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APPROXIMATION OF RECIPROCAL FUNCTION OF THREE VARIABLES

HEMEN DUTTA[1] AND P. NARASIMMAN[2]

Abstract. The significant role of rational functions in applied sciences motivated us to study a functional equation having a rational function as its solution. In this work, we introduce and analyze a reciprocal type functional equation defined on three variables. We find a solution to this functional equation and study its stability using several types of upper bounds introduced by Ulam, Hyers, Găvruta, T. Rassias, and J. Rassias.

1. Introduction

The problem of stability in functional equations originates from a problem raised by Ulam in [17], which dealt with the study of approximate homomorphisms. An important, though only partial, answer was provided in [6], where a theorem was established that many researchers now refer to as Ulam-Hyers stability. The technique adopted in [6] to provide a solution to the stability problem is termed as the direct method. Furthermore, the corresponding stability issue has been addressed in several works under different names: in [12] it appears as Hyers-Ulam-Rassias stability, in [11] it is described as Ulam-Găvruta-Rassias stability, and in [4] it is presented as a generalized form of Hyers-Ulam-Rassias stability. The interesting and compelling stability properties established for diverse classes of functional, difference, differential, integral, and fractional differential equations have motivated numerous mathematicians to investigate stability issues across a wide range of mathematical equations.

Key words and phrases. Rational functional equations, three-variable rational functional equation, Hyers-Ulam stability.

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Numerous significant solutions to stability problems have been obtained by several researchers [1, 2, 5, 8, 18].

In recent years, attention in this area of research has shifted toward the study of functional equations that incorporate rational expressions. The earliest and most fundamental rational functional equation in a single variable was presented in [15]. More specifically, when a rational functional equation admits a reciprocal function as its solution, it is commonly termed a reciprocal functional equation.

Throughout this paper, $\mathbb{R}^\star$ denotes the set of all nonzero real numbers and $\mathbb{R}$ the set of all real numbers.

Definition 1.1 ([15])**.** Let $\mu : \mathbb{R}^\star \to \mathbb{R}$ be a function such that $\mu(\ell) \neq 0$ for all $\ell \in \mathbb{R}^\star$. Suppose that μ satisfies the functional identity

$$\mu(\ell_1 + \ell_2)\,(\mu(\ell_1) + \mu(\ell_2)) = \mu(\ell_1)\mu(\ell_2), \tag{1.1}$$

for all $\ell_1, \ell_2 \in \mathbb{R}^\star$ for which $\ell_1 + \ell_2 \neq 0$ and $\mu(\ell_1) + \mu(\ell_2) \neq 0$. Then, $\mu(\ell) = \frac{c}{\ell}$ for some nonzero constant $c \in \mathbb{R}$.

The stability properties of the functional equation (1.1) were studied in [15], providing foundational results for subsequent studies. The following rational functional equation involving a radical was introduced in [9] and is given by

$$\mu\left(\sqrt{\ell_1^2 + \ell_2^2}\right) = \frac{\mu(\ell_1)\mu(\ell_2)}{\mu(\ell_1) + \mu(\ell_2)}, \quad \ell_1, \ell_2 \in (0, +\infty). \tag{1.2}$$

It has the reciprocal-quadratic function $\mu(\ell) = \frac{c}{\ell^2}$ as its solution. A detailed analysis of the general solution to the functional equation (1.2) and a study of its associated Ulam-type stability approximations were also presented in [9]. The reciprocal functional equation was first studied in a single-variable setting and later extended to two variables in [14]. In that setting, Ravi et al. [14] considered the mapping satisfying the reciprocal-type relation

$$T_r(\ell + \jmath)(T_r(\ell) + T_r(\jmath)) = T_r(\ell)T_r(\jmath), \tag{1.3}$$

for all admissible $\ell = (\ell_1, \ell_2)$ and $\jmath = (\jmath_1, \jmath_2) \in \mathbb{R}^\star \times \mathbb{R}^\star$ for which the denominator is nonzero. It was proved that all solutions are of the form

$$T_r(x, y) = \frac{ab}{bx + ay},$$

where a and b are nonzero real constants. Furthermore, the Hyers-Ulam stability of the functional equation (1.3) was established in [14]. Several noteworthy rational functional equations have been examined in recently published works; for further details, see [3, 7, 10, 13, 16]. Inspired by the interesting solutions and significant stability results of the functional equations (1.1) and (1.3), we further extend the functional equation (1.3) to a reciprocal functional equation of three variables as

