x \otimes z, z \otimes x \ll_{\boxplus} z \otimes ((x \otimes y) \boxplus y),$$

$$(3.20) \quad (\forall x, y, z \in X)((y \otimes x) \boxplus x) \otimes z \ll_{\otimes} x \otimes z, z \otimes x \ll_{\boxplus} z \otimes ((y \otimes x) \boxplus x).$$

*Proof.* Let  $X$  be a  $\otimes$ -transitive  $(\otimes, \boxplus)$ -pseudo GE-algebra. If  $x \ll_{\otimes} y$ , then  $x \otimes y = 1$  and thus

$$(y \otimes z) \otimes (x \otimes z) = 1 \boxplus ((y \otimes z) \otimes (x \otimes z)) = (x \otimes y) \boxplus ((y \otimes z) \otimes (x \otimes z)) = 1,$$

that is,  $y \otimes z \ll_{\otimes} x \otimes z$ . Since  $X$  satisfies (3.14) by Proposition 3.6, we have

$$(z \otimes x) \boxplus (z \otimes y) = 1 \otimes ((z \otimes x) \boxplus (z \otimes y)) = (x \otimes y) \otimes ((z \otimes x) \boxplus (z \otimes y)) = 1$$

and so,  $z \otimes x \ll_{\boxplus} z \otimes y$ . This proves (3.18). The combination of (3.9) and (3.18) induces (3.19). The combination of Proposition 3.4 (5) and (3.18) induces (3.20).  $\square$

**Proposition 3.10.** *Every  $\boxplus$ -transitive  $(\otimes, \boxplus)$ -pseudo GE-algebra  $X$  satisfies:*

$$(\forall x, y, z \in X)(x \ll_{\boxplus} y \Rightarrow z \boxplus x \ll_{\otimes} z \boxplus y, y \boxplus z \ll_{\boxplus} x \boxplus z).$$

*Proof.* Let  $X$  be a  $\boxplus$ -transitive  $(\otimes, \boxplus)$ -pseudo GE-algebra. If  $x \ll_{\boxplus} y$ , then  $x \boxplus y = 1$  and thus

$$(z \boxplus x) \otimes (z \boxplus y) = 1 \boxplus ((z \boxplus x) \otimes (z \boxplus y)) = (x \boxplus y) \boxplus ((z \boxplus x) \otimes (z \boxplus y)) = 1.$$

Thus,  $z \boxplus x \ll_{\otimes} z \boxplus y$ . By Proposition 3.7, we know that  $X$  satisfies (3.17). Hence,

$$(y \boxplus z) \boxplus (x \boxplus z) = 1 \otimes ((y \boxplus z) \boxplus (x \boxplus z)) = (x \boxplus y) \otimes ((y \boxplus z) \boxplus (x \boxplus z)) = 1,$$

and so  $y \boxplus z \ll_{\boxplus} x \boxplus z$ .  $\square$

**Corollary 3.3.** *Every transitive  $(\otimes, \boxplus)$ -pseudo GE-algebra  $X$  satisfies:*

$$(\forall x, y, z \in X) \left( x \ll y \Rightarrow \begin{cases} y \otimes z \ll_{\otimes} x \otimes z, z \otimes x \ll_{\boxplus} z \otimes y \\ z \boxplus x \ll_{\otimes} z \boxplus y, y \boxplus z \ll_{\boxplus} x \boxplus z \end{cases} \right).$$

**Definition 3.6.** A  $(\boxplus, \otimes)$ -pseudo GE-algebra  $X$  is said to be

- $\otimes$ -transitive if it satisfies (3.14);
- $\boxplus$ -transitive if it satisfies (3.17).

If a  $(\boxplus, \otimes)$ -pseudo GE-algebra  $X$  is both  $\otimes$ -transitive and  $\boxplus$ -transitive, we say  $X$  is a *transitive  $(\boxplus, \otimes)$ -pseudo GE-algebra*.

