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ON GENERALIZED COMMUTATIVE QUATERNION BALANCING-TYPE POLYNOMIALS

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Abstract. Generalized commutative quaternions are generalizations of elliptic, parabolic, and hyperbolic quaternions. They generalize bicomplex numbers, complex hyperbolic numbers, and hyperbolic complex numbers, too. In this paper, we introduce and study generalized commutative quaternion balancing polynomials and generalized commutative quaternion Lucas-balancing polynomials.

1. Introduction and preliminaries

The set of quaternions $\mathbb{H}$ was introduced by Hamilton in 1843 ([5]) in the following way:

$$\mathbb{H} = \{q\colon q = x_0 + x_1 i + x_2 j + x_3 k;\ x_0, x_1, x_2, x_3 \in \mathbb{R}\},$$

where

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

The multiplication of quaternions is not commutative. Hence, some quaternions algebra problems are not easy. In [15], Segre modified the definition of a quaternion in such a way that it admits a commutative property in multiplication, and he introduced commutative quaternions. The set of commutative quaternions is a 4-dimensional structure, contains zero divisor, and isotropic elements. Commutative quaternions have many applications in physics and mechanics (see [3, 10, 11]).

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Non-commutative quaternions and commutative quaternions were generalized and studied by Jafari and Yayli in [6]. Moreover, in [16], Szynal-Liana and Włoch introduced generalized commutative quaternions and studied them in the special subfamily of quaternions of Fibonacci type. Recall some necessary definitions.

Let $\mathbb{H}^c_{\alpha\beta}$ be the set of generalized commutative quaternions $\mathbf{x}$ of the form

$$\mathbf{x} = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3,$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and for $\alpha, \beta \in \mathbb{R}$ quaternionic units e_1, e_2, e_3 satisfy the equalities

$$e_1^2 = \alpha, \quad e_2^2 = \beta, \quad e_3^2 = \alpha\beta, \tag{1.1}$$

$$e_1 e_2 = e_2 e_1 = e_3, \quad e_2 e_3 = e_3 e_2 = \beta e_1 \quad \text{and} \quad e_3 e_1 = e_1 e_3 = \alpha e_2. \tag{1.2}$$

The generalized commutative quaternions generalize the following:

- elliptic quaternions for $\alpha < 0$ and $\beta = 1$,
- parabolic quaternions for $\alpha = 0$ and $\beta = 1$,
- hyperbolic quaternions for $\alpha > 0$ and $\beta = 1$,
- bicomplex numbers for $\alpha = -1$ and $\beta = -1$,
- complex hyperbolic numbers for $\alpha = -1$ and $\beta = 1$,
- hyperbolic complex numbers for $\alpha = 1$ and $\beta = -1$.

In [17], generalized commutative quaternion polynomials of Fibonacci type were considered. We will apply the concept of balancing polynomials and introduce generalized commutative quaternion balancing-type polynomials.

2. Balancing Numbers and Lucas-balancing Numbers

Balancing numbers were introduced in 1999 by Behera and Panda ([2]). Let n be a positive integer. Then, n is called a balancing number with balancer r, if it is the solution of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r).$$

The balancing sequence is denoted by $\{B_n\}$. In [2], the authors proved that the balancing numbers satisfy the recurrence relation

$$B_n = 6B_{n-1} - B_{n-2}, \quad \text{for } n \geq 2, \tag{2.1}$$

with initial conditions $B_0 = 0$, $B_1 = 1$.

The balancing numbers can be defined by the non-linear first order recurrence

$$B_n = 3B_{n-1} + \sqrt{8B_{n-1}^2 + 1}, \quad \text{for } n \geq 1,$$

with initial condition $B_0 = 0$.

It is well known that n is a balancing number if and only if n^2 is a triangular number, i.e., $8n^2 + 1$ is a perfect square (see [2]). In [8], the author introduced Lucas-balancing numbers, defined as follows: if B_n is a balancing number, then $C_n = \sqrt{8{B_n}^2 + 1}$ is called a Lucas-balancing number.

The Lucas-balancing numbers satisfy the same recurrence relation as that of the balancing numbers, but with different initial conditions:

$$C_n = 6C_{n-1} - C_{n-2}, \quad \text{for } n \geq 2,$$

with $C_0 = 1$, $C_1 = 3$.

Balancing numbers and Lucas-balancing numbers are given by the relations called Binet-type formulas:

$$B_n = \frac{{r_1}^n - {r_2}^n}{r_1 - r_2}, \quad C_n = \frac{{r_1}^n + {r_2}^n}{2},$$

where r_1, r_2 are the roots of the characteristic equation $r^2 - 6r + 1 = 0$, associated with the recurrence relation (2.1), i.e.,

$$r_1 = 3 + 2\sqrt{2}, \quad r_2 = 3 - 2\sqrt{2}.$$

The balancing numbers are defined for the negative subscripts by formula

$$B_{-n} = -B_n, \quad \text{for every integer } n.$$

In [12], the author, using a matrix representation of balancing and Lucas-balancing numbers, obtained several interesting identities for these numbers. We recall some of them.