$$T_r(\ell_1 + \jmath_1, \ell_2 + \jmath_2, \ell_3 + \jmath_3)(T_r(\ell_1, \ell_2, \ell_3) + T_r(\jmath_1, \jmath_2, \jmath_3)) = T_r(\ell_1, \ell_2, \ell_3)T_r(\jmath_1, \jmath_2, \jmath_3) \tag{1.4}$$

such that $\ell_i + \jmath_i \neq 0$ for $i = 1, 2, 3$. In addition, we assume that $T_r(\ell) \neq 0$, $T_r(\jmath) \neq 0$, and that their sum is also nonzero. It implies that

$$T_r(\ell_1, \ell_2, \ell_3) = \frac{\alpha\beta\gamma}{\beta\gamma\ell_1 + \alpha\gamma\ell_2 + \alpha\beta\ell_3},$$

where α, β, γ are nonzero constants and T_r satisfies the functional equation (1.4). We obtain the general solution of the functional equation (1.4) and study its stability properties within the framework of Ulam stability theory using a direct method.

A function $T_r : (\mathbb{R}^\star)^3 \to \mathbb{R}$ that satisfies condition (1.4) is called a reciprocal function of three variables.

Remark 1.1. Define a function

$$T_r'(\ell) = \frac{1}{T_r(\ell)}, \quad \text{for all } \ell = (\ell_1, \ell_2, \ell_3) \in (\mathbb{R}^\star)^3.$$

Substituting this representation into the functional equation (1.4), we have

$$\frac{1}{T_r'(\ell + \jmath)}\left(\frac{1}{T_r'(\ell)} + \frac{1}{T_r'(\jmath)}\right) = \frac{1}{T_r'(\ell)} \cdot \frac{1}{T_r'(\jmath)}.$$

Multiplying both sides by $T_r'(\ell)T_r'(\jmath)T_r'(\ell + \jmath)$, we have $T_r'(\ell + \jmath) = T_r'(\ell) + T_r'(\jmath)$. Therefore, the transformed equation reduces to a three-variable additive (Cauchy) functional equation. It is well known that an additive function is linear provided it satisfies a mild regularity condition, such as continuity, measurability, boundedness on a set of positive measure, or monotonicity in each variable. Under any such assumption, its solution can be written as $T_r'(\ell) = a\ell_1 + b\ell_2 + c\ell_3$ for some real constants a, b, c.

In the following theorem, we obtain another relationship between a reciprocal function of one variable and a reciprocal function of three variables.

2. Motivation for Studying the Functional Equation (1.4)

From Remark 1.1, we directly obtain that $T_r(\ell_1, \ell_2, \ell_3) = \frac{1}{a\ell_1 + b\ell_2 + c\ell_3}$ is a solution of the functional equation (1.4). If ℓ_1, ℓ_2, and ℓ_3 represent the efficiencies (work done per unit time) of three workers, then the total time required for them to complete a job together is the inverse of their combined efficiency. If the total work is considered 1 unit, the time taken is given by

$$\text{Time} = \frac{1}{\ell_1 + \ell_2 + \ell_3}.$$

Suppose the above expression is considered as a function of three variables. Then, we can write the total time is a function of the reciprocal of the sum of three variables. That is, $T_r(\ell_1, \ell_2, \ell_3) = \frac{1}{\ell_1 + \ell_2 + \ell_3}$. Hence, this function satisfies the functional equation (1.4). This idea motivated us to introduce a reciprocal functional equation with three variables of the type (1.4). Thus, we can use the functional equation (1.4) to find the

total time required to complete a particular job by three workers with different time durations.

3. Relationship Between the Reciprocal Functions of Three Variables and One Variable

In this section, we now examine the connections between the reciprocal functional equation in three variables and its classical one-variable counterpart. The following lemma follows directly from Theorem 2.1 in [14].

Lemma 3.1. *Let $T_r : (\mathbb{R}^\star)^3 \to \mathbb{R}$ be a mapping satisfying the functional equation* (1.4). *We associate to T_r a single-variable function $\mu : \mathbb{R}^\star \to \mathbb{R}$ defined by*

$$\mu(\ell) = T_r(\ell, \ell, \ell), \tag{3.1}$$

for all $\ell \in \mathbb{R}^\star$. Then, μ satisfies the functional equation (1.1).