*Example 3.8.* 1. Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	$a$	$b$	$c$	,	$\boxplus$	1	$a$	$b$	$c$
1	1	$a$	$b$	$c$		1	1	$a$	$b$	$c$
$a$	1	1	$a$	$b$		$a$	1	1	1	$c$
$b$	1	1	1	1		$b$	1	1	1	1
$c$	1	1	1	1		$c$	1	1	1	1

Then  $X$  is a  $(\boxplus, \otimes)$ -pseudo GE-algebra which is  $\otimes$ -transitive. But it is not  $\boxplus$ -transitive since

$$(a \boxplus b) \otimes ((b \boxplus c) \boxplus (a \boxplus c)) = 1 \otimes (1 \boxplus c) = 1 \otimes c = c \neq 1.$$

2. Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	$a$	$b$	$c$	,	$\boxplus$	1	$a$	$b$	$c$
1	1	$a$	$b$	$c$		1	1	$a$	$b$	$c$
$a$	1	1	$a$	1		$a$	1	1	$b$	1
$b$	1	1	1	1		$b$	1	1	1	1
$c$	1	$a$	$a$	1		$c$	1	$a$	$b$	1

Then  $X$  is a  $(\boxplus, \otimes)$ -pseudo GE-algebra which is  $\boxplus$ -transitive. But it is not  $\otimes$ -transitive since

$$(a \otimes b) \otimes ((1 \otimes a) \boxplus (1 \otimes b)) = a \otimes (a \boxplus b) = a \otimes b = a \neq 1.$$

3. Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	a	b	c	,	$\boxplus$	1	a	b	c
1	1	a	b	c		1	1	a	b	c
a	1	1	b	b		a	1	1	c	c
b	1	a	1	1		b	1	a	1	1
c	1	a	1	1		c	1	a	1	1

Then  $X$  is a transitive  $(\boxplus, \otimes)$ -pseudo GE-algebra.

**Proposition 3.11.** *Every  $\boxplus$ -transitive  $(\boxplus, \otimes)$ -pseudo GE-algebra  $X$  satisfies*

$$(3.21) \quad (\forall x, y, z \in X)(x \ll_{\boxplus} y \Rightarrow y \boxplus z \ll_{\boxplus} x \boxplus z, z \boxplus x \ll_{\otimes} z \boxplus y),$$

$$(3.22) \quad (\forall x, y, z \in X)((x \boxplus y) \otimes y) \boxplus z \ll_{\boxplus} x \boxplus z, z \boxplus x \ll_{\otimes} z \boxplus ((x \boxplus y) \otimes y),$$

$$(3.23) \quad (\forall x, y, z \in X)((y \boxplus x) \otimes x) \boxplus z \ll_{\boxplus} x \boxplus z, z \boxplus x \ll_{\otimes} z \boxplus ((y \boxplus x) \otimes x).$$

*Proof.* Let  $X$  be a  $\boxplus$ -transitive  $(\boxplus, \otimes)$ -pseudo GE-algebra. Let  $x, y \in X$  be such that  $x \ll_{\boxplus} y$ . Then  $x \boxplus y = 1$  and so

$$(y \boxplus z) \boxplus (x \boxplus z) = 1 \otimes ((y \boxplus z) \boxplus (x \boxplus z)) = (x \boxplus y) \otimes ((y \boxplus z) \boxplus (x \boxplus z)) = 1,$$

by (3.4) and (3.17). Hence,  $y \boxplus z \ll_{\boxplus} x \boxplus z$ . We know that  $X$  satisfies (3.16) by Proposition 3.8. Thus,

$$(z \boxplus x) \otimes (z \boxplus y) = 1 \boxplus ((z \boxplus x) \otimes (z \boxplus y)) = (x \boxplus y) \boxplus ((z \boxplus x) \otimes (z \boxplus y)) = 1$$

and so  $z \boxplus x \ll_{\otimes} z \boxplus y$ , which proves (3.21). If we combine (3.21) and (3.12), then we have (3.22). The result (3.23) follows from the combination of (3.21) and Proposition 3.4 (5). □