Theorem 2.1 ([12])**.** *Let a, b, c, d be integers such that $a + b = c + d$. Then,*

$$B_aB_b - B_cB_d = B_{a-k}B_{b-k} - B_{c-k}B_{d-k},$$

$$C_aC_b - C_cC_d = C_{a-k}C_{b-k} - C_{c-k}C_{d-k},$$

$$B_aC_b - B_cC_d = B_{a-k}C_{b-k} - B_{c-k}C_{d-k},$$

for $k = 0, 1, 2, \ldots$

Corollary 2.1 ([12])**.** *Let n, k be integers. Then,*

$$\begin{aligned} B^2_{n+k} - B^2_{n-k} &= B_{2n}B_{2k}, \\ B^2_n - B_{n-1}B_{n+1} &= 1, \\ B_{n+1}C_{n-1} - B_nC_n &= 3, \\ C_{n+1}C_{n-1} - C^2_n &= 8. \end{aligned}$$

Balancing polynomials were introduced in [13] in the following way:

$$B_n(x) = 6xB_{n-1}(x) - B_{n-2}(x), \quad \text{for } n \geq 2, \tag{2.2}$$

with initial conditions $B_0(x) = 0$, $B_1(x) = 1$.

In [9], Lucas-balancing polynomials were considered. They are defined as follows.

$$C_n(x) = 6xC_{n-1}(x) - C_{n-2}(x), \quad \text{for } n \geq 2, \tag{2.3}$$

with initial terms $C_0(x) = 1$, $C_1(x) = 3x$.

For $x = 1$, we obtain $B_n(x) = B_n$, $C_n(x) = C_n$.

Binet-type formulas for balancing and Lucas-balancing polynomials have the form

$$B_n(x) = \frac{r_1^n(x) - r_2^n(x)}{r_1(x) - r_2(x)} \tag{2.4}$$

and

$$C_n(x) = r_1^n(x) + r_2^n(x),$$

where

$$r_1(x) = 3x + \sqrt{9x^2-1}, \quad r_2(x) = 3x - \sqrt{9x^2-1}, \quad 9x^2 - 1 > 0. \tag{2.5}$$

By simple calculations we get

$$r_1(x) + r_2(x) = 6x, \quad r_1(x) - r_2(x) = 2\sqrt{9x^2-1}, \quad r_1(x) \cdot r_2(x) = 1. \tag{2.6}$$

The first few balancing numbers, Lucas-balancing numbers, and balancing-type polynomials are given in Table 1.

n	0	1	2	3	4	5
B_n	0	1	6	35	204	1189
C_n	1	3	17	99	577	3363
$B_n(x)$	0	1	$6x$	$36x^2-1$	$216x^3-12x$	$1296x^4-108x^2+1$
$C_n(x)$	1	$3x$	$18x^2-1$	$108x^3-9x$	$648x^4-72x^2+1$	$3888x^5-540x^3+15x$

TABLE 1. A few first words of balancing-type numbers and balancing-type polynomials

In [4], some interesting properties of the balancing polynomials and Lucas-balancing polynomials are presented.

Theorem 2.2 ([4])**.** *Let $n \geq 1$ be an integer. Then,*

$$\begin{aligned} C_n(x) =& B_{n+1}(x) - 3xB_n(x), \\ C_n(x) =& \frac{1}{2}(B_{n+1}(x) - B_{n-1}(x)), \\ C_n(x) =& 3xB_n(x) - B_{n-1}(x), \\ C_n(x) =& 3xC_{n-1}(x) + (9x^2-1)B_{n-1}(x). \end{aligned}$$

Theorem 2.3 ([4])**.** *For $n \geq 1$ we have*

$$B_n'(x) = \frac{3nC_n(x) - 9xB_n(x)}{9x^2-1}, \quad C_n'(x) = 3nB_n(x).$$

Theorem 2.4 ([4])**.** *The ordinary generating functions are given by*

$$f(x,z) = \sum_{n=0}^{+\infty} B_n(x)z^n = \frac{z}{1-6xz+z^2},$$

$$g(x,z) = \sum_{n=0}^{+\infty} C_n(x)z^n = \frac{1-3xz}{1-6xz+z^2}.$$

Let $n \geq 0$ be an integer. The nth generalized commutative balancing quaternion $gc\,\mathcal{B}_n$, the nth generalized commutative Lucas-balancing quaternion $gc\,\mathcal{C}_n$ are defined as

$$\begin{aligned} gc\,\mathcal{B}_n =& B_n + B_{n+1}e_1 + B_{n+2}e_2 + B_{n+3}e_3, \\ gc\,\mathcal{C}_n =& C_n + C_{n+1}e_1 + C_{n+2}e_2 + C_{n+3}e_3, \end{aligned}$$

where e_1, e_2, e_3 are units satisfying the rules (1.1) and (1.2).