Theorem 3.1. *Let α, β and γ be nonzero real constants, and let $\mu : \mathbb{R}^\star \to \mathbb{R}$ be a function that satisfies the functional equation* (1.1). *For $\ell = (\ell_1, \ell_2, \ell_3) \in (\mathbb{R}^\star)^3$. Define the function $T_r : (\mathbb{R}^\star)^3 \to \mathbb{R}$ by*

$$T_r(\ell) = \frac{\alpha\beta\gamma \;\; \mu(\ell_1)\mu(\ell_2)\mu(\ell_3)}{\beta\gamma\mu(\ell_2)\mu(\ell_3) + \alpha\gamma\mu(\ell_1)\mu(\ell_3) + \alpha\beta\mu(\ell_1)\mu(\ell_2)}. \tag{3.2}$$

Then, the function T_r satisfies the three-variable reciprocal functional equation (1.4).

Proof. Since each of $\mu(\ell_1), \mu(\ell_2), \mu(\ell_3)$ is nonzero and $\alpha\beta\gamma \neq 0$, the numerator of (3.2) is nonzero; hence $T_r(\ell) \neq 0$ wherever it is defined, and we may pass to its reciprocal. Dividing the denominator of (3.2) by its numerator gives

$$\frac{1}{T_r(\ell)} = \frac{\beta\gamma\,\mu(\ell_2)\mu(\ell_3) + \alpha\gamma\,\mu(\ell_1)\mu(\ell_3) + \alpha\beta\,\mu(\ell_1)\mu(\ell_2)}{\alpha\beta\gamma\,\mu(\ell_1)\mu(\ell_2)\mu(\ell_3)} = \frac{1}{\alpha\,\mu(\ell_1)} + \frac{1}{\beta\,\mu(\ell_2)} + \frac{1}{\gamma\,\mu(\ell_3)}. \tag{3.3}$$

On the other hand, since μ satisfies (1.1), for all $\ell, \jmath \in \mathbb{R}^\star$ with $\ell + \jmath \neq 0$ and $\mu(\ell) + \mu(\jmath) \neq 0$ we may divide (1.1) by $\mu(\ell + \jmath)\,\mu(\ell)\,\mu(\jmath)$ to obtain

$$\frac{1}{\mu(\ell + \jmath)} = \frac{1}{\mu(\ell)} + \frac{1}{\mu(\jmath)}, \tag{3.4}$$

that is, $1/\mu$ is additive on $\mathbb{R}^\star$.

Now let $\ell = (\ell_1, \ell_2, \ell_3)$ and $\jmath = (\jmath_1, \jmath_2, \jmath_3)$ be admissible triples, so that $\ell_i + \jmath_i \neq 0$ and $\mu(\ell_i) + \mu(\jmath_i) \neq 0$ for $i = 1, 2, 3$. Applying (3.3) at $\ell + \jmath$ and then (3.4) in each coordinate, we obtain

$$\begin{aligned}
\frac{1}{T_r(\ell + \jmath)} &= \frac{1}{\alpha\,\mu(\ell_1 + \jmath_1)} + \frac{1}{\beta\,\mu(\ell_2 + \jmath_2)} + \frac{1}{\gamma\,\mu(\ell_3 + \jmath_3)} \\
&= \left(\frac{1}{\alpha\,\mu(\ell_1)} + \frac{1}{\alpha\,\mu(\jmath_1)}\right) + \left(\frac{1}{\beta\,\mu(\ell_2)} + \frac{1}{\beta\,\mu(\jmath_2)}\right) + \left(\frac{1}{\gamma\,\mu(\ell_3)} + \frac{1}{\gamma\,\mu(\jmath_3)}\right) \\
&= \frac{1}{T_r(\ell)} + \frac{1}{T_r(\jmath)},
\end{aligned}$$

where the last equality uses (3.3) at ℓ and at $\jmath$. Recalling that $\ell + \jmath = (\ell_1 + \jmath_1, \ell_2 + \jmath_2, \ell_3 + \jmath_3)$ and multiplying both sides by $T_r(\ell + \jmath)\, T_r(\ell)\, T_r(\jmath)$, we obtain

$$T_r(\ell_1 + \jmath_1, \ell_2 + \jmath_2, \ell_3 + \jmath_3)\,(T_r(\ell_1, \ell_2, \ell_3) + T_r(\jmath_1, \jmath_2, \jmath_3)) = T_r(\ell_1, \ell_2, \ell_3)\, T_r(\jmath_1, \jmath_2, \jmath_3),$$

which is precisely (1.4). Therefore, the mapping defined by (3.2) satisfies the three-variable reciprocal functional equation (1.4), which completes the proof. □

4. Hyers-Ulam Stability of the Three-variable Reciprocal Functional Equation

In the following analysis, we examine the generalized Hyers-Ulam stability related to the functional equation (1.4). Our approach is based on a direct method to establish the conditions that ensure the existence of, and thereby rigorously ensure, this form of stability.

