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ON (m, h_1, h_2)-G-CONVEX DOMINATED STOCHASTIC PROCESSES

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Abstract. In this paper is introduced the concept of (m, h_1, h_2)-convexity for stochastic processes dominated by other stochastic processes with the same property, some mean square integral Hermite-Hadamard type inequalities for this kind of generalized convexity are established and from the founded results, other mean square integral inequalities for the classical convex, s-convex in the first and second sense, P-convex and MT-convex stochastic processes are deduced.

1. Introduction

In 1974, B. Nagy applied a characterization of measurable stochastic processes to solve a generalization of the (additive) Cauchy functional equation [15]. Later, in 1980 K. Nikodem [17] considered convex stochastic processes, and in 1995 A. Skowronski [27] obtained some further results on Wright convex stochastic processes, which generalize some known properties of convex stochastic processes. For a detailed study about this topic the following references are helpful [2, 3, 13, 24, 25].

Convexity is one of the hypotheses often used in optimization theory. It is generally used to give global validity for certain propositions, which otherwise would only be true locally. A function $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, is said to be a convex function on I if the inequality

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. If the reversed inequality in (1.1) holds, then f is concave.

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The convexity of functions and their generalized forms play an important role in many fields such as Economic Science, Biology, Optimization and other [6, 21].

About the concept of convexity, its evolution has had a great impact in the community of investigators. In recent years, for example, generalized concepts such as s-convexity, h-convexity, MT-convexity, log-convexity, P-convexity, η-convexity, quasi convexity and others, as well as combinations of these new concepts have been introduced. The following references give more information about the research in this area [1, 5, 11, 14, 16, 18, 22, 29].

Similarly, some recent studies have been introduced the following concepts: J-convex [26], Wright-convex [27], strongly convex [9], strongly Wright [10], p-convex [20], harmonically convex [19], s-convex in the first and second sense [12,23] stochastic process.

The well-known Hermite-Hadamard inequality establish that for every convex function $f : I \subset \mathbb{R} \to \mathbb{R}$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.2}$$

holds for every $a, b \in I$, with $a < b$.

In 2012, D. Kotrys presented the Hermite-Hadamard inequality for convex stochastic processes [8].

Theorem 1.1. *If $X : I \times \Omega \to \mathbb{R}$ is Jensen-convex and mean square continuous in the interval $T \times \Omega$, then for any $u, v \in T$, we have*

$$X\left(\frac{u+v}{2}, \cdot\right) \leq \frac{1}{u-v}\int_u^v X(t, \cdot)dt \leq \frac{X(u, \cdot)+X(v, \cdot)}{2} \tag{1.3}$$

almost everywhere for all $u, v \in I$.

Many researchers have developed works where they relate the concepts of generalized convexity and stochastic processes using the inequality (1.3). For example, E. Set et al. in [23] investigated Hermite-Hadamard type inequalities for stochastic processes in the second sense, and M. J. Vivas-Cortez and J. E. Hernández Hernández in [30] studied about (h_1, h_2, m)-GA-convexity for stochastic processes.

Following this line of research, this paper introduces the concept of (m, h_1, h_2)-convexity for stochastic processes dominated by other stochastic processes with the same property, some mean square integral Hermite-Hadamard type inequalities for this kind of generalized convexity are established, and from the founded results, other integral inequalities for stochastic processes with other types of convexity are deduced.

2. Preliminaries

The following references [8,13,27,28] contain the basic notions of stochastic processes used in this work.

Let $(\Omega, \mathcal{A}, \mu)$ be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is $\mathcal{A}$-measurable and $P\{w \in \Omega : X(w) \notin \mathbb{R}\} = 0$. Let $I \subset \mathbb{R}$

be time. A function $X : I \times \Omega \to \mathbb{R}$ is called a stochastic process if for all $t \in I$ the function $X(t, \cdot) : \Omega \to \mathbb{R}$ is a random variable.

In this work I is an interval and $X(t, \cdot)$ is called a stochastic process with continuous time.