Let x be a real variable. The nth generalized commutative quaternion balancing polynomial $gc\,\mathcal{B}_n(x)$, the nth generalized commutative quaternion Lucas-balancing polynomial $gc\,\mathcal{C}_n(x)$ are defined by

$$gc\,\mathcal{B}_n(x) = B_n(x) + B_{n+1}(x)e_1 + B_{n+2}(x)e_2 + B_{n+3}(x)e_3, \tag{2.7}$$

$$gc\,\mathcal{C}_n(x) = C_n(x) + C_{n+1}(x)e_1 + C_{n+2}(x)e_2 + C_{n+3}(x)e_3, \tag{2.8}$$

respectively. If $x = 1$, then $gc\,\mathcal{B}_n(1) = gc\,\mathcal{B}_n$, $gc\,\mathcal{C}_n(1) = gc\,\mathcal{C}_n$.

3. Some Identities for Generalized Commutative Quaternion Balancing-type Polynomials

Theorem 3.1. *Let $n \geq 2$ be an integer and x be a real variable. Then,*

(i) $gc\,\mathcal{B}_n(x) = 6x gc\,\mathcal{B}_{n-1}(x) - gc\,\mathcal{B}_{n-2}(x)$,
(ii) $gc\,\mathcal{C}_n(x) = 6x gc\,\mathcal{C}_{n-1}(x) - gc\,\mathcal{C}_{n-2}(x)$,

where

$$\begin{aligned} gc\,\mathcal{B}_0(x) &= e_1 + 6xe_2 + (36x^2 - 1)e_3, \\ gc\,\mathcal{B}_1(x) &= 1 + 6xe_1 + (36x^2 - 1)e_2 + (216x^3 - 12x)e_3, \\ gc\,\mathcal{C}_0(x) &= 1 + 3xe_1 + (18x^2 - 1)e_2 + (108x^3 - 9x)e_3, \\ gc\,\mathcal{C}_1(x) &= 3x + (18x^2 - 1)e_1 + (108x^3 - 9x)e_2 + (648x^4 - 72x^2 + 1)e_3. \end{aligned}$$

Proof. For $n = 2$ we get

$$\begin{aligned} gc\,\mathcal{B}_2(x) &= 6x gc\,\mathcal{B}_1(x) - gc\,\mathcal{B}_0(x) \\ &= 6x + 36x^2e_1 + (216x^3 - 6x)e_2 + (1296x^4 - 72x^2)e_3 \\ &\quad - e_1 - 6xe_2 - (36x^2 - 1)e_3 \\ &= 6x + (36x^2 - 1)e_1 + (216x^3 - 12x)e_2 + (1296x^4 - 108x^2 + 1)e_3. \end{aligned}$$

Let $n \geq 3$. By (2.7) and (2.2) we get

$$\begin{aligned} gc\,\mathcal{B}_n(x) =& B_n(x) + B_{n+1}(x)e_1 + B_{n+2}(x)e_2 + B_{n+3}(x)e_3 \\ =& 6xB_{n-1}(x) - B_{n-2}(x) + (6xB_n(x) - B_{n-1}(x))e_1 \\ &+ (6B_{n+1}(x) - B_n(x))e_2 + (6B_{n+2}(x) - B_{n+1}(x))e_3 \\ =& 6x(B_{n-1}(x) + B_n(x)e_1 + B_{n+1}(x)e_2 + B_{n+2}(x)e_3) \\ &- (B_{n-2}(x) + B_{n-1}(x)e_1 + B_n(x)e_2 + B_{n+1}(x)e_3) \\ =& 6x gc\,\mathcal{B}_{n-1}(x) - gc\,\mathcal{B}_{n-2}(x), \end{aligned}$$

which ends the proof of (i).

The second part can be proved similarly using (2.8) and (2.3). □

Theorem 3.2. *Assume that $n \geq 1$ is an integer and x is a real variable. Then,*

$$3xB_n(x) - B_{n-1}(x) = C_n(x). \tag{3.1}$$

Proof. (By induction on n) For $n = 1$ we have $3xB_1(x) - B_0(x) = 3x = C_1(x)$, for $n = 2$ we get $3xB_2(x) - B_1(x) = 3x \cdot 6x - 1 = 18x^2 - 1 = C_2(x)$. Assuming that formula (3.1) is true for $k = 1, 2, \ldots, n$, we will prove it for $n + 1$. By the definition of balancing polynomials we get

$$\begin{aligned} 3xB_{n+1}(x) - B_n(x) =& 3x(6xB_n(x) - B_{n-1}(x)) - 6xB_{n-1}(x) + B_{n-2}(x) \\ =& 6x(3xB_n(x) - B_{n-1}(x)) - (3xB_{n-1}(x) - B_{n-2}(x)). \end{aligned}$$

By the induction hypothesis, we have

$$3xB_{n+1}(x) - B_n(x) = 6xC_n(x) - C_{n-1}(x) = C_{n+1}(x).$$

□

Corollary 3.1. *Assume that $n \geq 1$ is an integer and x is a real variable. Then,*

$$B_{n+1}(x) - 3xB_n(x) = C_n(x).$$

By formula (3.1) we get the following result.