**Proposition 3.12.** *Every  $\otimes$ -transitive  $(\boxplus, \otimes)$ -pseudo GE-algebra  $X$  satisfies:*

$$(\forall x, y, z \in X)(x \ll_{\otimes} y \Rightarrow z \otimes x \ll_{\boxplus} z \otimes y, y \otimes z \ll_{\otimes} x \otimes z).$$

*Proof.* Let  $X$  be a  $\otimes$ -transitive  $(\boxplus, \otimes)$ -pseudo GE-algebra. If  $x \ll_{\otimes} y$ , then  $x \otimes y = 1$  and so

$$(z \otimes x) \boxplus (z \otimes y) = 1 \otimes ((z \otimes x) \boxplus (z \otimes y)) = (x \otimes y) \otimes ((z \otimes x) \boxplus (z \otimes y)) = 1,$$

which shows that  $z \otimes x \ll_{\boxplus} z \otimes y$ . Using Proposition 3.5, we know that  $X$  satisfies condition (3.15). Thus,

$$(y \otimes z) \otimes (x \otimes z) = 1 \boxplus ((y \otimes z) \otimes (x \otimes z)) = (x \otimes y) \boxplus ((y \otimes z) \otimes (x \otimes z)) = 1,$$

and therefore  $y \otimes z \ll_{\otimes} x \otimes z$ . □

**Corollary 3.4.** *Every transitive  $(\boxplus, \otimes)$ -pseudo GE-algebra  $X$  satisfies:*

$$(\forall x, y, z \in X) \left( x \ll y \Rightarrow \begin{cases} y \boxplus z \ll_{\boxplus} x \boxplus z, z \boxplus x \ll_{\otimes} z \boxplus y \\ z \otimes x \ll_{\boxplus} z \otimes y, y \otimes z \ll_{\otimes} x \otimes z \end{cases} \right).$$

4. RELATIONS BETWEEN PSEUDO BE-ALGEBRAS AND PSEUDO GE-ALGEBRAS

As an extension of BE-algebras, Borzooei et al. introduced the notion of pseudo BE-algebras, and investigated its properties.

**Definition 4.1** ([2]). Let  $X$  be a set with a constant 1 and two binary operations “ $\otimes$ ” and “ $\boxplus$ ”. A structure  $(X, \otimes, \boxplus, 1)$  is called a *pseudo BE-algebra* if it satisfies (3.3), (3.4) and

(4.1)  $(\forall x \in X)(x \otimes 1 = 1, x \boxplus 1 = 1),$

(4.2)  $(\forall x, y, z \in X)(x \otimes (y \boxplus z) = y \boxplus (x \otimes z)),$

(4.3)  $(\forall x, y \in X)(x \otimes y = 1 \Leftrightarrow x \boxplus y = 1).$

Pseudo GE-algebra and pseudo BE-algebra basically form no relationship. In other words, the pseudo GE-algebra may not be the pseudo BE-algebra, and vice versa as seen in the following example.

*Example 4.1.* 1. Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	a	b	c	,	$\boxplus$	1	a	b	c
1	1	a	b	c		1	1	a	b	c
a	1	1	b	b		a	1	1	c	c
b	1	a	1	1		b	1	a	1	1
c	1	a	1	1		c	1	a	1	1

Then  $X$  is a pseudo GE-algebra. But  $X$  is not a pseudo BE-algebra since

$$a \otimes (a \boxplus b) = a \otimes c = b \neq c = a \boxplus b = a \boxplus (a \otimes b).$$

2. Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	a	b	c	,	$\boxplus$	1	a	b	c
1	1	a	b	c		1	1	a	b	c
a	1	1	c	b		a	1	1	b	c
b	1	a	1	1		b	1	a	1	1
c	1	a	1	1		c	1	a	1	1

Then  $X$  is a pseudo BE-algebra. But  $X$  is not a pseudo GE-algebra since

$$a \otimes (a \boxplus b) = a \otimes b = c \neq b = a \otimes c = a \otimes (a \boxplus c) = a \otimes (a \boxplus (a \otimes b)).$$

The following example shows that when  $(X, \otimes, \boxplus, 1)$  is a pseudo BE-algebra,  $(X, \otimes, 1)$  or  $(X, \boxplus, 1)$  does not need to be BE-algebra.