It is said that the stochastic process $X : I \times \Omega \to \mathbb{R}$ is called

(a) continuous in probability on the interval I if for all $t_0 \in I$ it follows that

$$\mu - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot),$$

where $P - \lim$ denotes the limit in probability;

(b) mean-square continuous in the interval I if for all $t_0 \in I$

$$\mu - \lim_{t \to t_0} \mathbb{E}(X(t, \cdot) - X(t_0, \cdot)) = 0,$$

where $\mathbb{E}(X(t, \cdot))$ denote the expectation value of the random variable $X(t, \cdot)$;

(c) increasing (decreasing) if for all $u, v \in I$ such that $t < s$,

$$X(u, \cdot) \leq X(v, \cdot), \quad (X(u, \cdot) \geq X(v, \cdot)) \quad \text{(a.e.)};$$

(d) monotonic if it is increasing or decreasing;

(e) differentiable at a point $t \in I$ if there exists a random variable $X'(t, \cdot) : I \times \Omega \to \mathbb{R}$ such that

$$X'(t, \cdot) = \mu - \lim_{t \to t_0} \frac{X(t, \cdot) - X(t_0, \cdot)}{t - t_0}.$$

A stochastic process $X : I \times \Omega \to \mathbb{R}$ is continuous (differentiable) if it is continuous (differentiable) at every point of the interval I.

Definition 2.1. Let (Ω, A, P) be a probability space, $I \subset \mathbb{R}$ be an interval with $E(X(t, \cdot)^2) < \infty$ for all $t \in I$. Let $[a, b] \subset I$, $a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a, b]$ and $\theta_k \in [t_{k-1}, t_k]$ for $k = 1, 2, \ldots, n$. A random variable $Y : \Omega \to \mathbb{R}$ is called mean-square integral of the process $X(t, \cdot)$ on $[a, b]$ if the following identity holds

$$\lim_{n \to \infty} E \left[\sum_{k=0}^{n} X(\theta_k, \cdot)(t_k - t_{k-1}) - Y \right]^2 = 0,$$

then it can be written

$$\int_a^b X(t, \cdot) dt = Y(\cdot) \quad \text{(a.e.)}.$$

Also, mean square integral operator is increasing, that is,

$$\int_a^b X(t, \cdot) dt \leq \int_a^b Z(t, \cdot) dt \quad \text{(a.e.)},$$

where $X(t, \cdot) \leq Z(t, \cdot)$ in $[a, b]$ ([26]).

In throughout paper, we will consider the stochastic processes that is with continuous time and mean-square continuous.

In 1980, K. Nickoden introduced an important definition in which the property of convexity for stochastic processes is established [17].

Definition 2.2. Set $(\Omega, \mathcal{A}, P)$ to be a probability space and $I \subset \mathbb{R}$ be an interval. It is said that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is convex if the following inequality holds almost everywhere

$$X(\lambda u + (1-\lambda)v, \cdot) \leq \lambda X(u, \cdot) + (1-\lambda)X(v, \cdot), \tag{2.1}$$

for all $u, v \in I$ and $\lambda \in [0,1]$.

In the work of J. E. Hernández Hernández and J. F. Gómez [7] the following definition is introduced.

Definition 2.3. Let $h_1, h_2 : [0,1] \to \mathbb{R}$ be two non negative functions, $m \in (0,1]$ and $I \subset \mathbb{R}$ an interval. A stochastic process $X : I \times \Omega \to \mathbb{R}$ is (m, h_1, h_2)-convex if the following inequality holds almost everywhere

$$X\left(ta + m(1-t)b, \cdot\right) \leq h_1(t)X(a, \cdot) + mh_2(t)X\left(b, \cdot\right), \tag{2.2}$$

for all $a, b \in I$ and $t \in [0,1]$.

Some other kinds of generalized convexity for stochastic process, as s-convexity in the second sense and P-convexity are presented in the same work.

With the notion of dominated convexity introduced by S. S. Dragomir et al. in [4], the following definitions for stochastic processes are introduced.

Definition 2.4. Let $I \subset \mathbb{R}$ be an interval and $G : I \times \Omega \to \mathbb{R}$ be a non negative convex stochastic process. A stochastic process $X : I \times \Omega \to \mathbb{R}$ is called a convex dominated by G if the following inequality holds almost everywhere

$$\begin{aligned} &\left|tX(a, \cdot) + (1-t)X\left(b, \cdot\right) - X\left(ta + (1-t)b, \cdot\right)\right| \\ \leq &t(t)G(a, \cdot) + (1-t)G\left(b, \cdot\right) - G\left(ta + (1-t)b, \cdot\right), \end{aligned} \tag{2.3}$$

for all $a, b \in I$ and $t \in [0,1]$.