Theorem 3.3. *Assume that $n \geq 1$ is an integer and x is a real variable. Then,*

$$3xgc\,\mathcal{B}_n(x) - gc\,\mathcal{B}_{n-1}(x) = gc\,\mathcal{C}_n(x).$$

Corollary 3.2. *Assume that $n \geq 1$ is an integer and x is a real variable. Then,*

$$gc\,\mathcal{B}_{n+1}(x) - 3xgc\,\mathcal{B}_n(x) = gc\,\mathcal{C}_n(x).$$

Now, we will present Binet-type formulas for generalized commutative quaternion balancing-type polynomials.

Theorem 3.4. *Let $n \geq 0$ be an integer and x be a real variable. Assume that $9x^2 - 1 > 0$. Then,*

$$gc\,\mathcal{B}_n(x) = \frac{r_1^n(x)\widehat{r_1(x)} - r_2^n(x)\widehat{r_2(x)}}{r_1(x) - r_2(x)}, \tag{3.2}$$

$$gc\,\mathcal{C}_n(x) = r_1^n(x)\widehat{r_1(x)} + r_2^n(x)\widehat{r_2(x)}, \tag{3.3}$$

where $r_1(x)$, $r_2(x)$ are given by (2.5) and

$$\begin{aligned} \widehat{r_1(x)} &= 1 + r_1(x)e_1 + r_1^2(x)e_2 + r_1^3(x)e_3, \\ \widehat{r_2(x)} &= 1 + r_2(x)e_1 + r_2^2(x)e_2 + r_2^3(x)e_3. \end{aligned} \tag{3.4}$$

Proof. We give the proof of (3.2). The proof of (3.3) is similar.

By (2.7) and (2.4) we get

$$\begin{aligned} gc\,\mathcal{B}_n(x) =&B_n(x) + B_{n+1}(x)e_1 + B_{n+2}(x)e_2 + B_{n+3}(x)e_3 \\ =&\frac{r_1^n(x) - r_2^n(x)}{r_1(x) - r_2(x)} + \frac{r_1^{n+1}(x) - r_2^{n+1}(x)}{r_1(x) - r_2(x)}e_1 \\ &+ \frac{r_1^{n+2}(x) - r_2^{n+2}(x)}{r_1(x) - r_2(x)}e_2 + \frac{r_1^{n+3}(x) - r_2^{n+3}(x)}{r_1(x) - r_2(x)}e_3 \\ =&\frac{1}{2\sqrt{9x^2-1}}\Big(r_1^n(x)(1 + r_1(x)e_1 + r_1^2(x)e_2 + r_1^3(x)e_3) \\ &- r_2^n(x)(1 + r_2(x)e_1 + r_2^2(x)e_2 + r_2^3(x)e_3)\Big), \end{aligned}$$

which ends the proof. □

Using (3.4), we have

$$\begin{aligned} \widehat{r_1(x)}\widehat{r_2(x)} =&\widehat{r_2(x)}\widehat{r_1(x)} \\ =&1 + r_1(x)r_2(x)\alpha + r_1^2(x)r_2^2(x)\beta + r_1^3(x)r_2^3(x)\alpha\beta \\ &+ \Big(r_1(x) + r_2(x) + r_1^2(x)r_2^3(x)\beta + r_1^3(x)r_2^2(x)\beta\Big)\, e_1 \\ &+ \Big(r_1^2(x) + r_2^2(x) + r_1(x)r_2^3(x)\alpha + r_1^3(x)r_2(x)\alpha\Big)\, e_2 \\ &+ \Big(r_1^3(x) + r_2^3(x) + r_1(x)r_2^2(x) + r_1^2(x)r_2(x)\Big)\, e_3 \\ =&1 + r_1(x)r_2(x)\alpha + (r_1(x)r_2(x))^2\,\beta + (r_1(x)r_2(x))^3\,\alpha\beta \\ &+ (r_1(x) + r_2(x))\Big(1 + (r_1(x)r_2(x))^2\,\beta\Big)\, e_1 \\ &+ \Big(r_1^2(x) + r_2^2(x)\Big)(1 + r_1(x)r_2(x)\alpha)\, e_2 \\ &+ \Big(r_1^3(x) + r_2^3(x) + r_1(x)r_2(x)\,(r_2(x) + r_1(x))\Big)\, e_3. \end{aligned}$$

Hence, by (2.6), we get

$$\begin{aligned} \widehat{r_1(x)}\widehat{r_2(x)} =&1 + \alpha + \beta + \alpha\beta + 6x\,(1+\beta)\, e_1 \\ &+ (36x^2 - 2)\,(1+\alpha)\, e_2 + (216x^3 - 12x)e_3. \end{aligned} \tag{3.5}$$

The next theorem presents general bilinear index reduction formulas for generalized commutative quaternion balancing-type polynomials.

