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NEW CLASSES OF SIMULTANEOUS COSPECTRAL GRAPHS FOR ADJACENCY, LAPLACIAN AND NORMALIZED LAPLACIAN MATRICES

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Abstract. Butler [2] constructed simultaneous cospectral graphs for the adjacency and normalized Laplacian matrices, and asked the same for all three matrices, namely, adjacency, Laplacian and normalized Laplacian. In this paper, we determine the full adjacency, Laplacian and normalized Laplacian spectrum of the Q-vertex join and Q-edge join of a connected regular graph with an arbitrary regular graph in terms of their respective eigenvalues. Applying these results we construct some non-regular A-cospectral, L-cospectral and $\mathcal{L}$-cospectral graphs which gives a partial answer of the question asked by Butler [2]. Moreover, we determine the number of spanning trees and the Kirchhoff index of the newly constructed graphs.

1. Introduction

All graphs considered in this paper are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *adjacency matrix* of G, denoted by $A(G)$, is an $n \times n$ symmetric matrix such that $A(u, v) = 1$ if and only if vertex u is adjacent to vertex v and 0 otherwise. If $D(G)$ is the diagonal matrix of vertex degrees of G, then the *Laplacian matrix* $L(G)$ and *normalized Laplacian matrix* $\mathcal{L}(G)$ are defined as $L(G) = D(G) - A(G)$ and $\mathcal{L}(G) = I - D(G)^{-1/2}A(G)D(G)^{-1/2}$, respectively. For a given matrix M of size n, we denote the characteristic polynomial $\det(xI_n - M)$ of M by $f_M(x)$. The eigenvalues of $A(G)$ (respectively $L(G)$ and $\mathcal{L}(G)$) are denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ (respectively $0 = \mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$ and $0 = \delta_1(G) \leq \delta_2(G) \leq \cdots \leq \delta_n(G)$) and the multiset of these eigenvalues is called as the adjacency spectrum (respectively Laplacian spectrum and

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normalized Laplacian spectrum). Two graphs are said to be A-cospectral, L-cospectral and $\mathcal{L}$-cospectral if they have the same A-spectrum, L-spectrum and $\mathcal{L}$-spectrum respectively.

There are several kinds of graph operations in the literature. One of these is join of two graphs. The *join* [8] of two graphs is their disjoint union together with all the edges that connect all the vertices of the first graph with all the vertices of the second graph. The *Q-graph* $Q(G)$ [5] of a graph G is the graph obtained from G by inserting a new vertex into every edge of G and then joining by edges those pair of new vertices which lie on adjacent edges of G. The set of such new vertices is denoted by $I(G)$, i.e., $I(G) = V(Q(G))\backslash V(G)$. We define Q-vertex join and Q-edge join of graphs which are given below.

Definition 1.1. Let G_1 and G_2 be two vertex-disjoint graphs with number of vertices n_1 and n_2, and edges m_1 and m_2, respectively. Then

(i) the *Q-vertex join* of G_1 and G_2, denoted by $G_1\dot{\vee}_Q G_2$, is the graph obtained from $Q(G_1)$ and G_2 by joining each vertex of $V(G_1)$ with every vertex of $V(G_2)$. The graph $G_1\dot{\vee}_Q G_2$ has $n_1+n_2+m_1$ vertices.

(ii) the *Q-edge join* of G_1 and G_2, denoted by $G_1 \veebar_Q G_2$, is the graph obtained from $Q(G_1)$ and G_2 by joining each vertex of $I(G_1)$ with every vertex of $V(G_2)$. The graph $G_1 \veebar_Q G_2$ has $n_1+n_2+m_1$ vertices.

Example 1.1. Let us consider two graphs $G_1 = C_4$ and $G_2 = C_3$. The Q-vertex join and Q-edge join of G_1 and G_2 are given in Figure 1, respectively.

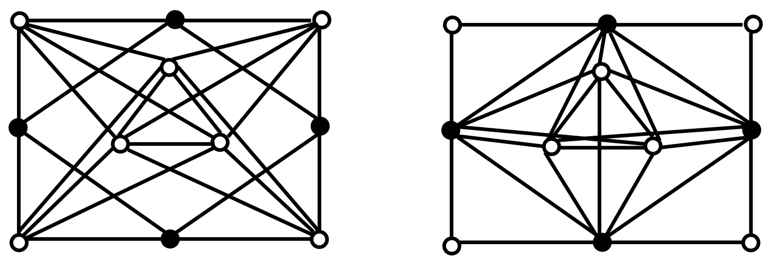

FIGURE 1. Q-vertex join and Q-edge join of C_4 and C_3

In [10], Indulal computed adjacency spectra of subdivision-vertex join and subdivision-edge join for two regular graphs in terms of their spectra. In [12], Liu et al. formulated the resistance distances and Kirchhoff index of R-vertex join and R-edge join respectively. Huang and Li [9] formulated the normalized laplacian characteristic polynomial of $Q(G)$ in terms of the normalized laplacian characteristic polynomial of G. Motivated by these works, here we determine the adjacency, Laplacian and normalized Laplacian spectrum of Q-vertex join and Q-edge join for a connected regular graph G_1

and an arbitrary regular graph G_2 in terms of the corresponding eigenvalues of G_1 and G_2. Using these spectra we construct some non-regular A-cospectral, L-cospectral and $\mathcal{L}$-cospectral graphs.

For two matrices A and B, of same size $m \times n$, the *Hadamard product* $A \bullet B$ of A and B is a matrix of the same size $m \times n$ with entries given by $(A \bullet B)_{ij} = (A)_{ij} \cdot (B)_{ij}$ (entrywise multiplication). Hadamard product is commutative, that is $A \bullet B = B \bullet A$.

To prove our results we need Lemma 1.1 and 1.2 below.

Lemma 1.1 (Schur Complement [5]). *Suppose that the order of all four matrices M, N, P and Q satisfy the rules of operations on matrices. Then we have,*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = \begin{cases} |Q||M - NQ^{-1}P|, & \text{if } Q \text{ is a non-singular square matrix,} \\ |M||Q - NM^{-1}P|, & \text{if } M \text{ is a non-singular square matrix.} \end{cases}$$

Lemma 1.2. [5] *Let A be an $n \times n$ real matrix, and $J_{s\times t}$ denote the $s \times t$ matrix with all entries equal to one. Then*

$$\det(A + \alpha J_{n\times n}) = \det(A) + \alpha \mathbf{1}_n^T \mathrm{adj}(A) \mathbf{1}_n,$$

where α is an real number and adj(A) is the adjugate matrix of A.

The following results are also useful in the sequel.

Lemma 1.3. *For any real numbers $c, d > 0$, we have*

$$(cI_n - dJ_{n\times n})^{-1} = \frac{1}{c} I_n + \frac{d}{c(c - nd)} J_{n\times n}.$$

Proof.

$$\begin{aligned} (cI_n - dJ_{n\times n})^{-1} &= \frac{\mathrm{adj}(cI_n - dJ_{n\times n})}{\det(cI_n - dJ_{n\times n})} = \frac{c^{n-2}(c - nd)I_n + c^{n-2} d J_{n\times n}}{c^{n-1}(c - nd)} \\ &= \frac{1}{c} I_n + \frac{d}{c(c - nd)} J_{n\times n}. \end{aligned}$$

□

For a graph G with n vertices and m edges, the *vertex-edge incidence matrix* $R(G)$ [6] is a matrix of order $n \times m$, with entry $r_{ij} = 1$ if the i^{th} vertex is incident to the j^{th} edge, and 0 otherwise. The *line graph* [6] of a graph G is the graph $l(G)$, whose vertices are the edges of G and two of these are adjacent in $l(G)$ if and only if they are incident on a common end vertex in G. It is well known [5] that $R(G)^T R(G) = A(l(G)) + 2I_m$. In particular, if G is an r-regular graph then $R(G)R(G)^T = A(G) + rI_n = 2rI_n - L(G) = r(2I_n - \mathcal{L}(G))$.

Lemma 1.4. [5] *Let G be an r-regular graph. Then the eigenvalues of $A(l(G))$ are the eigenvalues of $A(G) + (r - 2)I_n$ and -2 repeated $m - n$ times.*

If G is an r-regular graph, then $L(G) = rI_n - A(G)$ and $\mathcal{L}(G) = I_n - \frac{1}{r}A(G)$. Therefore, by Lemma 1.4, we get representations of $A(l(G))$ in terms of $L(G)$ and $\mathcal{L}(G)$ as given below.

Lemma 1.5. *For an r-regular graph G,*

(a) *the eigenvalues of $A(l(G))$ are the eigenvalues of $2(r-1)I_n - L(G)$ and -2 repeated $m-n$ times;*

(b) *the eigenvalues of $A(l(G))$ are the eigenvalues of $2(r-1)I_n - r\mathcal{L}(G)$ and -2 repeated $m-n$ times.*

Let $t(G)$ denote the number of spanning trees of G. It is well known [4] that if G is a connected graph on n vertices with Laplacian spectrum $0 = \mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$, then $t(G) = \frac{\mu_2(G)\cdots\mu_n(G)}{n}$.