Definition 2.5. Let $h_1, h_2 : [0,1] \to \mathbb{R}$ be two non negative functions, $m \in (0,1]$, $I \subset \mathbb{R}$ an interval and $G : I \times \Omega \to \mathbb{R}$ be a non negative (m, h_1, h_2)-convex stochastic process. A stochastic process $X : I \times \Omega \to \mathbb{R}$ is called a (m, h_1, h_2)-convex dominated by G if the following inequality holds almost everywhere

$$\begin{aligned} &\left|h_1(t)X(a, \cdot) + mh_2(t)X\left(b, \cdot\right) - X\left(ta + m(1-t)b, \cdot\right)\right| \\ \leq &h_1(t)G(a, \cdot) + mh_2(t)G\left(b, \cdot\right) - G\left(ta + m(1-t)b, \cdot\right), \end{aligned} \tag{2.4}$$

for all $a, b \in I$ and $t \in [0,1]$.

Note that if $m = 1$, $h_1(t) = t$ and $h_2(t) = 1-t$ for all $t \in [0,1]$ the Definition 2.4 is obtained, if $m = 1$, $h_1(t) = t^s$ and $h_2(t) = 1 - t^s$ for all $t \in [0,1]$ and some $s \in (0,1]$ we have the definition of s-convex stochastic process in the first sense [12]; if $m = 1$, $h_1(t) = t^s$ and $h_2(t) = (1-t)^s$ for all $t \in [0,1]$ and some $s \in (0,1]$ we have the definition of s-convex stochastic process in the second sense [23]; if $m = 1$, $h_1(t) = h_2(t) = 1$ for all $t \in [0,1]$ then the definition of P-convex stochastic process follows [7] and also, if $m = 1$, $h_1(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$ and $h_2(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$ for all $t \in (0,1)$ the definition of MT-convex stochastic process is obtained.

3. Main Results

Henceforth, $h_1, h_2 : [0,1] \to \mathbb{R}$ are considered non-negative functions and $m \in (0,1]$.

Proposition 3.1. *Let $G : I \times \Omega \to \mathbb{R}$ and $X : I \times \Omega \to \mathbb{R}$ be a non negative (m, h_1, h_2)-convex stochastic processes. The following statements are equivalent:*

i) X is a (m, h_1, h_2)-convex dominated by G;
ii) the stochastic processes $(G - X)$ and $(G + X)$ are (m, h_1, h_2)-convex;
iii) there exist two (m, h_1, h_2)-convex stochastic processes $H, K : I \times \Omega \to \mathbb{R}$ such that $X = \frac{1}{2}(H - K)$ and $G = \frac{1}{2}(H + K)$.

Proof. $i) \Leftrightarrow ii)$ The condition (2.4) is equivalent to

$$\begin{aligned}
&G\left(ta + m(1-t)b, \cdot\right) - h_1(t)G(a, \cdot) - mh_2(t)G\left(b, \cdot\right) \\
\leq &h_1(t)X(a, \cdot) + mh_2(t)X\left(b, \cdot\right) - X\left(ta + m(1-t)b, \cdot\right) \\
\leq &h_1(t)G(a, \cdot) + mh_2(t)G\left(b, \cdot\right) - G\left(ta + m(1-t)b, \cdot\right),
\end{aligned}$$

and, from this double inequality, making a correct rearrange it follows that

$$\left(G + X\right)\left(ta + m(1-t)b, \cdot\right) \leq h_1(t)\left(G + X\right)(a, \cdot) + mh_2(t)\left(G + X\right)(b, \cdot)$$

and

$$\left(G - X\right)\left(ta + m(1-t)b, \cdot\right) \leq h_1(t)\left(G - X\right)(a, \cdot) + mh_2(t)\left(G - X\right)(b, \cdot).$$

$iii) \Rightarrow ii)$ Lets define $X = \frac{1}{2}(H - K)$ and $G = \frac{1}{2}(H + K)$. Adding and subtracting we have $(G + X) = H$ and $(G - X) = K$, so, both are (m, h_1, h_2)-convex stochastic processes.