Theorem 3.5. *Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a + b = c + d$. Then, for a real variable x and $9x^2 - 1 > 0$ we have*

$$\begin{aligned} &gc\,\mathcal{B}_a(x) \cdot gc\,\mathcal{B}_b(x) - gc\,\mathcal{B}_c(x) \cdot gc\,\mathcal{B}_d(x) \\ =&\frac{\Big(r_1^c(x)r_2^d(x) - r_1^a(x)r_2^b(x) + r_2^c(x)r_1^d(x) - r_2^a(x)r_1^b(x)\Big)\,\widehat{r_1(x)}\widehat{r_2(x)}}{36x^2 - 4}, \end{aligned} \tag{3.6}$$

(3.7)
$$gc\,\mathcal{C}_a(x)\cdot gc\,\mathcal{C}_b(x) - gc\,\mathcal{C}_c(x)\cdot gc\,\mathcal{C}_d(x) = \left(r_1^a(x)r_2^b(x) + r_2^a(x)r_1^b(x) - r_1^c(x)r_2^d(x) - r_2^c(x)r_1^d(x)\right)\widehat{r_1(x)}\widehat{r_2(x)},$$

where $r_1(x)$, $r_2(x)$ *are given by* (2.5) *and* $\widehat{r_1(x)}\widehat{r_2(x)}$ *is given by* (3.5).

Proof. Using (3.2) and the fact that $a+b=c+d$ we have

$$\begin{aligned}
&gc\,\mathcal{B}_a(x)\cdot gc\,\mathcal{B}_b(x) - gc\,\mathcal{B}_c(x)\cdot gc\,\mathcal{B}_d(x)\\
=&\frac{-r_1^a(x)r_2^b(x)\widehat{r_1(x)}\widehat{r_2(x)} - r_2^a(x)r_1^b(x)\widehat{r_2(x)}\widehat{r_1(x)}}{36x^2-4}\\
&+\frac{r_1^c(x)r_2^d(x)\widehat{r_1(x)}\widehat{r_2(x)} + r_2^c(x)r_1^d(x)\widehat{r_2(x)}\widehat{r_1(x)}}{36x^2-4}\\
=&\frac{\left(r_1^c(x)r_2^d(x) - r_1^a(x)r_2^b(x) + r_2^c(x)r_1^d(x) - r_2^a(x)r_1^b(x)\right)\widehat{r_1(x)}\widehat{r_2(x)}}{36x^2-4},
\end{aligned}$$

which ends the proof of (3.6). The formula (3.7) can be proved similarly. □

For special values of a, b, c, and d, we obtain the Catalan, Cassini, Vajda, d'Ocagne, and Halton-type identities, respectively.

Assume that $r_1(x)$, $r_2(x)$ are given by (2.5), $\widehat{r_1(x)}\widehat{r_2(x)}$ is given by (3.5) and x is a real variable. Let $9x^2-1>0$.

Corollary 3.3 (Catalan-type identities for generalized commutative quaternion balancing-type polynomials)**.** *Let* $n\geq 0$, $k\geq 0$ *be integers such that* $n\geq k$. *Then,*

$$gc\,\mathcal{B}_{n+k}(x)\cdot gc\,\mathcal{B}_{n-k}(x) - (gc\,\mathcal{B}_n(x))^2 = \left(2-\left(\frac{r_1(x)}{r_2(x)}\right)^k - \left(\frac{r_2(x)}{r_1(x)}\right)^k\right)\frac{\widehat{r_1(x)}\widehat{r_2(x)}}{36x^2-4},$$

$$gc\,\mathcal{C}_{n+k}(x)\cdot gc\,\mathcal{C}_{n-k}(x) - (gc\,\mathcal{C}_n(x))^2 = \left(\left(\frac{r_1(x)}{r_2(x)}\right)^k + \left(\frac{r_2(x)}{r_1(x)}\right)^k - 2\right)\widehat{r_1(x)}\widehat{r_2(x)}.$$

Corollary 3.4 (Cassini-type identities for generalized commutative quaternion balancing-type polynomials)**.** *Let* $n\geq 1$ *be an integer. Then,*

$$gc\,\mathcal{B}_{n+1}(x)\cdot gc\,\mathcal{B}_{n-1}(x) - (gc\,\mathcal{B}_n(x))^2 = -\widehat{r_1(x)}\widehat{r_2(x)},$$
$$gc\,\mathcal{C}_{n+1}(x)\cdot gc\,\mathcal{C}_{n-1}(x) - (gc\,\mathcal{C}_n(x))^2 = (36x^2-4)\widehat{r_1(x)}\widehat{r_2(x)}.$$

Corollary 3.5 (Vajda-type identities for generalized commutative quaternion balancing-type polynomials)**.** *Let* $n\geq 0$, $m\geq 0$, $k\geq 0$ *be integers such that* $n\geq k$. *Then,*

$$\begin{aligned}
&gc\,\mathcal{B}_{m+k}(x)\cdot gc\,\mathcal{B}_{n-k}(x) - gc\,\mathcal{B}_m(x)\cdot gc\,\mathcal{B}_n(x)\\
=&\frac{\left(r_1^m(x)r_2^n(x)\left(1-\left(\frac{r_1(x)}{r_2(x)}\right)^k\right) + r_1^n(x)r_2^m(x)\left(1-\left(\frac{r_2(x)}{r_1(x)}\right)^k\right)\right)\widehat{r_1(x)}\widehat{r_2(x)}}{36x^2-4},
\end{aligned}$$

$$\begin{aligned}
&gc\,\mathcal{C}_{m+k}(x)\cdot gc\,\mathcal{C}_{n-k}(x)-gc\,\mathcal{C}_{m}(x)\cdot gc\,\mathcal{C}_{n}(x)\\
=&\left(r_1^m(x)r_2^n(x)\left(\left(\frac{r_1(x)}{r_2(x)}\right)^k-1\right)+r_1^n(x)r_2^m(x)\left(\left(\frac{r_2(x)}{r_1(x)}\right)^k-1\right)\right)\widehat{r_1(x)}\widehat{r_2(x)}.
\end{aligned}$$