The *Kirchhoff index* of a graph G, denoted by $Kf(G)$, is defined as the sum of resistances between all pairs of vertices [1,11] in G. For a connected graph G on n vertices, the Kirchhoff index [7] can be expressed as $Kf(G) = n\sum_{i=2}^{n}\frac{1}{\mu_i(G)}$.

In this paper we compute full adjacency, Laplacian and normalized Laplacian spectra of Q-vertex join and Q-edge join of a connected regular graph with an arbitrary regular graph. Applying these results we answer partially a question "Is there an example of two non-regular graphs which are cospectral with respect to the adjacency, combinatorial Laplacian and normalized Laplacian at the same time?" asked by Butler [2]. We also find Kirchhoff index and Spanning tree of the newly constructed graphs.

2. Our Results

Throughout the paper for any integer k, I_k denotes the identity matrix of size k, $\mathbf{1}_k$ denotes the column vector of size k whose all entries are 1 and $J_{n_1\times n_2}$ denotes $n_1 \times n_2$ matrix whose all entries are 1. The M-coronal $\Gamma_M(x)$ of an $n\times n$ matrix M is defined [3,13] to be the sum of the entries of the matrix $(xI_n - M)^{-1}$, that is, $\Gamma_M(x) = \mathbf{1}_n^T(xI_n - M)^{-1}\mathbf{1}_n$.

Lemma 2.1. [3] *If M is an $n\times n$ matrix with each row sum equal to a constant t, then $\Gamma_M(x) = \frac{n}{x-t}$.*

For an n vertex graph G, matrices B and C of sizes $n\times n$ and $n\times 1$ respectively, and a parameter x, we have the notation: $\chi_G(B,C,x) = C^T(xI_n - (\mathcal{L}(G)\bullet B))^{-1}C$. We note that the notation is similar to the notion 'coronal'. Let G_i be a graph with n_i vertices and m_i edges. Let $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$, $I(G_1) = \{e_1, e_2, \ldots, e_{m_1}\}$, $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$. Then $V(G_1)\cup I(G_1)\cup V(G_2)$ is a partition of $V(G_1\dot{\vee}_Q G_2)$ and $V(G_1 \veebar_Q G_2)$.

2.1. Spectra of Q-vertex join.

Let G_i be an r_i-regular graph on n_i vertices and m_i edges for $i = 1,2$. Then the degrees of the vertices of $G_1\dot{\vee}_Q G_2$ are $d_{G_1\dot{\vee}_Q G_2}(v_i) = r_1 + n_2$, $d_{G_1\dot{\vee}_Q G_2}(e_i) = 2r_1$ and $d_{G_1\dot{\vee}_Q G_2}(u_i) = r_2 + n_1$.

2.1.1. *A-spectra of Q-vertex join.* The adjacency matrix of $G_1\dot{\vee}_Q G_2$ can be expressed as:

$$A(G_1\dot{\vee}_Q G_2) = \begin{pmatrix} O_{n_1} & R(G_1) & J_{n_1\times n_2} \\ R(G_1)^T & A(l(G_1)) & O_{m_1\times n_2} \\ J_{n_2\times n_1} & O_{n_2\times m_1} & A(G_2) \end{pmatrix}.$$

Theorem 2.1. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then the adjacency spectrum of $G_1\dot{\vee}_Q G_2$ consists of:*

(i) *the eigenvalue $\lambda_j(G_2)$ for every eigenvalue λ_j, $j = 2, 3, \ldots, n_2$, of $A(G_2)$;*
(ii) *the eigenvalue -2 with multiplicity $m_1 - n_1$;*
(iii) *two roots of the equation $x^2 - (r_1 - 2 + \lambda_i(G_1))x - r_1 - \lambda_i(G_1) = 0$ for each eigenvalue λ_i, $i = 2, 3, \ldots, n_1$, of $A(G_1)$;*
(iv) *three roots of the equation*
$x^3 - (2r_1 + r_2 - 2)x^2 - (2r_1 + 2r_2 + n_1n_2 - 2r_1r_2)x + 2r_1r_2 + 2r_1n_1n_2 - 2n_1n_2 = 0.$

Proof. The adjacency characteristic polynomial of $G_1\dot{\vee}_Q G_2$ is

$$\begin{aligned}
f_{A(G_1\dot{\vee}_Q G_2)}(x) &= \det(xI_{n_1+n_2+m_1} - A(G_1\dot{\vee}_Q G_2)) \\
&= \det\begin{pmatrix} xI_{n_1} & -R(G_1) & -J_{n_1\times n_2} \\ -R(G_1)^T & xI_{m_1} - A(l(G_1)) & O_{m_1\times n_2} \\ -J_{n_2\times n_1} & O_{n_2\times m_1} & xI_{n_2} - A(G_2) \end{pmatrix} \\
&= \det(xI_{n_2} - A(G_2))\det(S) = \prod_{i=1}^{n_2}\{x - \lambda_j(G_2)\}\det(S),
\end{aligned}$$

where

$$\begin{aligned}
S &= \begin{pmatrix} xI_{n_1} & -R(G_1) \\ -R(G_1)^T & xI_{m_1} - A(l(G_1)) \end{pmatrix} \\
&\quad - \begin{pmatrix} -J_{n_1\times n_2} \\ O_{m_1\times n_2} \end{pmatrix}(xI_{n_2} - A(G_2))^{-1}\begin{pmatrix} -J_{n_2\times n_1} & O_{n_2\times m_1} \end{pmatrix} \\
&= \begin{pmatrix} xI_{n_1} - \Gamma_{A(G_2)}(x)J_{n_1\times n_1} & -R(G_1) \\ -R(G_1)^T & xI_{m_1} - A(l(G_1)) \end{pmatrix}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\det(S) =& \det(xI_{n_1} - \Gamma_{A(G_2)}(x)J_{n_1\times n_1}) \\
&\times \det\left(xI_{m_1} - A(l(G_1)) - R(G_1)^T(xI_{n_1} - \Gamma_{A(G_2)}(x)J_{n_1\times n_1})^{-1}R(G_1)\right) \\
=& x^{n_1}\left(1 - \Gamma_{A(G_2)}(x)\frac{n_1}{x}\right)\det\Bigg[xI_{m_1} - A(l(G_1)) \\
&- R(G_1)^T\left\{\frac{1}{x}I_{n_1} + \frac{\Gamma_{A(G_2)}(x)}{x(x - n_1\Gamma_{A(G_2)}(x))}J_{n_1\times n_1}\right\}R(G_1)\Bigg] \\
=& x^{n_1}\left(1 - \Gamma_{A(G_2)}(x)\frac{n_1}{x}\right)\det\Big[xI_{m_1} - A(l(G_1)) \\
&- \frac{1}{x}R(G_1)^TR(G_1) - \frac{\Gamma_{A(G_2)}(x)}{x(x - n_1\Gamma_{A(G_2)}(x))}R(G_1)^TJ_{n_1\times n_1}R(G_1)\Big] \\
=& x^{n_1}\left(1 - \Gamma_{A(G_2)}(x)\frac{n_1}{x}\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \det\left[xI_{m_1} - A(l(G_1)) - \frac{1}{x}R(G_1)^T R(G_1) - \frac{4\Gamma_{A(G_2)}(x)}{x(x-n_1\Gamma_{A(G_2)}(x))}J_{m_1\times m_1}\right]\\
=&x^{n_1}\left(1-\Gamma_{A(G_2)}(x)\frac{n_1}{x}\right)\left[\det\left(xI_{m_1} - A(l(G_1)) - \frac{1}{x}R(G_1)^T R(G_1)\right)\right.\\
&\left. - \frac{4\Gamma_{A(G_2)}(x)}{x(x-n_1\Gamma_{A(G_2)}(x))}\mathbf{1}_{m_1}^T adj\left(xI_{m_1} - A(l(G_1)) - \frac{1}{x}R(G_1)^T R(G_1)\right)\mathbf{1}_{m_1}\right]\\
=&x^{n_1}\left(1-\Gamma_{A(G_2)}(x)\frac{n_1}{x}\right)\det\left(xI_{m_1} - A(l(G_1)) - \frac{1}{x}R(G_1)^T R(G_1)\right)\\
&\times\left[1 - \frac{4\Gamma_{A(G_2)}(x)}{x(x-n_1\Gamma_{A(G_2)}(x))}\mathbf{1}_{m_1}^T\right.\\
&\left.\times\left(xI_{m_1} - A(l(G_1)) - \frac{1}{x}R(G_1)^T R(G_1)\right)^{-1}\mathbf{1}_{m_1}\right]\\
=&x^{n_1}\left(1-\Gamma_{A(G_2)}(x)\frac{n_1}{x}\right)\det\left(\left(x-\frac{2}{x}\right)I_{m_1} - \left(1+\frac{1}{x}\right)A(l(G_1))\right)\\
&\times\left[1 - \frac{4\Gamma_{A(G_2)}(x)\Gamma_{A(l(G_1))+\frac{1}{x}R(G_1)^T R(G_1)}(x)}{x(x-n_1\Gamma_{A(G_2)}(x))}\right]\\
=&x^{n_1}(x+2)^{m_1-n_1}\left(1-\frac{n_1n_2}{x(x-r_2)}\right)\\
&\times\det\left(\left(x-\frac{2}{x}\right)I_{m_1} - \left(1+\frac{1}{x}\right)(r_1-2+A(G_1))\right)\\
&\times\left[1-\frac{4m_1n_2}{x(x-r_2)(x-\frac{n_1n_2}{x-r_2})(x+2-2r_1-\frac{2r_1}{x})}\right]\\
=&x^{n_1}(x+2)^{m_1-n_1}\left(\frac{x^2-r_2x-n_1n_2}{x(x-r_2)}\right)\\
&\times\prod_{i=1}^{n_1}\left\{x-\frac{2}{x}-\left(1+\frac{1}{x}\right)(r_1-2+\lambda_i(G_1))\right\}\\
&\times\left[1-\frac{4m_1n_2}{(x^2-r_2x-n_1n_2)(x^2+2x-2r_1x-2r_1)}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
f_{A(G_1\dot{\vee}_Q G_2)}(x) =&x^{n_1}(x+2)^{m_1-n_1}\left(\frac{x^2-r_2x-n_1n_2}{x(x-r_2)}\right)\prod_{j=1}^{n_2}\{x-\lambda_j(G_2)\}\\
&\times\prod_{i=1}^{n_1}\left\{x-\frac{2}{x}-\left(1+\frac{1}{x}\right)(r_1-2+\lambda_i(G_1))\right\}\\
&\times\left[1-\frac{4m_1n_2}{(x^2-r_2x-n_1n_2)(x^2+2x-2r_1x-2r_1)}\right]
\end{aligned}
$$