Theorem 4.1. *Let $\Phi : (\mathbb{R}^\star)^3 \times (\mathbb{R}^\star)^3 \to [0, +\infty)$ be a control function satisfying*

$$\lim_{n\to+\infty} \frac{1}{2^n} \Phi\left(\Psi^{(n)}\right) = 0, \tag{4.1}$$

where $\Psi^{(n)} = \left(\frac{\ell_1}{2^n}, \frac{\ell_1}{2^n}, \frac{\ell_2}{2^n}, \frac{\ell_2}{2^n}, \frac{\ell_3}{2^n}, \frac{\ell_3}{2^n}\right)$ for all $\ell_1, \ell_2, \ell_3 \in \mathbb{R}^\star$ and further suppose that $\sum_{n=0}^{+\infty} \frac{1}{2^{n+1}} \Phi\left(\Psi^{(n+1)}\right)$ converges. Let $F_r : (\mathbb{R}^\star)^3 \to \mathbb{R}$ be a function such that

$$|F_r(\ell + \jmath)\,(F_r(\ell) + F_r(\jmath)) - F_r(\ell)F_r(\jmath)| \le \frac{1}{2}\Phi(\ell, \jmath)\,|F_r(\ell) + F_r(\jmath)|\,, \tag{4.2}$$

for all admissible triples $\ell = (\ell_1, \ell_2, \ell_3)$, $\jmath = (\jmath_1, \jmath_2, \jmath_3) \in (\mathbb{R}^\star)^3$. Then, there exists $G_r : (\mathbb{R}^\star)^3 \to \mathbb{R}$ satisfying (1.4). Moreover, this mapping is uniquely determined by

$$G_r(\ell) = \lim_{n\to+\infty} \frac{1}{2^n} F_r\left(\frac{\ell_1}{2^n}, \frac{\ell_2}{2^n}, \frac{\ell_3}{2^n}\right)$$

and

$$|F_r(\ell) - G_r(\ell)| \le \sum_{i=0}^{+\infty} \frac{1}{2^{(i+1)}} \Phi\left(\Psi^{(i+1)}\right), \tag{4.3}$$

for all $\ell = (\ell_1, \ell_2, \ell_3) \in (\mathbb{R}^\star)^3$.

Proof. From (4.2), define $\ell = \left(\frac{\ell_1}{2}, \frac{\ell_2}{2}, \frac{\ell_3}{2}\right)$ and $\jmath = \left(\frac{\jmath_1}{2}, \frac{\jmath_2}{2}, \frac{\jmath_3}{2}\right)$. Since $\ell + \jmath = (\ell_1 + \jmath_1, \ell_2 + \jmath_2, \ell_3 + \jmath_3)$, (4.2) gives

$$\left|F_r(\ell) - \frac{1}{2} F_r\left(\frac{\ell}{2}\right)\right| \le \frac{1}{2}\Phi\left(\Psi^{(1)}\right), \tag{4.4}$$

for all $\ell = (\ell_1, \ell_2, \ell_3) \in (\mathbb{R}^\star)^3$. Now, replacing ℓ by $(\frac{\ell_1}{2}, \frac{\ell_2}{2}, \frac{\ell_3}{2})$ in (4.4), dividing by 2 and adding the result to (4.4), we have

$$\left|F_r(\ell) - \frac{1}{2^2} F_r\left(\frac{\ell}{2^2}\right)\right| \le \sum_{i=0}^{n-1} \frac{1}{2^{i+1}} \Phi\left(\Psi^{(i+1)}\right),$$

for all $\ell = (\ell_1, \ell_2, \ell_3) \in (\mathbb{R}^\star)^3$. By iterating the above inequality, we obtain, for each integer $n > 1$,

$$\left| F_r(\ell) - \frac{1}{2^n} F_r\left(\frac{\ell}{2^n}\right) \right| \leq \sum_{i=0}^{n-1} \frac{1}{2^{i+1}} \Phi\left(\Psi^{(i+1)}\right).$$