Corollary 3.6 (d'Ocagne-type identities for generalized commutative quaternion balancing-type polynomials)**.** *Let $n\geq 0$, $m\geq 0$ be integers such that $n\geq m$. Then,*

$$\begin{aligned}
&gc\,\mathcal{B}_{n}(x)\cdot gc\,\mathcal{B}_{m+1}(x)-gc\,\mathcal{B}_{n+1}(x)\cdot gc\,\mathcal{B}_{m}(x)\\
=&\frac{\left(r_1^n(x)r_2^m(x)-r_2^n(x)r_1^m(x)\right)\widehat{r_1(x)}\widehat{r_2(x)}}{2\sqrt{9x^2-1}},\\
&gc\,\mathcal{C}_{n}(x)\cdot gc\,\mathcal{C}_{m+1}(x)-gc\,\mathcal{C}_{n+1}(x)\cdot gc\,\mathcal{C}_{m}(x)\\
=&\left(r_1^n(x)r_2^m(x)-r_2^n(x)r_1^m(x)\right)\left(r_2(x)-r_1(x)\right)\widehat{r_1(x)}\widehat{r_2(x)}.
\end{aligned}$$

Corollary 3.7 (The first Halton-type identities for generalized commutative quaternion balancing-type polynomials)**.** *Let $n\geq 0$, $m\geq 0$, $k\geq 0$ be integers such that $n\geq k$. Then,*

$$\begin{aligned}
&gc\,\mathcal{B}_{m+k}(x)\cdot gc\,\mathcal{B}_{n}(x)-gc\,\mathcal{B}_{k}(x)\cdot gc\,\mathcal{B}_{m+n}(x)\\
=&\left(r_1(x)^{n-k}-r_2(x)^{n-k}\right)\left(r_1(x)^m-r_2(x)^m\right)\frac{\widehat{r_1(x)}\widehat{r_2(x)}}{2\sqrt{9x^2-1}},\\
&gc\,\mathcal{C}_{m+k}(x)\cdot gc\,\mathcal{C}_{n}(x)-gc\,\mathcal{C}_{k}(x)\cdot gc\,\mathcal{C}_{m+n}(x)\\
=&\left(r_2(x)^{n-k}-r_1(x)^{n-k}\right)\left(r_1(x)^m-r_2(x)^m\right)\widehat{r_1(x)}\widehat{r_2(x)}.
\end{aligned}$$

Corollary 3.8 (The second Halton-type identities for generalized commutative quaternion balancing-type polynomials)**.** *Let $n\geq 0$, $k\geq 0$, $s\geq 0$ be integers such that $n\geq k$, $n\geq s$. Then,*

$$\begin{aligned}
&gc\,\mathcal{B}_{n+k}(x)\cdot gc\,\mathcal{B}_{n-k}(x)-gc\,\mathcal{B}_{n+s}(x)\cdot gc\,\mathcal{B}_{n-s}(x)\\
=&\left(\left(\frac{r_1(x)}{r_2(x)}\right)^s+\left(\frac{r_2(x)}{r_1(x)}\right)^s-\left(\frac{r_1(x)}{r_2(x)}\right)^k-\left(\frac{r_2(x)}{r_1(x)}\right)^k\right)\frac{\widehat{r_1(x)}\widehat{r_2(x)}}{9x^2-1},\\
&gc\,\mathcal{C}_{n+k}(x)\cdot gc\,\mathcal{C}_{n-k}(x)-gc\,\mathcal{C}_{n+s}(x)\cdot gc\,\mathcal{C}_{n-s}(x)\\
=&\left(\left(\frac{r_1(x)}{r_2(x)}\right)^k+\left(\frac{r_2(x)}{r_1(x)}\right)^k-\left(\frac{r_1(x)}{r_2(x)}\right)^s-\left(\frac{r_2(x)}{r_1(x)}\right)^s\right)\widehat{r_1(x)}\widehat{r_2(x)}.
\end{aligned}$$

Theorem 3.6. *Let $a\geq 0$, $b\geq 0$, $c\geq 0$, $d\geq 0$ be integers such that $a+b=c+d$. Then, for a real variable x such that $9x^2-1>0$, we have*

$$\begin{aligned}
&gc\,\mathcal{B}_{a}(x)\cdot gc\,\mathcal{C}_{b}(x)-gc\,\mathcal{B}_{c}(x)\cdot gc\,\mathcal{C}_{d}(x)\\
=&\left(r_1^a(x)r_2^b(x)-r_2^a(x)r_1^b(x)-r_1^c(x)r_2^d(x)+r_2^c(x)r_1^d(x)\right)\frac{\widehat{r_1(x)}\widehat{r_2(x)}}{2\sqrt{9x^2-1}},
\end{aligned}$$

where $r_1(x)$, $r_2(x)$, $\widehat{r_1(x)}\widehat{r_2(x)}$ *are given by* (2.5), (3.5), *respectively.*