$ii) \Rightarrow iii)$ By condition $ii)$, $(G + X)$ and $(G - X)$ are (m, h_1, h_2)-convex stochastic processes, so $H = G + K$ and $K = G - X$ are (m, h_1, h_2)-convex stochastic processes. □

Proposition 3.2. *Let $X : I \times \Omega \to \mathbb{R}$ be a (m, h_1, h_2)-convex stochastic process and $A : \Omega \to \mathbb{R}$ a random variable, then the stochastic process defined by $A(\cdot)X(t, \cdot)$ is (m, h_1, h_2)-convex.*

Proof. Using Definition 2.3 we have the desired result. □

Proposition 3.3. *Let $G : I \times \Omega \to \mathbb{R}$ be a (m, h_1, h_2)-convex stochastic process and $X, Y : I \times \Omega \to \mathbb{R}$ two (m, h_1, h_2)-convex stochastic process dominated by G, then we have that $X + Y$ is a (m, h_1, h_2)-convex stochastic process dominated by $2G$. Also, if $A : \Omega \to \mathbb{R}$ is a random variable, then the (m, h_1, h_2)-convex stochastic process defined by $A(\cdot)X(t, \cdot)$ is dominated by $|A(\cdot)|\, G$.*

Proof. With the help of Definition 2.5 and the triangular inequality we have that

$$\begin{aligned}
&|h_1(t)(X+Y)(u,\cdot)+mh_2(t)(X+Y)(v,\cdot)-(X+Y)(tu+(1-t)v,\cdot)|\\
=&|h_1X(u,\cdot)+mh_2(t)X(v,\cdot)-X(tu+(1-t)v,\cdot)\\
&+h_1Y(u,\cdot)+mh_2(t)Y(v,\cdot)-Y(tu+(1-t)v,\cdot)|\\
\le&|h_1X(u,\cdot)+mh_2(t)X(v,\cdot)-X(tu+(1-t)v,\cdot)|\\
&+|h_1Y(u,\cdot)+mh_2(t)Y(v,\cdot)-Y(tu+(1-t)v,\cdot)|\\
\le&2(h_1(t)G(u,\cdot)+mh_2(t)G(v,\cdot)-G(tu+(1-t)v,\cdot))
\end{aligned}$$

and

$$\begin{aligned}
&|h_1(t)A(\cdot)X(u,\cdot)+mh_2(t)A(\cdot)X(v,\cdot)-A(\cdot)X(tu+(1-t)v,\cdot)|\\
\le&|A(\cdot)|\,(h_1(t)G(u,\cdot)+mh_2(t)G(v,\cdot)-G(tu+(1-t)v,\cdot)).
\end{aligned}$$

The proof is complete. □

Remark 3.1. The previous proposition is also valid for the case of subtraction of stochastic processes, and it is easily proved that the algebraic sum of n (m,h_1,h_2)-convex stochastic processes, each one dominated by the same (m,h_1,h_2)-convex stochastic process G is a (m,h_1,h_2)-convex stochastic process dominated by nG.

Proposition 3.4. *Let $G : I\times\Omega\to\mathbb{R}$ be a (m,h_1,h_2)-convex stochastic process, $\{X_k\}_{k=1}^n$ be a finite collection of (m,h_1,h_2)-convex stochastic process dominated by G, and $\{A_k\}_{k=1}^n$ a finite collection of random variables. Then $\sum_{k=1}^n A_k(\cdot)X_k(t,\cdot)$ is dominated by $\sum_{k=1}^n |A_k|\,G$.*