$$
\begin{aligned}
=&(x+2)^{m_1-n_1}\prod_{j=2}^{n_2}\{x-\lambda_j(G_2)\}\\
&\times\prod_{i=2}^{n_1}\{x^2-(r_1-2+\lambda_i(G_1))x-r_1-\lambda_i(G_1)\}\{x^3-(2r_1+r_2-2)x^2\\
&-(2r_1+2r_2+n_1n_2-2r_1r_2)x+2r_1r_2+2r_1n_1n_2-2n_1n_2\}. \qquad \square
\end{aligned}
$$

2.1.2. *L-spectra of Q-vertex join.* The Laplacian matrix of $G_1\dot{\vee}_Q G_2$ can be written as:

$$
L(G_1\dot{\vee}_Q G_2)=\begin{pmatrix}(r_1+n_2)I_{n_1} & -R(G_1) & -J_{n_1\times n_2}\\ -R(G_1)^T & 2r_1I_{m_1}-A(l(G_1)) & O_{m_1\times n_2}\\ -J_{n_2\times n_1} & O_{n_2\times m_1} & n_1I_{n_2}+L(G_2)\end{pmatrix}.
$$

Theorem 2.2. *For $i=1,2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then the Laplacian spectrum of $G_1\dot{\vee}_Q G_2$ consists of:*

(i) *the eigenvalue $n_1+\mu_j(G_2)$ for every eigenvalue μ_j, $j=2,3,\dots,n_2$, of $L(G_2)$;*
(ii) *the eigenvalue $2+2r_1$ with multiplicity m_1-n_1;*
(iii) *two roots of the equation $x^2-(2+r_1+n_2+\mu_i(G_1))x+2n_2+\mu_i(G_1)+r_1\mu_i(G_1)+n_2\mu_i(G_1)=0$ for each eigenvalue μ_i, $i=2,3,\dots,n_1$, of $L(G_1)$;*
(iv) *three roots of the equation $x^3-(2+r_1+n_1+n_2)x^2-(2n_1+2n_2+r_1n_1)x=0$.*

Proof. The Laplacian characteristic polynomial of $G_1\dot{\vee}_Q G_2$ is

$$
\begin{aligned}
f_{L(G_1\dot{\vee}_Q G_2)}(x)&=\det(xI_{n_1+n_2+m_1}-L(G_1\dot{\vee}_Q G_2))\\
&=\det\begin{pmatrix}(x-r_1-n_2)I_{n_1} & R(G_1) & J_{n_1\times n_2}\\ R(G_1)^T & (x-2r_1)I_{m_1}+A(l(G_1)) & O_{m_1\times n_2}\\ J_{n_2\times n_1} & O_{n_2\times m_1} & (x-n_1)I_{n_2}-L(G_2)\end{pmatrix}\\
&=\det((x-n_1)I_{n_2}-L(G_2))\det(S)=\prod_{j=1}^{n_2}\{x-n_1-\mu_j(G_2)\}\det(S),
\end{aligned}
$$

where

$$
\begin{aligned}
S=&\begin{pmatrix}(x-r_1-n_2)I_{n_1} & R(G_1)\\ R(G_1)^T & (x-2r_1)I_{m_1}+A(l(G_1))\end{pmatrix}\\
&-\begin{pmatrix}J_{n_1\times n_2}\\ O_{m_1\times n_2}\end{pmatrix}((x-n_1)I_{n_2}-L(G_2))^{-1}\begin{pmatrix}J_{n_2\times n_1} & O_{n_2\times m_1}\end{pmatrix}\\
=&\begin{pmatrix}(x-r_1-n_2)I_{n_1}-\Gamma_{L(G_2)}(x-n_1)J_{n_1\times n_1} & R(G_1)\\ R(G_1)^T & (x-2r_1)I_{m_1}+A(l(G_1))\end{pmatrix}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\det(S)=&\det((x-r_1-n_2)I_{n_1}-\Gamma_{L(G_2)}(x-n_1)J_{n_1\times n_1})\det\Big((x-2r_1)I_{m_1}+A(l(G_1))\\
&-R(G_1)^T((x-r_1-n_2)I_{n_1}-\Gamma_{L(G_2)}(x-n_1)J_{n_1\times n_1})^{-1}R(G_1)\Big)
\end{aligned}
$$