Therefore,

$$\left| F_r(\ell) - \frac{1}{2^n} F_r\left(\frac{\ell}{2^n}\right) \right| \leq \sum_{i=0}^{+\infty} \frac{1}{2^{i+1}} \Phi\left(\Psi^{(i+1)}\right). \tag{4.5}$$

Define

$$\mathrm{T}_n(\ell) = \frac{1}{2^n} F_r\left(\frac{\ell_1}{2^n}, \frac{\ell_2}{2^n}, \frac{\ell_3}{2^n}\right).$$

Using (4.5) with a scaled argument and multiplying appropriately by 2^{-p}, we have

$$|\mathrm{T}_{n+p}(\ell) - \mathrm{T}_p(\ell)| \leq \sum_{i=p}^{n+p-1} \frac{1}{2^{i+1}} \Phi\left(\Psi^{(i+1)}\right).$$

Since the series converges, the sequence $\{\mathrm{T}_n(\ell)\}$ is Cauchy. Therefore, the limit

$$G_r(\ell) = \lim_{n \to +\infty} \mathrm{T}_n(\ell)$$

exists for all $\ell \in (\mathbb{R}^\star)^3$. Now, by taking the limit as $n \to +\infty$ in the inequality above, we obtain (4.5) as the limiting relation. Next, we show that G_r satisfies (1.4). Replacing $(\ell, \jmath)$ in (4.2) by $\left(\frac{\ell}{2^n}, \frac{\jmath}{2^n}\right)$ and multiplying by 2^n, the right-hand side converges to 0 as $n \to +\infty$. Hence, taking the limit on the left-hand side and using the definition of G_r, we obtain $G_r(\ell + \jmath)\,(G_r(\ell) + G_r(\jmath)) = G_r(\ell) G_r(\jmath)$. Therefore, G_r satisfies (1.4). We now establish the uniqueness of this solution. Suppose there exists a different reciprocal mapping $S_r : (\mathbb{R}^\star)^3 \to \mathbb{R}$ that satisfies (1.4) and also satisfies (4.3). Then, both G_r and S_r solve the same reciprocal functional equation, and applying relation (4.3) we obtain

$$\left| S_r(\ell) - G_r(\ell) \right| \leq \frac{1}{2^n} \left(\left| S_r\left(\frac{\ell}{2^n}\right) - F_r\left(\frac{\ell}{2^n}\right) \right| + \left| F_r\left(\frac{\ell}{2^n}\right) - G_r\left(\frac{\ell}{2^n}\right) \right| \right),$$

which simplifies to

$$|S_r(\ell) - G_r(\ell)| \leq 2 \sum_{i=0}^{+\infty} \frac{1}{2^{n+i+1}} \Phi\left(\Psi^{(n+i+1)}\right), \tag{4.6}$$

for every choice of $\ell = (\ell_1, \ell_2, \ell_3) \in (\mathbb{R}^\star)^3$. Letting $n \to +\infty$ in (4.6) and using (4.1), we have $\sum_{i=0}^{+\infty} \frac{1}{2^{n+i+1}} \Phi\left(\Psi^{(n+i+1)}\right) \to 0$, so $S_r = G_r$. Therefore, G_r is uniquely determined, which completes the proof of the theorem. □

In the corollaries below, we present several stability results using different types of upper bounds, corresponding respectively to Hyers (a fixed positive constant), T. Rassias, and J. Rassias. Throughout, $F_r : (\mathbb{R}^\star)^3 \to \mathbb{R}$ denotes a fixed function.

Corollary 4.1. *For a positive constant ε, let F_r satisfy the following condition*

$$\left|F_r(\ell+\jmath)-\frac{F_r(\ell)F_r(\jmath)}{F_r(\ell)+F_r(\jmath)}\right|\leq\varepsilon,$$

for all $\ell=(\ell_1,\ell_2,\ell_3)$, $\jmath=(\jmath_1,\jmath_2,\jmath_3)\in(\mathbb{R}^\star)^3$. Consequently, there exists a unique reciprocal function $G_r:(\mathbb{R}^\star)^3\to\mathbb{R}$ which satisfies the functional equation (1.4) *with the condition that $|F_r(\ell)-G_r(\ell)|\leq\varepsilon$, for all $\ell=(\ell_1,\ell_2,\ell_3)\in(\mathbb{R}^\star)^3$.*