*Example 4.2.* Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	a	b	c		$\boxplus$	1	a	b	c
1	1	a	b	c	,	1	1	a	b	c
a	1	1	1	a		a	1	1	1	b
b	1	a	1	a		b	1	a	1	c
c	1	1	1	1		c	1	1	1	1

Then  $X$  is a pseudo BE-algebra. But  $(X, \otimes, 1)$  is not a BE-algebra since

$$a \otimes (b \otimes c) = a \otimes a = 1 \neq a = b \otimes a = b \otimes (a \otimes c).$$

Also,  $(X, \boxplus, 1)$  is not a BE-algebra since

$$a \boxplus (b \boxplus c) = a \boxplus c = b \neq 1 = b \boxplus b = b \boxplus (a \boxplus c).$$

**Definition 4.2.** A pseudo BE-algebra  $(X, \otimes, \boxplus, 1)$  is said to be *strong* if  $(X, \otimes, 1)$  and  $(X, \boxplus, 1)$  are BE-algebras.

*Example 4.3.* Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	a	b	c		$\boxplus$	1	a	b	c
1	1	a	b	c	,	1	1	a	b	c
a	1	1	a	1		a	1	1	c	1
b	1	1	1	1		b	1	1	1	1
c	1	1	a	1		c	1	1	a	1

Then  $X$  is a strong pseudo BE-algebra.

We have the following question.

*Question 4.1.* Does pseudo GE-algebra  $X$  satisfy the following conditions

$$(4.4) \quad (\forall x, y, z \in X) \left( \begin{array}{l} x \otimes (y \boxplus z) = x \otimes (y \boxplus (x \boxplus z)) \\ x \boxplus (y \otimes z) = x \boxplus (y \otimes (x \otimes z)) \end{array} \right)?$$

The following example shows that the answer to Question 4.1 is negative.

*Example 4.4.* Let  $X = \{1, a, b, c, d\}$  be a set with binary operations  $\otimes, \boxplus$  given in the following table:

$\otimes$	1	a	b	c	d		$\boxplus$	1	a	b	c	d
1	1	a	b	c	d	,	1	1	a	b	c	d
a	1	1	c	c	1		a	1	1	d	1	d
b	1	1	1	1	1	,	b	1	a	1	1	a
c	1	1	1	1	1		c	1	a	a	1	a
d	1	1	1	1	1		d	1	a	a	1	1

Then  $X$  is a pseudo GE-algebra. But it does not satisfy (4.4) since

$$a \otimes (1 \boxplus b) = a \otimes b = c \neq 1 = a \otimes d = a \otimes (1 \boxplus d) = a \otimes (1 \boxplus (a \boxplus b)),$$

$$a \boxplus (1 \otimes b) = a \boxplus b = d \neq 1 = a \boxplus c = a \boxplus (1 \otimes c) = a \boxplus (1 \otimes (a \otimes b)).$$

**Definition 4.3.** Let  $X$  be a set with a constant 1 and two binary operations “ $\otimes$ ” and “ $\boxplus$ ”. A structure  $(X, \otimes, \boxplus, 1)$  is called a *good pseudo GE-algebra* if it satisfies (3.3), (3.4) and (4.4).