Proof. Using (3.2) and (3.3), we have

$$\begin{aligned}
&gc\,\mathcal{B}_a(x)\cdot gc\,\mathcal{C}_b(x) - gc\,\mathcal{B}_c(x)\cdot gc\,\mathcal{C}_d(x)\\
=&\frac{1}{2\sqrt{9x^2-1}}\Big(r_1^{a+b}(x) - r_1^{c+d}(x)\widehat{r_1^2(x)} + (r_2^{c+d}(x) - r_2^{a+b}(x)\widehat{r_1^2(x)}\\
&+\widehat{r_1(x)}\widehat{r_2(x)}(r_1^a(x)r_2^b(x) - r_2^a(x)r_1^b(x) - r_1^c(x)r_2^d(x) + r_1^d(x)r_2^c(x)\Big).
\end{aligned}$$

Since $a+b=c+d$, we get

$$\begin{aligned}
&gc\,\mathcal{B}_a(x)\cdot gc\,\mathcal{C}_b(x) - gc\,\mathcal{B}_c(x)\cdot gc\,\mathcal{C}_d(x)\\
=&\frac{1}{2\sqrt{9x^2-1}}\Big(\widehat{r_1(x)}\widehat{r_2(x)}(r_1^a(x)r_2^b(x) - r_2^a(x)r_1^b(x) - r_1^c(x)r_2^d(x) + r_1^d(x)r_2^c(x)\Big),
\end{aligned}$$

which ends the proof. □

Corollary 3.9. *Let* $n\geq 0$, $m\geq 0$, $k\geq 0$ *be integers. Then,*

$$\begin{aligned}
&gc\,\mathcal{B}_k(x)\cdot gc\,\mathcal{C}_{n+m}(x) - gc\,\mathcal{B}_m(x)\cdot gc\,\mathcal{C}_{n+k}(x)\\
=&\Big(r_1^n(x)+r_2^n(x)\Big)\Big(r_1^k(x)r_2^m(x) - r_1^m(x)r_2^k(x)\Big)\frac{\widehat{r_1(x)}\widehat{r_2(x)}}{2\sqrt{9x^2-1}},
\end{aligned}$$

where $r_1(x)$, $r_2(x)$, $\widehat{r_1(x)}\widehat{r_2(x)}$ *are given by* (2.5), (3.5), *respectively.*

4. Generating Functions and Matrix Generators

Theorem 4.1. *The generating function of the generalized commutative quaternion balancing polynomial has the following form*

$$g(t) = \frac{e_1 + 6xe_2 + (36x^2-1)e_3 + (1-e_2-6xe_3)t}{1-6xt+t^2}.$$

Proof. Let

$$g(t) = gc\,\mathcal{B}_0(x) + tgc\,\mathcal{B}_1(x) + t^2gc\,\mathcal{B}_2(x) + \cdots + t^ngc\,\mathcal{B}_n(x) + \cdots$$

be the generating function of the generalized commutative quaternion balancing polynomial. Then,

$$\begin{aligned}
6xtg(t) &= 6txgc\,\mathcal{B}_0(x) + 6t^2xgc\,\mathcal{B}_1(x) + 6t^3xgc\,\mathcal{B}_2(x) + \cdots + 6t^nxgc\,\mathcal{B}_n(x) + \cdots,\\
t^2g(t) &= t^2gc\,\mathcal{B}_0(x) + t^3gc\,\mathcal{B}_1(x) + t^4gc\,\mathcal{B}_2(x) + \cdots + t^ngc\,\mathcal{B}_{n-2}(x) + \cdots.
\end{aligned}$$

Hence, by the recurrence $gc\,\mathcal{B}_n(x) = 6xgc\,\mathcal{B}_{n-1}(x) - gc\,\mathcal{B}_{n-2}(x)$, we get

$$\begin{aligned}
&g(t) - 6xtg(t) + t^2g(t)\\
=&gc\,\mathcal{B}_0(x) + \Big(gc\,\mathcal{B}_1(x) - 6xgc\,\mathcal{B}_0(x)\Big)t + \Big(gc\,\mathcal{B}_0(x) + gc\,\mathcal{B}_2(x) - 6xgc\,\mathcal{B}_1(x)\Big)t^2 + \cdots\\
=&gc\,\mathcal{B}_0(x) + \Big(gc\,\mathcal{B}_1(x) - 6xgc\,\mathcal{B}_0(x)\Big)t.
\end{aligned}$$

Thus,

$$g(t)=\frac{gc\,\mathcal{B}_0(x)+\big(gc\,\mathcal{B}_1(x)-6xgc\,\mathcal{B}_0(x)\big)t}{1-6xt+t^2}.$$

After simple calculations we obtain

$$g(t)=\frac{e_1+6xe_2+(36x^2-1)e_3+(1-e_2-6xe_3)t}{1-6xt+t^2}.$$

□

In the same way, we can prove the following theorem.