Proof. The proof follows by letting $\Phi(\ell_1,\jmath_1,\ell_2,\jmath_2,\ell_3,\jmath_3)=\varepsilon$ in Theorem 4.1. □

Corollary 4.2. *Let υ_1 be a fixed positive constant and $p>-1$. Suppose that F_r satisfies*

$$\left|F_r(\ell+\jmath)-\frac{F_r(\ell)F_r(\jmath)}{F_r(\ell)+F_r(\jmath)}\right|\leq\frac{\upsilon_1}{2}\left(|\ell_1|^p+|\jmath_1|^p+|\ell_2|^p+|\jmath_2|^p+|\ell_3|^p+|\jmath_3|^p\right),$$

for all $\ell_1,\jmath_1,\ell_2,\jmath_2,\ell_3,\jmath_3\in\mathbb{R}^\star$. Then, there exists a unique reciprocal function of three variables $G_r:(\mathbb{R}^\star)^3\to\mathbb{R}$ which satisfies the functional equation (1.4) *and, in addition,*

$$|F_r(\ell)-G_r(\ell)|\leq\frac{2\upsilon_1}{2^{p+1}-1}\left(|\ell_1|^p+|\ell_2|^p+|\ell_3|^p\right)$$

for all $\ell=(\ell_1,\ell_2,\ell_3)\in(\mathbb{R}^\star)^3$.

Proof. Consider the control function

$$\Phi(\ell_1,\jmath_1,\ell_2,\jmath_2,\ell_3,\jmath_3)=\upsilon_1\left(|\ell_1|^p+|\jmath_1|^p+|\ell_2|^p+|\jmath_2|^p+|\ell_3|^p+|\jmath_3|^p\right).$$

For each pair in which the arguments are the same, this reduces to

$$\Phi\left(\ell_1,\ell_1,\ell_2,\ell_2,\ell_3,\ell_3\right)=2\upsilon_1\left(|\ell_1|^p+|\ell_2|^p+|\ell_3|^p\right).$$

From (4.3), we get

$$|F_r(\ell)-G_r(\ell)|\leq\sum_{i=0}^{+\infty}\frac{2\upsilon_1}{2^{i+1}}\left(\left|\frac{\ell_1}{2^{i+1}}\right|^p+\left|\frac{\ell_2}{2^{i+1}}\right|^p+\left|\frac{\ell_3}{2^{i+1}}\right|^p\right),$$

which simplifies to

$$|F_r(\ell)-G_r(\ell)|\leq\left(|\ell_1|^p+|\ell_2|^p+|\ell_3|^p\right)\frac{2\upsilon_1}{2^{p+1}}\sum_{i=0}^{+\infty}2^{-(p+1)i}.$$

Since the series $\sum_{i=0}^{+\infty}2^{-(p+1)i}=\frac{1}{1-2^{-(p+1)}}$, we obtain the bound

$$|F_r(\ell_1,\ell_2,\ell_3)-G_r(\ell_1,\ell_2,\ell_3)|\leq\frac{2\upsilon_1}{2^{p+1}-1}\left(|\ell_1|^p+|\ell_2|^p+|\ell_3|^p\right),$$

for all $\ell=(\ell_1,\ell_2,\ell_3)\in(\mathbb{R}^\star)^3$. □

Corollary 4.3. *Let $a,b\in\mathbb{R}$ and let $\rho=a+b$. Assume that $\rho>-1$. Also, assume that there exists a constant $\upsilon_2>0$ such that*

$$\left|F_r(\ell+\jmath)-\frac{F_r(\ell)F_r(\jmath)}{F_r(\ell)+F_r(\jmath)}\right|\leq\frac{\upsilon_2}{2}|\ell_1|^{\frac{a}{3}}|\jmath_1|^{\frac{b}{3}}|\ell_2|^{\frac{a}{3}}|\jmath_2|^{\frac{b}{3}}|\ell_3|^{\frac{a}{3}}|\jmath_3|^{\frac{b}{3}},\tag{4.7}$$

for all $\ell_1, \jmath_1, \ell_2, \jmath_2, \ell_3, \jmath_3 \in \mathbb{R}^\star$. *Then there exists a unique reciprocal function with three variables* $G_r : \mathbb{R}^\star \times \mathbb{R}^\star \times \mathbb{R}^\star \to \mathbb{R}$ *satisfying the functional equation* (1.4) *under the condition that*

$$|F_r(\ell) - G_r(\ell)| \leq \frac{\upsilon_2}{2^{\rho+1}-1}\left(|\ell_1|^{\frac{\rho}{3}}|\ell_2|^{\frac{\rho}{3}}|\ell_3|^{\frac{\rho}{3}}\right),$$

for all $\ell = (\ell_1, \ell_2, \ell_3) \in (\mathbb{R}^\star)^3$.