*Example 4.5.* Let  $X = \{1, a, b, c, d\}$  be a set with binary operations  $\otimes, \boxplus$  given in the following table:

$\otimes$	1	$a$	$b$	$c$	$d$	,	$\boxplus$	1	$a$	$b$	$c$	$d$
1	1	$a$	$b$	$c$	$d$		1	1	$a$	$b$	$c$	$d$
$a$	1	1	1	$c$	$c$		$a$	1	1	1	$c$	$c$
$b$	1	$a$	1	$c$	$c$		$b$	1	$a$	1	$d$	$d$
$c$	1	$a$	1	1	1		$c$	1	$a$	1	1	1
$d$	1	$a$	1	1	1		$d$	1	$a$	1	1	1

Then  $X$  is a good pseudo GE-algebra.

**Theorem 4.1.** *Every good pseudo GE-algebra is a pseudo GE-algebra. But the converse is not true.*

*Proof.* Example 4.4 shows that any pseudo GE-algebra may not be a good pseudo GE-algebra. Let  $X$  be a good pseudo GE-algebra and let  $x, y, z \in X$ . Then

$$x \otimes y = x \otimes (1 \boxplus y) = x \otimes (1 \boxplus (x \boxplus y)) = x \otimes (x \boxplus y)$$

and

$$x \boxplus y = x \boxplus (1 \otimes y) = x \boxplus (1 \otimes (x \otimes y)) = x \boxplus (x \otimes y).$$

It follows from (4.4) that

$$x \otimes (y \boxplus (x \otimes z)) = x \otimes (y \boxplus (x \boxplus (x \otimes z))) = x \otimes (y \boxplus (x \boxplus z)) = x \otimes (y \boxplus z)$$

and

$$x \boxplus (y \otimes (x \boxplus z)) = x \boxplus (y \otimes (x \otimes (x \boxplus z))) = x \boxplus (y \otimes (x \otimes z)) = x \boxplus (y \otimes z).$$

Therefore,  $X$  is a pseudo GE-algebra. □

The following example shows that any pseudo BE-algebra  $X$  does not satisfy the condition

$$(4.5) \quad (\forall x, y, z \in X) \left( \begin{array}{l} x \otimes (y \boxplus z) = (x \otimes y) \boxplus (x \otimes z) \\ x \boxplus (y \otimes z) = (x \boxplus y) \otimes (x \boxplus z) \end{array} \right).$$

*Example 4.6.* Let  $X = \{1, a, b, c, d\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	$a$	$b$	$c$	$d$	,	$\boxplus$	1	$a$	$b$	$c$	$d$
1	1	$a$	$b$	$c$	$d$		1	1	$a$	$b$	$c$	$d$
$a$	1	1	$a$	$c$	1		$a$	1	1	$d$	$c$	1
$b$	1	1	1	1	1		$b$	1	1	1	1	1
$c$	1	$a$	$a$	1	1		$c$	1	$a$	$b$	1	1
$d$	1	$a$	$a$	$c$	1		$d$	1	$a$	$b$	$c$	1



Then  $X$  is a pseudo BE-algebra. But  $X$  does not satisfy (4.5) since

$$a \otimes (b \boxplus c) = a \otimes 1 = 1 \neq c = a \boxplus c = (a \otimes b) \boxplus (a \otimes c)$$

and

$$a \boxplus (b \otimes c) = a \boxplus 1 = 1 \neq c = d \otimes c = (a \boxplus b) \otimes (a \boxplus c).$$

*Question 4.2.* Does pseudo BE-algebra  $X$  satisfy the following conditions

$$(4.6) \quad (\forall x, y, z \in X) \left( \begin{array}{l} x \otimes (y \otimes z) = y \otimes (x \otimes z) \\ x \boxplus (y \boxplus z) = y \boxplus (x \boxplus z) \end{array} \right)?$$

The answer to Question 4.2 is negative as seen in the following example.