Theorem 4.2. *The generating function of the generalized commutative quaternion Lucas-balancing polynomial has the following form*

$$f(t)=\frac{gc\,\mathcal{C}_0(x)+\big(gc\,\mathcal{C}_1(x)-6xgc\,\mathcal{C}_0(x)\big)t}{1-6xt+t^2},$$

where $gc\,\mathcal{C}_0(x)=1+3xe_1+(18x^2-1)e_2+(108x^3-9x)e_3$, $gc\,\mathcal{C}_1(x)-6xgc\,\mathcal{C}_0(x)=-3x-e_1-3xe_2+(1-18x^2)e_3$.

The following theorems give some matrix representations of the generalized commutative quaternion balancing-type polynomials.

Theorem 4.3. *Let* $n\geq 1$ *be an integer and* x *be a real variable. Then,*

$$\begin{pmatrix} gc\,\mathcal{B}_{n+1}(x) & -gc\,\mathcal{B}_n(x)\\ gc\,\mathcal{B}_n(x) & -gc\,\mathcal{B}_{n-1}(x)\end{pmatrix}=\begin{pmatrix} gc\,\mathcal{B}_2(x) & -gc\,\mathcal{B}_1(x)\\ gc\,\mathcal{B}_1(x) & -gc\,\mathcal{B}_0(x)\end{pmatrix}\cdot\begin{pmatrix} 6x & -1\\ 1 & 0\end{pmatrix}^{n-1}. \tag{4.1}$$

Proof. (By induction on n). If $n=1$, then the result is obvious. Assuming (4.1) holds for n, we will prove it for $n+1$. By the inductions hypothesis, we get

$$\begin{aligned}&\begin{pmatrix} gc\,\mathcal{B}_2(x) & -gc\,\mathcal{B}_1(x)\\ gc\,\mathcal{B}_1(x) & -gc\,\mathcal{B}_0(x)\end{pmatrix}\cdot\begin{pmatrix} 6x & -1\\ 1 & 0\end{pmatrix}^{n-1}\cdot\begin{pmatrix} 6x & -1\\ 1 & 0\end{pmatrix}\\ &=\begin{pmatrix} gc\,\mathcal{B}_{n+1}(x) & -gc\,\mathcal{B}_n(x)\\ gc\,\mathcal{B}_n(x) & -gc\,\mathcal{B}_{n-1}(x)\end{pmatrix}\cdot\begin{pmatrix} 6x & -1\\ 1 & 0\end{pmatrix}\\ &=\begin{pmatrix} 6xgc\,\mathcal{B}_{n+1}(x)-gc\,\mathcal{B}_n(x) & -gc\,\mathcal{B}_{n+1}(x)\\ 6xgc\,\mathcal{B}_n(x)-gc\,\mathcal{B}_{n-1}(x) & -gc\,\mathcal{B}_n(x)\end{pmatrix}=\begin{pmatrix} gc\,\mathcal{B}_{n+2}(x) & -gc\,\mathcal{B}_{n+1}(x)\\ gc\,\mathcal{B}_{n+1}(x) & -gc\,\mathcal{B}_n(x)\end{pmatrix}.\end{aligned}$$

□

In the same way, using Theorem 3.3 and Corollary 3.2, one can easily prove the next result.

Theorem 4.4. *Let* $n\geq 1$ *be an integer and* x *be a real variable. Then,*

$$\begin{aligned}&\begin{pmatrix} gc\,\mathcal{C}_{n+1}(x) & -gc\,\mathcal{C}_n(x)\\ gc\,\mathcal{C}_n(x) & -gc\,\mathcal{C}_{n-1}(x)\end{pmatrix}\\ &=\begin{pmatrix} 3x & -1\\ 1 & -3x\end{pmatrix}\cdot\begin{pmatrix} gc\,\mathcal{B}_2(x) & -gc\,\mathcal{B}_1(x)\\ gc\,\mathcal{B}_1(x) & -gc\,\mathcal{B}_0(x)\end{pmatrix}\cdot\begin{pmatrix} 6x & -1\\ 1 & 0\end{pmatrix}^{n-1}.\end{aligned}$$

Concluding Remarks

In this paper, we define generalized commutative balancing and Lucas-balancing polynomials of one variable. A sequence b_n is the binomial transform of the sequence a_n if $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$. The definitions and properties of the binomial transforms of balancing and Lucas-balancing polynomials of one variable are presented in [18]. Applications of the binomial transforms of Fibonacci-type sequences in the theory of hypercomplex numbers can be found, for example, in [7].

The bivariate balancing, Lucas-balancing polynomials, and hybrid generalizations of these polynomials were studied in [1, 19] and [14], respectively.

It would be interesting to continue this research by examining generalized commutative balancing and Lucas-balancing polynomials of two variables, also in combination with their binomial transforms.

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