$$
\begin{aligned}
=&(x-r_1-n_2)^{n_1}\left\{1-\Gamma_{L(G_2)}(x-n_1)\frac{n_1}{x-r_1-n_2}\right\}\\
&\times\det\Bigg[(x-2r_1)I_{m_1}+A(l(G_1))-R(G_1)^T\Bigg\{\frac{1}{x-r_1-n_2}I_{n_1}\\
&+\frac{\Gamma_{L(G_2)}(x-n_1)}{(x-r_1-n_2)(x-r_1-n_2-n_1\Gamma_{L(G_2)}(x-n_1))}J_{n_1\times n_1}\Bigg\}R(G_1)\Bigg]\\
=&(x-r_1-n_2)^{n_1}\left\{1-\Gamma_{L(G_2)}(x-n_1)\frac{n_1}{x-r_1-n_2}\right\}\\
&\times\det\Bigg[(x-2r_1)I_{m_1}+A(l(G_1))-\frac{1}{x-r_1-n_2}R(G_1)^TR(G_1)\\
&-\frac{\Gamma_{L(G_2)}(x-n_1)}{(x-r_1-n_2)(x-r_1-n_2-n_1\Gamma_{L(G_2)}(x-n_1))}R(G_1)^TJ_{n_1\times n_1}R(G_1)\Bigg]\\
=&(x-r_1-n_2)^{n_1}\left\{1-\Gamma_{L(G_2)}(x-n_1)\frac{n_1}{x-r_1-n_2}\right\}\\
&\times\det\Bigg[(x-2r_1)I_{m_1}+A(l(G_1))-\frac{1}{x-r_1-n_2}R(G_1)^TR(G_1)\\
&-\frac{4\Gamma_{L(G_2)}(x-n_1)}{(x-r_1-n_2)(x-r_1-n_2-n_1\Gamma_{L(G_2)}(x-n_1))}J_{m_1\times m_1}\Bigg]\\
=&(x-r_1-n_2)^{n_1}\left\{1-\Gamma_{L(G_2)}(x-n_1)\frac{n_1}{x-r_1-n_2}\right\}\\
&\times\Bigg[\det((x-2r_1)I_{m_1}+A(l(G_1))-\frac{1}{x-r_1-n_2}R(G_1)^TR(G_1))\\
&-\frac{4\Gamma_{L(G_2)}(x-n_1)}{(x-r_1-n_2)(x-r_1-n_2-n_1\Gamma_{L(G_2)}(x-n_1))}\\
&\times\mathbf{1}_{m_1}^T\operatorname{adj}\left((x-2r_1)I_{m_1}+A(l(G_1))-\frac{1}{x-r_1-n_2}R(G_1)^TR(G_1)\right)\mathbf{1}_{m_1}\Bigg]\\
=&(x-r_1-n_2)^{n_1}\left\{1-\Gamma_{L(G_2)}(x-n_1)\frac{n_1}{x-r_1-n_2}\right\}\\
&\times\det\left((x-2r_1)I_{m_1}+A(l(G_1))-\frac{1}{x-r_1-n_2}R(G_1)^TR(G_1)\right)\\
&\times\Bigg[1-\frac{4\Gamma_{L(G_2)}(x-n_1)}{(x-r_1-n_2)(x-r_1-n_2-n_1\Gamma_{L(G_2)}(x-n_1))}\\
&\times\mathbf{1}_{m_1}^T\left((x-2r_1)I_{m_1}+A(l(G_1))-\frac{1}{x-r_1-n_2}R(G_1)^TR(G_1)\right)^{-1}\mathbf{1}_{m_1}\Bigg]\\
=&(x-r_1-n_2)^{n_1}\left\{1-\Gamma_{L(G_2)}(x-n_1)\frac{n_1}{x-r_1-n_2}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\times \det\left(\left(x-2r_1-\frac{2}{x-r_1-n_2}\right)I_{m_1}+\left(1-\frac{1}{x-r_1-n_2}\right)A(l(G_1))\right)\\
&\times\left[1-\frac{4\Gamma_{L(G_2)}(x-n_1)\Gamma_{\frac{1}{x-r_1-n_2}R(G_1)^TR(G_1)-A(l(G_1))}(x-2r_1)}{(x-r_1-n_2)(x-r_1-n_2-n_1\Gamma_{L(G_2)}(x-n_1))}\right]\\
=&(x-r_1-n_2)^{n_1}(x-2r_1-2)^{m_1-n_1}\left\{1-\frac{n_1n_2}{(x-n_1)(x-r_1-n_2)}\right\}\\
&\times \det\left(\left(x-2r_1-\frac{2}{x-r_1-n_2}\right)I_{m_1}\right.\\
&\left.+\left(1-\frac{1}{x-r_1-n_2}\right)(2(r_1-1)I_{n_1}-L(G_1))\right)\\
&\times\left[1-\frac{4m_1n_2}{(x-n_1)(x-r_1-n_2)(x-r_1-n_2-\frac{n_1n_2}{x-n_1})(x-2-\frac{2r_1}{x-r_1-n_2})}\right]\\
=&(x-r_1-n_2)^{n_1}(x-2r_1-2)^{m_1-n_1}\left\{\frac{x^2-(r_1+n_1+n_2)x+r_1n_1}{(x-n_1)(x-r_1-n_2)}\right\}\\
&\times\prod_{i=1}^{n_1}\left\{x-2r_1-\frac{2}{x-r_1-n_2}+\left(1-\frac{1}{x-r_1-n_2}\right)(2r_1-2-\mu_i(G_1))\right\}\\
&\times\left[1-\frac{4m_1n_2}{\{x^2-(r_1+n_1+n_2)x+r_1n_1\}\{x^2-(2+r_1+n_2)x+2n_2\}}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
f_{L(G_1\dot{\vee}_Q G_2)}(x)=&(x-r_1-n_2)^{n_1}(x-2r_1-2)^{m_1-n_1}\\
&\times\left\{\frac{x^2-(r_1+n_1+n_2)x+r_1n_1}{(x-n_1)(x-r_1-n_2)}\right\}\prod_{j=1}^{n_2}\{x-n_1-\mu_j(G_2)\}\\
&\times\prod_{i=1}^{n_1}\left\{x-2r_1-\frac{2}{x-r_1-n_2}+\left(1-\frac{1}{x-r_1-n_2}\right)\right.\\
&\left.\times(2r_1-2-\mu_i(G_1))\right\}\\
&\times\left[1-\frac{4m_1n_2}{\{x^2-(r_1+n_1+n_2)x+r_1n_1\}\{x^2-(2+r_1+n_2)x+2n_2\}}\right]\\
=&(x-2r_1-2)^{m_1-n_1}\prod_{j=2}^{n_2}\{x-n_1-\mu_j(G_2)\}\\
&\times\prod_{i=2}^{n_1}\{x^2-(2+r_1+n_2+\mu_i(G_1))x+2n_2+\mu_i(G_1)+r_1\mu_i(G_1)\\
&+n_2\mu_i(G_1)\}\{x^3-(2+r_1+n_1+n_2)x^2-(2n_1+2n_2+r_1n_1)x\}. \qquad \square
\end{aligned}
$$

Corollary 2.1. *For $i=1,2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then*

$$Kf(G_1\dot{\vee}_Q G_2)=(n_1+n_2+m_1)\times\Bigg(\frac{m_1-n_1}{2+2r_1}+\frac{2+r_1+n_1+n_2}{2n_1+2n_2+r_1n_1}+\sum_{i=2}^{n_1}\frac{2+r_1+n_2+\mu_i(G_1)}{2n_2+\mu_i(G_1)+r_1\mu_i(G_1)+n_2\mu_i(G_1)}+\sum_{j=2}^{n_2}\frac{1}{n_1+\mu_j(G_2)}\Bigg).$$

Corollary 2.2. *For $i=1,2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then*

$$t(G_1\dot{\vee}_Q G_2)=\frac{(2+2r_1)^{m_1-n_1}\cdot(2n_1+2n_2+r_1n_1)\cdot\prod_{i=2}^{n_1}(2n_2+\mu_i(G_1)+r_1\mu_i(G_1)+n_2\mu_i(G_1))\cdot\prod_{j=2}^{n_2}(n_1+\mu_j(G_2))}{n_1+n_2+m_1}.$$

2.1.3. *$\mathcal{L}$-spectra of Q-vertex join.* The normalized Laplacian matrix of $G_1\dot{\vee}_Q G_2$ can be obtained as:

$$\mathcal{L}(G_1\dot{\vee}_Q G_2)=\begin{pmatrix} I_{n_1} & -cR(G_1) & -K_{n_1\times n_2}\\ -cR(G_1)^T & I_{m_1}-\frac{1}{2r_1}A(l(G_1)) & O_{m_1\times n_2}\\ -K_{n_2\times n_1} & O_{n_2\times m_1} & \mathcal{L}(G_2)\bullet B(G_2)\end{pmatrix},$$

where $K_{n_1\times n_2}$ is the matrix of size $n_1\times n_2$ with all entries equal to $\frac{1}{\sqrt{(r_1+n_2)(r_2+n_1)}}$, $B(G_2)$ is the $n_2\times n_2$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_2}{r_2+n_1}$, c is the constant whose value is $\frac{1}{\sqrt{2r_1(r_1+n_2)}}$.

Theorem 2.3. *The normalized Laplacian spectrum of $G_1\dot{\vee}_Q G_2$ consists of:*

(i) *the eigenvalue $\frac{n_1+r_2\delta_j}{r_2+n_1}$, for every eigenvalue δ_j, $j=2,3,\ldots,n_2$, of $\mathcal{L}(G_2)$;*
(ii) *the eigenvalue $1+\frac{1}{r_1}$ with multiplicity m_1-n_1;*
(iii) *two roots of the equation $2r_1(r_1+n_2)x^2-(2r_1^2+2r_1+2n_2+2r_1n_2+r_1^2\delta_i(G_1)+r_1n_2\delta_i(G_1))x+2n_2+r_1\delta_i(G_1)+r_1^2\delta_i(G_1)+r_1n_2\delta_i(G_1)=0$, for each eigenvalue δ_i, $i=2,3,\ldots,n_1$ of $\mathcal{L}(G_1)$;*
(iv) *three roots of the equation $(r_1^2r_2+r_1^2n_1+r_1r_2n_2+r_1n_1n_2)x^3-(r_1^2r_2+2r_1^2n_1+r_1r_2+r_2n_2+r_1n_1+n_1n_2+r_1r_2n_2+2r_1n_1n_2)x^2+(r_1^2n_1+r_1n_1+r_2n_2+2n_1n_2)x=0$.*

Proof. The normalized Laplacian characteristic polynomial of $G_1\dot{\vee}_Q G_2$ is

$$\begin{aligned} f_{\mathcal{L}(G_1\dot{\vee}_Q G_2)}(x)&=\det(xI_{n_1+n_2+m_1}-\mathcal{L}(G_1\dot{\vee}_Q G_2))\\ &=\det\begin{pmatrix}(x-1)I_{n_1} & cR(G_1) & K_{n_1\times n_2}\\ cR(G_1)^T & (x-1)I_{m_1}+\frac{1}{2r_1}A(l(G_1)) & O_{m_1\times n_2}\\ K_{n_2\times n_1} & O_{n_2\times m_1} & xI_{n_2}-(\mathcal{L}(G_2)\bullet B(G_2))\end{pmatrix}\\ &=\det(xI_{n_2}-(\mathcal{L}(G_2)\bullet B(G_2)))\det(S),\end{aligned}$$

where

$$S=\begin{pmatrix}(x-1)I_{n_1} & cR(G_1)\\ cR(G_1)^T & (x-1)I_{m_1}+\frac{1}{2r_1}A(l(G_1))\end{pmatrix}$$