*Example 4.7.* Let  $X = \{1, a, b, c\}$  be a set with binary operations “ $\otimes$ ” and “ $\boxplus$ ” given in the following tables:

$\otimes$	1	$a$	$b$	$c$	,	$\boxplus$	1	$a$	$b$	$c$
1	1	$a$	$b$	$c$	,	1	1	$a$	$b$	$c$
$a$	1	1	1	$a$	,	$a$	1	1	1	$b$
$b$	1	$a$	1	$a$	,	$b$	1	$a$	1	$c$
$c$	1	1	1	1	,	$c$	1	1	1	1

Then  $X$  is a pseudo BE-algebra. But it does not satisfy (4.6) since

$$a \otimes (b \otimes c) = a \otimes a = 1 \neq a = b \otimes a = b \otimes (a \otimes c)$$

and

$$a \boxplus (b \boxplus c) = a \boxplus c = b \neq 1 = b \boxplus b = b \boxplus (a \boxplus c).$$

**Definition 4.4.** Let  $X$  be a set with a constant 1 and two binary operations “ $\otimes$ ” and “ $\boxplus$ ”. A structure  $(X, \otimes, \boxplus, 1)$  is called a *good pseudo BE-algebra* if it satisfies (3.3), (3.4), (4.1), (4.3) and (4.6).

*Example 4.8.* Let  $X = \{1, a, b, c\}$  be a set with binary operations  $\otimes, \boxplus$  given in the following tables:

$\otimes$	1	$a$	$b$	$c$	,	$\boxplus$	1	$a$	$b$	$c$
1	1	$a$	$b$	$c$	,	1	1	$a$	$b$	$c$
$a$	1	1	$a$	1	,	$a$	1	1	$b$	1
$b$	1	1	1	1	,	$b$	1	1	1	1
$c$	1	$a$	$b$	1	,	$c$	1	$a$	$b$	1

Then  $X$  is a good pseudo BE-algebra. But  $X$  is not pseudo BE-algebra since

$$a \otimes (a \boxplus b) = a \otimes b = a \neq 1 = a \boxplus a = a \boxplus (a \otimes b).$$

We now consider conditions for a pseudo BE-algebra to be a pseudo GE-algebra.

**Theorem 4.2.** *If a good pseudo BE-algebra  $X$  satisfies the condition (4.5), then it is a pseudo GE-algebra.*

*Proof.* Let  $X$  be a good pseudo BE-algebra that satisfies the conditions (4.5). It is sufficient to show that  $X$  satisfies the condition (4.4). Let  $x, y, z \in X$ . Then  $x \otimes (x \boxplus y) = (x \otimes x) \boxplus (x \otimes y) = 1 \boxplus (x \otimes y) = x \otimes y$  and  $x \boxplus (x \otimes y) = (x \boxplus x) \otimes (x \boxplus y) = 1 \otimes (x \boxplus y) = x \boxplus y$  by (3.3), (3.4) and (4.5). It follows that  $x \otimes (y \boxplus z) = x \otimes (x \boxplus (y \boxplus z)) = x \otimes (y \boxplus (x \boxplus z))$  and  $x \boxplus (y \otimes z) = x \boxplus (x \otimes (y \otimes z)) = x \boxplus (y \otimes (x \otimes z))$ . Hence  $X$  is a good pseudo GE-algebra, and therefore it is a pseudo GE-algebra by Theorem 4.1.  $\square$

**Corollary 4.1.** *Every strong pseudo BE-algebra  $X$  satisfying the condition (4.5) is a (good) pseudo GE-algebra.*

We finally pose the following question.

*Question 4.3.* What conditions will be required to make pseudo GE-algebra into pseudo BE-algebra?

## 5. CONCLUDING REMARKS

In this paper, we generalized GE-algebras to the case of pseudo GE-algebras and studied some basic of those properties. In the last section we investigated among relation between pseudo BE-algebras and pseudo GE-algebras. Starting from these notions, one can define and investigate commutative pseudo GE-algebras, involutive pseudo-GE algebras and Smarandache pseudo GE-algebras. Another topic of research could be to define and investigate state and monadic on pseudo GE-algebras.

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