Proof. Considering $\Phi(\ell_1, \jmath_1, \ell_2, \jmath_2, \ell_3, \jmath_3) = \upsilon_2|\ell_1|^{\frac{a}{3}}|\jmath_1|^{\frac{b}{3}}|\ell_2|^{\frac{a}{3}}|\jmath_2|^{\frac{b}{3}}|\ell_3|^{\frac{a}{3}}|\jmath_3|^{\frac{b}{3}}$, in Theorem 4.1, we have $\Phi(\ell_1, \ell_1, \ell_2, \ell_2, \ell_3, \ell_3) = \upsilon_2|\ell_1|^{\frac{\rho}{3}}|\ell_2|^{\frac{\rho}{3}}|\ell_3|^{\frac{\rho}{3}}$. From (4.3), we obtain

$$\begin{aligned}
|F_r(\ell) - G_r(\ell)| &\leq \sum_{i=0}^{+\infty} \frac{1}{2^{i+1}}\upsilon_2 \left|\frac{\ell_1}{2^{i+1}}\right|^{\frac{\rho}{3}} \left|\frac{\ell_2}{2^{i+1}}\right|^{\frac{\rho}{3}} \left|\frac{\ell_3}{2^{i+1}}\right|^{\frac{\rho}{3}} \\
&\leq \frac{\upsilon_2}{2^{\rho+1}} \sum_{i=0}^{+\infty} \left(\frac{1}{2^{(\rho+1)i}}\right) |\ell_1|^{\frac{\rho}{3}}|\ell_2|^{\frac{\rho}{3}}|\ell_3|^{\frac{\rho}{3}} \\
&\leq \frac{\upsilon_2}{2^{\rho+1}-1}|\ell_1|^{\frac{\rho}{3}}|\ell_2|^{\frac{\rho}{3}}|\ell_3|^{\frac{\rho}{3}},
\end{aligned}$$

for all $\ell = (\ell_1, \ell_2, \ell_3) \in (\mathbb{R}^\star)^3$. □

Corollary 4.4. *Let* $\upsilon_3 > 0$ *and* $\alpha > -\frac{1}{3}$ *be real numbers, and let the mapping* $F_r : (\mathbb{R}^\star)^3 \to \mathbb{R}$ *satisfy*

$$\begin{aligned}
\left|F_r(\ell + \jmath) - \frac{F_r(\ell)F_r(\jmath)}{F_r(\ell) + F_r(\jmath)}\right| \leq & \frac{\upsilon_3}{2}\Big(|\ell_1|^{\frac{\alpha}{2}}|\jmath_1|^{\frac{\alpha}{2}}|\ell_2|^{\frac{\alpha}{2}}|\jmath_2|^{\frac{\alpha}{2}}|\ell_3|^{\frac{\alpha}{2}}|\jmath_3|^{\frac{\alpha}{2}} \\
& + \frac{1}{2}\left(|\ell_1|^{3\alpha} + |\jmath_1|^{3\alpha} + |\ell_2|^{3\alpha} + |\jmath_2|^{3\alpha} + |\ell_3|^{3\alpha} + |\jmath_3|^{3\alpha}\right)\Big),
\end{aligned} \tag{4.8}$$

for all $\ell_1, \jmath_1, \ell_2, \jmath_2, \ell_3, \jmath_3 \in \mathbb{R}^\star$. *Then, there exists a unique reciprocal function* $G_r : (\mathbb{R}^\star)^3 \to \mathbb{R}$ *which satisfies the functional equation* (1.4) *with the condition that*

$$|F_r(\ell) - G_r(\ell)| \leq \frac{\upsilon_3}{2^{3\alpha+1}-1}\left(|\ell_1|^\alpha|\ell_2|^\alpha|\ell_3|^\alpha + |\ell_1|^{3\alpha} + |\ell_2|^{3\alpha} + |\ell_3|^{3\alpha}\right), \tag{4.9}$$

for all $\ell = (\ell_1, \ell_2, \ell_3) \in (\mathbb{R}^\star)^3$.