$$
\begin{aligned}
&- \begin{pmatrix} K_{n_1\times n_2} \\ O_{m_1\times n_2} \end{pmatrix} (xI_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2)))^{-1} \begin{pmatrix} K_{n_2\times n_1} & O_{n_2\times m_1} \end{pmatrix} \\
=& \begin{pmatrix} (x-1)I_{n_1} - \chi_{G_2}(B(G_2), C_{n_2}, x)J_{n_1\times n_1} & cR(G_1) \\ cR(G_1)^T & (x-1)I_{m_1} + \frac{1}{2r_1}A(l(G_1)) \end{pmatrix}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\det(S) =& \det((x-1)I_{n_1} - \chi_{G_2}(B(G_2), C_{n_2}, x)J_{n_1\times n_1}) \det\Big[(x-1)I_{m_1} + \frac{1}{2r_1}A(l(G_1)) \\
&- c^2R(G_1)^T((x-1)I_{n_1} - \chi_{G_2}(B(G_2), C_{n_2}, x)J_{n_1\times n_1})^{-1}R(G_1)\Big] \\
=&(x-1)^{n_1}\left\{1 - \chi_{G_2}(B(G_2), C_{n_2}, x)\frac{n_1}{x-1}\right\} \\
&\times \det\Big[(x-1)I_{m_1} + \frac{1}{2r_1}A(l(G_1)) - c^2R(G_1)^T\Big\{\frac{1}{x-1}I_{n_1} \\
&+ \frac{\chi_{G_2}(B(G_2), C_{n_2}, x)}{(x-1)(x-1-n_1\chi_{G_2}(B(G_2), C_{n_2}, x))}J_{n_1\times n_1}\Big\}R(G_1)\Big] \\
=&(x-1)^{n_1}\left\{1 - \chi_{G_2}(B(G_2), C_{n_2}, x)\frac{n_1}{x-1}\right\} \\
&\times \det\Big[(x-1)I_{m_1} + \frac{1}{2r_1}A(l(G_1)) - \frac{c^2}{x-1}R(G_1)^TR(G_1) \\
&- c^2\frac{\chi_{G_2}(B(G_2), C_{n_2}, x)}{(x-1)(x-1-n_1\chi_{G_2}(B(G_2), C_{n_2}, x))}R(G_1)^TJ_{n_1\times n_1}R(G_1)\Big] \\
=&(x-1)^{n_1}\left\{1 - \chi_{G_2}(B(G_2), C_{n_2}, x)\frac{n_1}{x-1}\right\}\det\Big[(x-1)I_{m_1} + \frac{1}{2r_1}A(l(G_1)) \\
&- \frac{c^2}{x-1}R(G_1)^TR(G_1) - c^2\frac{4\chi_{G_2}(B(G_2), C_{n_2}, x)}{(x-1)(x-1-n_1\chi_{G_2}(B(G_2), C_{n_2}, x))}J_{m_1\times m_1}\Big] \\
=&(x-1)^{n_1}\left\{1 - \chi_{G_2}(B(G_2), C_{n_2}, x)\frac{n_1}{x-1}\right\}\Bigg[\det\Big((x-1)I_{m_1} + \frac{1}{2r_1}A(l(G_1)) \\
&- \frac{c^2}{x-1}R(G_1)^TR(G_1)\Big) - \frac{4c^2\chi_{G_2}(B(G_2), C_{n_2}, x)}{(x-1)(x-1-n_1\chi_{G_2}(B(G_2), C_{n_2}, x))} \\
&\times \mathbf{1}_{m_1}^T\mathrm{adj}\left\{(x-1)I_{m_1} + \frac{1}{2r_1}A(l(G_1)) - \frac{c^2}{x-1}R(G_1)^TR(G_1)\right\}\mathbf{1}_{m_1}\Bigg] \\
=&(x-1)^{n_1}\left\{1 - \chi_{G_2}(B(G_2), C_{n_2}, x)\frac{n_1}{x-1}\right\}\det\Big((x-1)I_{m_1} + \frac{1}{2r_1}A(l(G_1)) \\
&- \frac{c^2}{x-1}R(G_1)^TR(G_1)\Big)\Big[1 - \frac{4c^2\chi_{G_2}(B(G_2), C_{n_2}, x)}{(x-1)(x-1-n_1\chi_{G_2}(B(G_2), C_{n_2}, x))}
\end{aligned}
$$

$$\times \mathbf{1}_{m_1}^T \left\{(x-1)I_{m_1} + \frac{1}{2r_1}A(l(G_1)) - \frac{c^2}{x-1}R(G_1)^T R(G_1)\right\}^{-1} \mathbf{1}_{m_1}\Bigg]$$

$$= (x-1)^{n_1}\left\{1 - \chi_{G_2}(B(G_2), C_{n_2}, x)\frac{n_1}{x-1}\right\}$$

$$\times \det\left(\left(x - 1 - \frac{2c^2}{x-1}\right)I_{m_1} + \left(\frac{1}{2r_1} - \frac{c^2}{x-1}\right)A(l(G_1))\right)$$

$$\times \left[1 - \frac{4c^2\chi_{G_2}(B(G_2), C_{n_2}, x)\Gamma_{\frac{c^2}{x-1}R(G_1)^T R(G_1) - \frac{1}{2r_1}A(l(G_1))}(x-1)}{(x-1)(x-1-n_1\chi_{G_2}(B(G_2), C_{n_2}, x))}\right]$$

$$= (x-1)^{n_1}\left(x - 1 - \frac{1}{r_1}\right)^{m_1 - n_1}\left\{1 - \frac{n_1 n_2}{(x-1)(r_1+n_2)(r_2+n_1)(x - \frac{n_1}{r_2+n_1})}\right\}$$

$$\times \det\left(\left(x - 1 - \frac{2c^2}{x-1}\right)I_{n_1} + \left(\frac{1}{2r_1} - \frac{c^2}{x-1}\right)(2(r_1-1)I_{n_1} - r_1\mathcal{L}(G_1))\right)$$

$$\times \left[1 - \frac{4c^2\chi_{G_2}(B(G_2), C_{n_2}, x)\Gamma_{\frac{c^2}{x-1}R(G_1)^T R(G_1) - \frac{1}{2r_1}A(l(G_1))}(x-1)}{(x-1)(x-1-n_1\chi_{G_2}(B(G_2), C_{n_2}, x))}\right].$$

Since $\mathcal{L}(G_2) \bullet B(G_2) = I_{n_2} - \frac{1}{r_2+n_1}A(G_2)$, we get, $\mathcal{L}(G_2) \bullet B(G_2) = \frac{1}{r_2+n_1}(n_1 I_{n_2} + r_2\mathcal{L}(G_2))$. As G_2 is regular, the sum of all entries on every row of its normalized Laplacian matrix is zero. That means, $\mathcal{L}(G_2)C_{n_2} = (1 - \frac{r_2}{r_2})C_{n_2} = 0C_{n_2}$. Then $(\mathcal{L}(G_2) \bullet B(G_2))C_{n_2} = (1 - \frac{r_2}{r_2+n_1})C_{n_2} = \frac{n_1}{r_2+n_1}C_{n_2}$ and $(xI_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2)))C_{n_2} = (x - \frac{n_1}{r_2+n_1})C_{n_2}$. Also, $C_{n_2}^T C_{n_2} = \frac{n_2}{(r_1+n_2)(r_2+n_1)}$.

Now,

$$\chi_{G_2}(B(G_2), C_{n_2}, x) = \frac{n_2}{(r_1+n_2)(r_2+n_1)(x - \frac{n_1}{r_2+n_1})}$$

and

$$\Gamma_{\frac{c^2}{x-1}R(G_1)^T R(G_1) - \frac{1}{2r_1}A(l(G_1))}(x-1) = \frac{m_1}{x - \frac{1}{r_1} - \frac{2r_1}{(x-1)2r_1(r_1+n_2)}},$$

$$f_{\mathcal{L}(G_1 \dot{\vee}_Q G_2)}(x) = \left(x - 1 - \frac{1}{r_1}\right)^{m_1 - n_1}\prod_{j=2}^{n_2}\left(x - \frac{n_1 + r_2\delta_j(G_2)}{r_2 + n_1}\right)$$
$$\times \prod_{i=2}^{n_1}\{2r_1(r_1+n_2)x^2 - (2r_1^2 + 2r_1 + 2n_2 + 2r_1n_2 + r_1^2\delta_i(G_1)$$
$$+ r_1n_2\delta_i(G_1))x + 2n_2 + r_1\delta_i(G_1) + r_1^2\delta_i(G_1) + r_1n_2\delta_i(G_1)\}$$
$$\times [(r_1^2r_2 + r_1^2n_1 + r_1r_2n_2 + r_1n_1n_2)x^3 - (r_1^2r_2 + 2r_1^2n_1 + r_1r_2 + r_2n_2$$
$$+ r_1n_1 + n_1n_2 + r_1r_2n_2 + 2r_1n_1n_2)x^2$$
$$+ (r_1^2n_1 + r_1n_1 + r_2n_2 + 2n_1n_2)x].$$

□

2.2. **Spectra of Q-edge join.** Let G_i be an r_i-regular graph on n_i vertices and m_i edges for $i = 1, 2$. Then the degrees of the vertices of $G_1 \veebar_Q G_2$ are $d_{G_1 \veebar_Q G_2}(v_i) = r_1$, $d_{G_1 \veebar_Q G_2}(e_i) = 2r_1 + n_2$ and $d_{G_1 \veebar_Q G_2}(u_i) = r_2 + m_1$.

2.2.1. *A-spectra of Q-edge join.* The adjacency matrix of $G_1 \veebar_Q G_2$ can be expressed as:

$$A(G_1 \veebar_Q G_2) = \begin{pmatrix} O_{n_1} & R(G_1) & O_{m_1 \times n_2} \\ R(G_1)^T & A(l(G_1)) & J_{n_1 \times n_2} \\ O_{n_2 \times m_1} & J_{n_2 \times n_1} & A(G_2) \end{pmatrix}.$$

Theorem 2.4. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then the adjacency spectrum of $G_1 \veebar_Q G_2$ consists of:*

(i) *the eigenvalue $\lambda_j(G_2)$, for every eigenvalue λ_j, $j = 2, 3, \ldots, n_2$, of $A(G_2)$;*
(ii) *the eigenvalue -2 with multiplicity $m_1 - n_1$;*
(iii) *two roots of the equation $x^2 - (r_1 - 2 + \lambda_i(G_1))x - r_1 - \lambda_i(G_1) = 0$ for each eigenvalue λ_i, $i = 2, 3, \ldots, n_1$, of $A(G_1)$;*
(iv) *three roots of the equation $x^3 - (2r_1 + r_2 - 2)x^2 - (2r_1 + m_1 n_2 + 2r_2 - 2r_1 r_2)x + 2r_1 r_2 = 0$.*

Proof. The adjacency characteristic polynomial of $G_1 \veebar_Q G_2$ is

$$\begin{aligned} f_{A(G_1 \veebar_Q G_2)}(x) &= \det(xI_{n_1+n_2+m_1} - A(G_1 \veebar_Q G_2)) \\ &= \det \begin{pmatrix} xI_{n_1} & -R(G_1) & O_{n_1 \times n_2} \\ -R(G_1)^T & xI_{m_1} - A(l(G_1)) & -J_{m_1 \times n_2} \\ O_{n_2 \times n_1} & -J_{n_2 \times m_1} & xI_{n_2} - A(G_2) \end{pmatrix} \\ &= \det(xI_{n_2} - A(G_2)) \det(S) = \prod_{j=1}^{n_2} \{x - \lambda_j(G_2)\} \det(S), \end{aligned}$$

where

$$\begin{aligned} S = &\begin{pmatrix} xI_{n_1} & -R(G_1) \\ -R(G_1)^T & xI_{m_1} - A(l(G_1)) \end{pmatrix} \\ &- \begin{pmatrix} O_{n_1 \times n_2} \\ -J_{m_1 \times n_2} \end{pmatrix} (xI_{n_2} - A(G_2))^{-1} \begin{pmatrix} O_{n_2 \times n_1} & -J_{n_2 \times m_1} \end{pmatrix} \\ = &\begin{pmatrix} xI_{n_1} & -R(G_1) \\ -R(G_1)^T & xI_{m_1} - A(l(G_1)) - \Gamma_{A(G_2)}(x) J_{m_1 \times m_1} \end{pmatrix}. \end{aligned}$$

Then, we have

$$\begin{aligned} \det(S) =& x^{n_1} \det \left(xI_{m_1} - A(l(G_1)) - \Gamma_{A(G_2)}(x) J_{m_1 \times m_1} - \frac{1}{x} R(G_1)^T R(G_1) \right) \\ =& x^{n_1} \Bigg[\det(xI_{m_1} - A(l(G_1)) - \frac{1}{x} R(G_1)^T R(G_1)) \\ &- \Gamma_{A(G_2)}(x) \mathbf{1}_{m_1}^T adj \left\{ xI_{m_1} - A(l(G_1)) - \frac{1}{x} R(G_1)^T R(G_1) \right\} \mathbf{1}_{m_1} \Bigg] \end{aligned}$$

$$
\begin{aligned}
=&x^{n_1}\det\left(xI_{m_1}-A(l(G_1))-\frac{1}{x}R(G_1)^TR(G_1)\right)\\
&\times\left[1-\Gamma_{A(G_2)}(x)\mathbf{1}_{m_1}^T\left\{xI_{m_1}-A(l(G_1))-\frac{1}{x}R(G_1)^TR(G_1)\right\}^{-1}\mathbf{1}_{m_1}\right]\\
=&x^{n_1}\det\left(\left(x-\frac{2}{x}\right)I_{m_1}-\left(1+\frac{1}{x}\right)A(l(G_1))\right)\\
&\times\left[1-\Gamma_{A(G_2)}(x)\Gamma_{A(l(G_1))+\frac{1}{x}R(G_1)^TR(G_1)}(x)\right]\\
=&x^{n_1}(x+2)^{m_1-n_1}\det\left(\left(x-\frac{2}{x}\right)I_{m_1}-\left(1+\frac{1}{x}\right)(r_1-2+A(G_1))\right)\\
&\times\left[1-\frac{n_2}{(x-r_2)}\frac{m_1}{(x+2-2r_1-\frac{2r_1}{x})}\right]\\
=&x^{n_1}(x+2)^{m_1-n_1}\prod_{i=1}^{n_1}\left\{x-\frac{2}{x}-\left(1+\frac{1}{x}\right)(r_1-2+\lambda_i(G_1))\right\}\\
&\times\left[1-\frac{m_1n_2x}{(x-r_2)\{x^2+(2-2r_1)x-2r_1\}}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
f_{A(G_1\veebar_Q G_2)}(x)=&x^{n_1}(x+2)^{m_1-n_1}\prod_{j=1}^{n_2}\{x-\lambda_j(G_2)\}\\
&\times\prod_{i=1}^{n_1}\left\{x-\frac{2}{x}-\left(1+\frac{1}{x}\right)(r_1-2+\lambda_i(G_1))\right\}\\
&\times\left[1-\frac{m_1n_2x}{(x-r_2)\{x^2+(2-2r_1)x-2r_1\}}\right]\\
=&(x+2)^{m_1-n_1}\prod_{j=2}^{n_2}\{x-\lambda_j(G_2)\}\\
&\times\prod_{i=2}^{n_1}\{x^2-(r_1-2+\lambda_i(G_1))x-r_1-\lambda_i(G_1)\}\\
&\times\{x^3-(2r_1+r_2-2)x^2-(2r_1+m_1n_2+2r_2-2r_1r_2)x+2r_1r_2\}.\square
\end{aligned}
$$

2.2.2. *L-spectra of Q-edge join.*
The Laplacian matrix of $G_1\veebar_Q G_2$ can be written as:

$$
L(G_1\veebar_Q G_2)=\begin{pmatrix} r_1I_{n_1} & -R(G_1) & O_{m_1\times n_2}\\ -R(G_1)^T & (2r_1+n_2)I_{m_1}-A(l(G_1)) & -J_{n_1\times n_2}\\ O_{n_2\times m_1} & -J_{n_2\times n_1} & m_1I_{n_2}+L(G_2)\end{pmatrix}.
$$

Theorem 2.5. *For $i=1,2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then the Laplacian spectrum of $G_1\veebar_Q G_2$ consists of:*

(i) *the eigenvalue $m_1+\mu_j(G_2)$ for every eigenvalue μ_j, $j=2,3,\ldots,n_2$, of $L(G_2)$;*

(ii) *the eigenvalue* $2+2r_1+n_2$ *with multiplicity* m_1-n_1;
(iii) *two roots of the equation* $x^2-(2+r_1+n_2+\mu_i(G_1))x+r_1n_2+r_1\mu_i(G_1)+\mu_i(G_1)=0$ *for each eigenvalue* μ_i, $i=2,3,\dots,n_1$, *of* $L(G_1)$;
(iv) *three roots of the equation* $x^3-(2+r_1+m_1+n_2)x^2+(2m_1+r_1n_2+r_1m_1)x=0$.

Proof. The Laplacian characteristic polynomial of $G_1 \veebar_Q G_2$ is

$$\begin{aligned}
f_{L(G_1\veebar_Q G_2)}(x) &= \det(xI_{n_1+n_2+m_1} - L(G_1 \veebar_Q G_2))\\
&= \det\begin{pmatrix}(x-r_1)I_{n_1} & R(G_1) & O_{n_1\times n_2}\\ R(G_1)^T & (x-2r_1-n_2)I_{m_1}+A(l(G_1)) & J_{m_1\times n_2}\\ O_{n_2\times n_1} & J_{n_2\times m_1} & (x-m_1)I_{n_2}-L(G_2)\end{pmatrix}\\
&= \det((x-m_1)I_{n_2}-L(G_2))\det(S) = \prod_{j=1}^{n_2}\{x-m_1-\mu_j(G_2)\}\det(S),
\end{aligned}$$