Proof. Choosing

$$\begin{aligned}
\Phi(\ell_1, \jmath_1, \ell_2, \jmath_2, \ell_3, \jmath_3) = \upsilon_3 \Big(&|\ell_1|^{\frac{\alpha}{2}}|\jmath_1|^{\frac{\alpha}{2}}|\ell_2|^{\frac{\alpha}{2}}|\jmath_2|^{\frac{\alpha}{2}}|\ell_3|^{\frac{\alpha}{2}}|\jmath_3|^{\frac{\alpha}{2}} \\
&+ \frac{1}{2}\left(|\ell_1|^{3\alpha} + |\jmath_1|^{3\alpha} + |\ell_2|^{3\alpha} + |\jmath_2|^{3\alpha} + |\ell_3|^{3\alpha} + |\jmath_3|^{3\alpha}\right)\Big),
\end{aligned}$$

in Theorem 4.1, we have

$$\Phi(\ell_1, \ell_1, \ell_2, \ell_2, \ell_3, \ell_3) = \upsilon_3\left(|\ell_1|^\alpha|\ell_2|^\alpha|\ell_3|^\alpha + |\ell_1|^{3\alpha} + |\ell_2|^{3\alpha} + |\ell_3|^{3\alpha}\right).$$

From (4.3), we get

$$\left|F_r(\ell) - G_r(\ell)\right| \leq \sum_{i=0}^{+\infty} \frac{\upsilon_3}{2^{i+1}} \left(\left|\frac{\ell_1}{2^{i+1}}\right|^\alpha \left|\frac{\ell_2}{2^{i+1}}\right|^\alpha \left|\frac{\ell_3}{2^{i+1}}\right|^\alpha + \sum_{j=1}^{3} \left|\frac{\ell_j}{2^{i+1}}\right|^{3\alpha}\right),$$

which gives

$$\left|F_r(\ell) - G_r(\ell)\right| \leq \upsilon_3 \left(|\ell_1|^\alpha |\ell_2|^\alpha |\ell_3|^\alpha + |\ell_1|^{3\alpha} + |\ell_2|^{3\alpha} + |\ell_3|^{3\alpha}\right) \sum_{i=0}^{+\infty} 2^{-(3\alpha+1)(i+1)}.$$

Consequently,

$$\left|F_r(\ell) - G_r(\ell)\right| \leq \frac{\upsilon_3}{2^{3\alpha+1}-1}\left(|\ell_1|^\alpha |\ell_2|^\alpha |\ell_3|^\alpha + \sum_{j=1}^{3} |\ell_j|^{3\alpha}\right),$$

for all $\ell = (\ell_1, \ell_2, \ell_3) \in (\mathbb{R}^\star)^3$. □

5. Concluding Remarks

In this work, we extend the previously investigated two-variable reciprocal functional equation given in (1.3) to a more general framework involving three variables, thereby obtaining and analyzing the corresponding three-variable reciprocal functional equation presented in (1.4). We found a relationship between the functional equations (1.1) and (1.4). It is worth noting that the function $T_r(\ell_1, \ell_2, \ell_3) = \frac{\alpha\beta\gamma}{\beta\gamma\ell_1 + \alpha\gamma\ell_2 + \alpha\beta\ell_3}$; for nonzero $\alpha, \beta, \gamma \in \mathbb{R}$ is a solution of the functional equation (1.4). Within the classical setting of stability theory, our investigation focuses on the approximation of three-variable reciprocal-type functions that are constrained by the functional equation (1.4). In this context, we systematically analyze their behavior under several well-established notions of functional stability, namely, Ulam-Hyers stability, Hyers-Ulam-Rassias stability, Ulam-Găvruta-Rassias stability, as well as the mixed product-sum stability introduced by J. Rassias. We aim to explain how these stability frameworks relate to one another when applied to such three-variable reciprocal maps.

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[1]Department of Mathematics,
Gauhati University,
Guwahati-781014, India
Email address: hemen_dutta08@rediffmail.com
ORCID iD: https://orcid.org/0000-0003-2765-2386

[2]Department of Mathematics,
Government Arts and Science College,
Tirupattur-635 901, Tamil Nadu, India
Email address: drpnarasimman@gmail.com
ORCID iD: https://orcid.org/0000-0002-8541-7838