where

$$\begin{aligned}
S &= \begin{pmatrix}(x-r_1)I_{n_1} & R(G_1)\\ R(G_1)^T & (x-2r_1-n_2)I_{m_1}+A(l(G_1))\end{pmatrix}\\
&\quad - \begin{pmatrix}O_{n_1\times n_2}\\ -J_{m_1\times n_2}\end{pmatrix}((x-m_1)I_{n_2}-L(G_2))^{-1}\begin{pmatrix}O_{n_2\times n_1} & -J_{n_2\times m_1}\end{pmatrix}\\
&= \begin{pmatrix}(x-r_1)I_{n_1} & R(G_1)\\ R(G_1)^T & (x-2r_1-n_2)I_{m_1}+A(l(G_1))-\Gamma_{L(G_2)}(x-m_1)J_{m_1\times m_1}\end{pmatrix}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\det(S) =&(x-r_1)^{n_1}\det\Big((x-2r_1-n_2)I_{m_1}+A(l(G_1))\\
&-\Gamma_{L(G_2)}(x-m_1)J_{m_1\times m_1}-\frac{1}{x-r_1}R(G_1)^TR(G_1)\Big)\\
=&(x-r_1)^{n_1}\Bigg[\det((x-2r_1-n_2)I_{m_1}+A(l(G_1))\\
&-\frac{1}{x-r_1}R(G_1)^TR(G_1))-\Gamma_{L(G_2)}(x-m_1)\\
&\times \mathbf{1}_{m_1}^T\operatorname{adj}\left\{(x-2r_1-n_2)I_{m_1}+A(l(G_1))-\frac{1}{x-r_1}R(G_1)^TR(G_1)\right\}\mathbf{1}_{m_1}\Bigg]\\
=&(x-r_1)^{n_1}\det\Big((x-2r_1-n_2)I_{m_1}+A(l(G_1))\\
&-\frac{1}{x-r_1}R(G_1)^TR(G_1)\Big)\Bigg[1-\Gamma_{L(G_2)}(x-m_1)\\
&\times \mathbf{1}_{m_1}^T\left\{(x-2r_1-n_2)I_{m_1}+A(l(G_1))-\frac{1}{x-r_1}R(G_1)^TR(G_1)\right\}^{-1}\mathbf{1}_{m_1}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
=&(x-r_1)^{n_1}\det\left(\left(x-2r_1-n_2-\frac{2}{x-r_1}\right)I_{m_1}+\left(1-\frac{1}{x-r_1}\right)A(l(G_1))\right)\\
&\times\left[1-\Gamma_{L(G_2)}(x-m_1)\Gamma_{\frac{1}{x-r_1}R(G_1)^TR(G_1)-A(l(G_1))}(x-2r_1-n_2)\right]\\
=&(x-r_1)^{n_1}(x-2-2r_1-n_2)^{m_1-n_1}\det\left(\left(x-2r_1-n_2-\frac{2}{x-r_1}\right)I_{n_1}\right.\\
&\left.+\left(1-\frac{1}{x-r_1}\right)(2(r_1-1)I_{n_1}-L(G_1))\right)\\
&\times\left[1-\frac{n_2}{(x-m_1)}\frac{m_1}{(x-2-n_2-\frac{2r_1}{x-r_1})}\right]\\
=&(x-r_1)^{n_1}(x-2-2r_1-n_2)^{m_1-n_1}\\
&\times\prod_{i=1}^{n_1}\left\{x-2r_1-n_2-\frac{2}{x-r_1}+\left(1-\frac{1}{x-r_1}\right)(2r_1-2-\mu_i(G_1))\right\}\\
&\times\left[1-\frac{m_1n_2(x-r_1)}{(x-m_1)(x^2-(2+r_1+n_2)x+r_1n_2)}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
f_{L(G_1\veebar_Q G_2)}(x)=&(x-r_1)^{n_1}(x-2-2r_1-n_2)^{m_1-n_1}\prod_{j=1}^{n_2}\{x-m_1-\mu_j(G_2)\}\\
&\times\prod_{i=1}^{n_1}\left\{x-2r_1-n_2-\frac{2}{x-r_1}+\left(1-\frac{1}{x-r_1}\right)(2r_1-2-\mu_i(G_1))\right\}\\
&\times\left[1-\frac{m_1n_2(x-r_1)}{(x-m_1)(x^2-(2+r_1+n_2)x+r_1n_2)}\right]\\
=&(x-2-2r_1-n_2)^{m_1-n_1}\prod_{j=2}^{n_2}\{x-m_1-\mu_j(G_2)\}\\
&\times\prod_{i=2}^{n_1}\{x^2-(2+r_1+n_2+\mu_i(G_1))x+r_1n_2+r_1\mu_i(G_1)+\mu_i(G_1)\}\\
&\times\{x^3-(2+r_1+m_1+n_2)x^2+(2m_1+r_1n_2+r_1m_1)x\}. \qquad \square
\end{aligned}
$$

Corollary 2.3. *For $i=1,2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then*

$$
\begin{aligned}
Kf(G_1\veebar_Q G_2)=(n_1+n_2+m_1)\times&\left(\frac{m_1-n_1}{2+2r_1+n_2}+\frac{2+r_1+m_1+n_2}{2m_1+r_1n_2+r_1m_1}\right.\\
&\left.+\sum_{i=2}^{n_1}\frac{2+r_1+n_2+\mu_i(G_1)}{r_1n_2+r_1\mu_i(G_1)+\mu_i(G_1)}+\sum_{j=2}^{n_2}\frac{1}{m_1+\mu_j(G_2)}\right).
\end{aligned}
$$

Corollary 2.4. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then*

$$t(G_1 \veebar_Q G_2) = \frac{(2+2r_1+n_2)^{m_1-n_1}\cdot(2m_1+r_1n_2+r_1m_1)\cdot\prod_{i=2}^{n_1}(r_1n_2+r_1\mu_i(G_1)+\mu_i(G_1))\cdot\prod_{j=2}^{n_2}(m_1+\mu_j(G_2))}{n_1+n_2+m_1}.$$

2.2.3. *$\mathcal{L}$-spectra of Q-edge join.*

The normalized Laplacian matrix of $G_1 \veebar_Q G_2$ can be obtained as:

$$\mathcal{L}(G_1 \veebar_Q G_2) = \begin{pmatrix} I_{n_1} & -cR(G_1) & O_{n_1\times n_2} \\ -cR(G_1)^T & I_{m_1} - \frac{1}{2r_1+n_2}A(l(G_1)) & -K_{m_1\times n_2} \\ O_{n_2\times n_1} & -K_{n_2\times m_1} & \mathcal{L}(G_2)\bullet B(G_2) \end{pmatrix},$$

where $K_{m_1\times n_2}$ is the matrix of size $m_1 \times n_2$ with all entries equal to $\frac{1}{\sqrt{(2r_1+n_2)(r_2+m_1)}}$, $B(G_2)$ is the $n_2 \times n_2$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_2}{r_2+m_1}$, c is the constant whose value is $\frac{1}{\sqrt{r_1(2r_1+n_2)}}$.

Theorem 2.6. *The normalized Laplacian spectrum of $G_1 \veebar_Q G_2$ consists of:*

(i) *the eigenvalue $\frac{m_1+r_2\delta_j(G_2)}{r_2+m_1}$, for every eigenvalue $\delta_j(G_2)$, $j = 2, 3, \dots, n_2$ of $\mathcal{L}(G_2)$;*
(ii) *the eigenvalue $1 + \frac{2}{2r_1+n_2}$ with multiplicity $m_1 - n_1$;*
(iii) *two roots of the equation $(2r_1 + n_2)x^2 - (2 + 2r_1 + 2n_2 + r_1\delta_i(G_1))x + n_2 + \delta_i(G_1) + r_1\delta_i(G_1) = 0$, for each eigenvalue $\delta_i(G_1)$, $i = 2, 3, \dots, n_1$ of $\mathcal{L}(G_1)$;*
(iv) *three roots of the equation $(2r_1r_2 + 2r_1m_1 + r_2n_2 + m_1n_2)x^3 - (2r_1r_2 + 2r_2 + 4r_1m_1 + 2m_1 + 2r_2n_2 + 3m_1n_2)x^2 + (2m_1 + 2r_1m_1 + r_2n_2 + 2m_1n_2)x = 0$.*

Proof. The normalized Laplacian characteristic polynomial of $G_1 \veebar_Q G_2$ is

$$\begin{aligned} f_{\mathcal{L}(G_1\veebar_Q G_2)}(x) &= \det(xI_{n_1+n_2+m_1} - \mathcal{L}(G_1 \veebar_Q G_2)) \\ &= \det\begin{pmatrix} (x-1)I_{n_1} & cR(G_1) & O_{n_1\times n_2} \\ cR(G_1)^T & (x-1)I_{m_1} + \frac{1}{2r_1+n_2}A(l(G_1)) & K_{m_1\times n_2} \\ O_{n_2\times n_1} & K_{n_2\times m_1} & xI_{n_2} - (\mathcal{L}(G_2)\bullet B(G_2)) \end{pmatrix} \\ &= \det(xI_{n_2} - (\mathcal{L}(G_2)\bullet B(G_2)))\det(S), \end{aligned}$$

where

$$\begin{aligned} S = &\begin{pmatrix} (x-1)I_{n_1} & cR(G_1) \\ cR(G_1)^T & (x-1)I_{m_1} + \frac{1}{2r_1+n_2}A(l(G_1)) \end{pmatrix} \\ &- \begin{pmatrix} O_{n_1\times n_2} \\ K_{m_1\times n_2} \end{pmatrix} (xI_{n_2} - (\mathcal{L}(G_2)\bullet B(G_2)))^{-1} \begin{pmatrix} O_{n_2\times n_1} & K_{n_2\times m_1} \end{pmatrix} \\ = &\begin{pmatrix} (x-1)I_{n_1} & cR(G_1) \\ cR(G_1)^T & (x-1)I_{m_1} + \frac{1}{2r_1+n_2}A(l(G_1)) - \chi_{G_2}(B(G_2), C_{n_2}, x)J_{m_1\times m_1} \end{pmatrix}. \end{aligned}$$

Hence,

$$\det(S) = (x-1)^{n_1}\det\Big((x-1)I_{m_1} + \frac{1}{2r_1+n_2}A(l(G_1)) - \chi_{G_2}(B(G_2), C_{n_2}, x)J_{m_1\times m_1}$$

$$
\begin{aligned}
&- \frac{c^2}{x-1}R(G_1)^TR(G_1)\Big)\\
=&(x-1)^{n_1}\Bigg[\det((x-1)I_{m_1}+\frac{1}{2r_1+n_2}A(l(G_1))\\
&- \frac{c^2}{x-1}R(G_1)^TR(G_1)) - \chi_{G_2}(B(G_2),C_{n_2},x)\\
&\times \mathbf{1}_{m_1}^T\operatorname{adj}\left\{(x-1)I_{m_1}+\frac{1}{2r_1+n_2}A(l(G_1))-\frac{c^2}{x-1}R(G_1)^TR(G_1)\right\}\mathbf{1}_{m_1}\Bigg]\\
=&(x-1)^{n_1}\det\Big((x-1)I_{m_1}+\frac{1}{2r_1+n_2}A(l(G_1))\\
&- \frac{c^2}{x-1}R(G_1)^TR(G_1)\Big)\Bigg[1-\chi_{G_2}(B(G_2),C_{n_2},x)\\
&\times \mathbf{1}_{m_1}^T\left\{(x-1)I_{m_1}+\frac{1}{2r_1+n_2}A(l(G_1))-\frac{c^2}{x-1}R(G_1)^TR(G_1)\right\}^{-1}\mathbf{1}_{m_1}\Bigg]\\
=&(x-1)^{n_1}\det\left(\left(x-1-\frac{2c^2}{x-1}\right)I_{m_1}+\left(\frac{1}{2r_1+n_2}-\frac{c^2}{x-1}\right)A(l(G_1))\right)\\
&\times\left[1-\chi_{G_2}(B(G_2),C_{n_2},x)\Gamma_{\frac{c^2}{x-1}R(G_1)^TR(G_1)-\frac{1}{2r_1+n_2}A(l(G_1))}(x-1)\right]\\
=&(x-1)^{n_1}\left(x-1-\frac{2}{2r_1+n_2}\right)^{m_1-n_1}\det\Bigg(\left(x-1-\frac{2c^2}{x-1}\right)I_{n_1}\\
&+\left(\frac{1}{2r_1+n_2}-\frac{c^2}{x-1}\right)(2(r_1-1)I_{n_1}-r_1\mathcal{L}(G_1))\Bigg)\\
&\times\left[1-\chi_{G_2}(B(G_2),C_{n_2},x)\Gamma_{\frac{c^2}{x-1}R(G_1)^TR(G_1)-\frac{1}{2r_1+n_2}A(l(G_1))}(x-1)\right]\\
=&(x-1)^{n_1}\left(x-1-\frac{2}{2r_1+n_2}\right)^{m_1-n_1}\\
&\times\det\Bigg(\left(x-1-\frac{2}{r_1(x-1)(2r_1+n_2)}\right)I_{n_1}\\
&+\left(\frac{1}{2r_1+n_2}-\frac{1}{r_1(x-1)(2r_1+n_2)}\right)((2r_1-2)I_{n_1}-r_1\mathcal{L}(G_1))\Bigg)\\
&\times\left[1-\frac{m_1n_2}{(r_2+m_1)(2r_1+n_2)\Big(x-\frac{m_1}{r_2+m_1}\Big)\Big(x-1+\frac{2r_1-2}{2r_1+n_2}-\frac{2r_1}{r_1(x-1)(2r_1+n_2)}\Big)}\right].
\end{aligned}
$$

As $\mathcal{L}(G_2)\bullet B(G_2)=I_{n_2}-\frac{1}{r_2+m_1}A(G_2)$, $\mathcal{L}(G_2)\bullet B(G_2)=\frac{1}{r_2+m_1}(m_1I_{n_2}+r_2\mathcal{L}(G_2))$.

Since G_2 is regular, the sum of all entries on every row of its normalized Laplacian matrix is zero. That means, $\mathcal{L}(G_2)C_{n_2} = (1-\frac{r_2}{r_2})C_{n_2} = 0C_{n_2}$. Therefore,

$$(\mathcal{L}(G_2)\bullet B(G_2))C_{n_2} = \left(1-\frac{r_2}{r_2+m_1}\right)C_{n_2} = \frac{m_1}{r_2+m_1}C_{n_2}$$

and

$$(xI_{n_2}-(\mathcal{L}(G_2)\bullet B(G_2)))C_{n_2} = \left(x-\frac{m_1}{r_2+m_1}\right)C_{n_2}.$$

Also, $C_{n_2}^T C_{n_2} = \frac{n_2}{(2r_1+n_2)(r_2+m_1)}$.

Now,

$$\chi_{G_2}(B(G_2),C_{n_2},x) = \frac{n_2}{(2r_1+n_2)(r_2+m_1)\left(x-\frac{m_1}{r_2+m_1}\right)}$$

and

$$\Gamma_{\frac{c^2}{x-1}R(G_1)^TR(G_1)-\frac{1}{2r_1+n_2}A(l(G_1))}(x-1) = \frac{m_1}{x-1+\frac{2r_1-2}{2r_1+n_2}-\frac{2r_1}{r_1(x-1)(2r_1+n_2)}}.$$

Then

$$\begin{aligned}
&f_{\mathcal{L}(G_1\veebar_Q G_2)}(x)\\
=&(x-1)^{n_1}\left(x-1-\frac{2}{2r_1+n_2}\right)^{m_1-n_1}\prod_{j=2}^{n_2}\left(x-\frac{m_1+r_2\delta_j(G_2)}{r_2+m_1}\right)\\
&\times\prod_{i=2}^{n_1}\left\{x-1-\frac{2}{r_1(x-1)(2r_1+n_2)}\right.\\
&\left.+\left(\frac{1}{2r_1+n_2}-\frac{1}{r_1(x-1)(2r_1+n_2)}\right)(2r_1-2-r_1\delta_i(G_1))\right\}\\
&\times\left[1-\frac{m_1n_2}{(r_2+m_1)(2r_1+n_2)(x-\frac{m_1}{r_2+m_1})(x-1+\frac{2r_1-2}{2r_1+n_2}-\frac{2r_1}{r_1(x-1)(2r_1+n_2)})}\right]\\
=&\left(x-1-\frac{2}{2r_1+n_2}\right)^{m_1-n_1}\prod_{j=2}^{n_2}\left(x-\frac{m_1+r_2\delta_j(G_2)}{r_2+m_1}\right)\prod_{i=2}^{n_1}\{(2r_1+n_2)x^2\\
&\times(2+2r_1+2n_2+r_1\delta_i(G_1))x+n_2+\delta_i(G_1)+r_1\delta_i(G_1)\}\\
&\times[(2r_1r_2+2r_1m_1+r_2n_2+m_1n_2)x^3-(2r_1r_2+2r_2+4r_1m_1\\
&+2m_1+2r_2n_2+3m_1n_2)x^2+(2m_1+2r_1m_1+r_2n_2+2m_1n_2)x].
\end{aligned}$$ □

3. Simultaneous Cospectral Graphs

Butler [2] constructed non-regular bipartite graphs which are cospectral with respect to both the adjacency and normalized Laplacian matrices, and then asked for existence of non-regular graphs which are cospectral with respect to all the three matrices, namely, adjacency, Laplacian and normalized Laplacian. In this section we construct several classes of non-regular graphs which are cospectral with respect to all the above mentioned three matrices. For the construction of these graphs we consider two pairs of A-cospectral regular graphs (for example see [14]). Then we take Q-join of graphs belong to different pairs.

The following Lemma is immediate from the definition of Laplacian and normalized Laplacian matrices.

Lemma 3.1. (i) *If G is an r-regular graph then $L(G) = rI_n - A(G)$ and $\mathcal{L}(G) = I_n - \frac{1}{r}A(G)$.*

(ii) *If G_1 and G_2 are A-cospectral regular graphs then they are also cospectral with respect to the Laplacian and normalized Laplacian matrices.*

Observation. From all the theorems given in the previous section we observe that the adjacency, Laplacian and normalized Lpalacian spectra of all the Q-join graphs $G_1\dot{\vee}_Q G_2$ and $G_1 \veebar_Q G_2$, depend only on the number of vertices, number of edges, degree of regularities, and the corresponding spectrum of G_1 and G_2. Furthermore, we note that, although G_1 and G_2 are regular graphs, $G_1\dot{\vee}_Q G_2$ and $G_1 \veebar_Q G_2$ are non-regular graphs.

Theorem 3.1. *Let G_i, H_i be r_i-regular graphs, $i = 1, 2$, where G_1 need not be different from H_1. If G_1 and H_1 are A-cospectral, and G_2 and H_2 are A-cospectral then $G_1\dot{\vee}_Q G_2$ (respectively $G_1 \veebar_Q G_2$) and $H_1\dot{\vee}_Q H_2$ (respectively $H_1 \veebar_Q H_2$) are simultaneously A-cospectral, L-cospectral and $\mathcal{L}$-cospectral.*

Proof. Follows from the Lemma 3.1 and the above observation. □

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