Theorem 3.1. *Let $X : I\times\Omega\to\mathbb{R}$ be a mean square integrable stochastic process on the interval $[0,b/m]$ and (m,h_1,h_2)-convex. Then the following inequalities hold almost everywhere*

$$X\left(\frac{a+b}{2},\cdot\right)\le\frac{h_1(1/2)}{b-a}\int_a^b X(u,\cdot)du+\frac{m^2h_2(1/2)}{b-a}\int_{a/m}^{b/m}X(u,\cdot)du \tag{3.1}$$

and

$$\frac{1}{b-a}\int_a^b X(u,\cdot)\,dt\le\frac{(X(a,\cdot)+X(b,\cdot))}{2}I(h_1) \tag{3.2}$$

$$+\frac{m\left(X\left(\frac{a}{m},\cdot\right)+X\left(\frac{b}{m},\cdot\right)\right)}{2}I(h_2), \tag{3.3}$$

where

$$I(h_1)=\int_0^1 h_1(t)dt \quad \text{and} \quad I(h_2)=\int_0^1 h_2(t)dt.$$

Proof. Let $a,b\in I$ and $m\in(0,1]$. Then for $t\in[0,1]$ we have

$$X\left(\frac{a+b}{2},\cdot\right)=X\left(\frac{ta+(1-t)b+(1-t)a+tb}{2},\cdot\right),$$

and using the (m, h_1, h_2)-convexity of X we obtain

$$X\left(\frac{a+b}{2}, \cdot\right) \leq h_1(1/2)X\left(ta + (1-t)b, \cdot\right) + mh_2(1/2)X\left(t\frac{a}{m} + (1-t)\frac{b}{m}, \cdot\right).$$

Integrating over $t \in [0, 1]$ it follows that

$$\begin{aligned} X\left(\frac{a+b}{2}, \cdot\right) \leq & h_1(1/2)\int_0^1 X\left(ta + (1-t)b, \cdot\right) dt \\ & + mh_2(1/2)\int_0^1 X\left(t\frac{a}{m} + (1-t)\frac{b}{m}, \cdot\right) dt, \end{aligned}$$

and with the change of variable $u = ta + (1-t)b$ and $v = t\frac{a}{m} + (1-t)\frac{b}{m}$ we achieve the inequality (3.1).

Now, using the $(m, h_1, h_2)-$convexity of X we have

$$X\left(ta + (1-t)b, \cdot\right) \leq h_1(t)X\left(a, \cdot\right) + mh_2(t)X\left(\frac{b}{m}, \cdot\right) \tag{3.4}$$

and

$$X\left((1-t)a + tb, \cdot\right) \leq h_1(t)X\left(b, \cdot\right) + mh_2(t)X\left(\frac{a}{m}, \cdot\right). \tag{3.5}$$

Adding (3.4) and (3.5) and integrating over $t \in [0, 1]$ it follows that

$$\begin{aligned} &\int_0^1 X\left(ta + (1-t)b, \cdot\right) dt + \int_0^1 X\left((1-t)a + tb, \cdot\right) dt \\ &\leq \left(X\left(a, \cdot\right) + X\left(b, \cdot\right)\right)\int_0^1 h_1(t)dt + m\left(X\left(\frac{a}{m}, \cdot\right) + X\left(\frac{b}{m}, \cdot\right)\right)\int_0^1 h_2(t)dt. \end{aligned}$$

So, with the above change of variable and doing

$$I(h_1) = \int_0^1 h_1(t)dt \quad \text{and} \quad I(h_2) = \int_0^1 h_2(t)dt,$$

the inequality (3.3) is attained.

The proof is complete. □

Corollary 3.1. *Let $X : I \times \Omega \to \mathbb{R}$ be an mean square integrable on the interval I and convex stochastic process. Then the following inequalities hold almost everywhere*

$$X\left(\frac{a+b}{2}, \cdot\right) \leq \frac{1}{b-a}\int_a^b X(u, \cdot)du \leq \frac{X\left(a, \cdot\right) + X\left(b, \cdot\right)}{2}.$$

Proof. Letting $m = 1$, $h_1(t) = t$ and $h_2(t) = 1 - t$, $t \in [0, 1]$, in Theorem 3.1, we obtain the desired result. □

Corollary 3.2. *Let $X : I \times \Omega \to \mathbb{R}$ be an mean square integrable on the interval I and s-convex stochastic process in the second sense. Then the following inequalities* *hold almost everywhere*

$$2^{s-1}X\left(\frac{a+b}{2},\cdot\right)\leq\frac{1}{(b-a)}\int_a^b X(u,\cdot)du\leq\frac{X(a,\cdot)+X(b,\cdot)}{(s+1)}.$$

Proof. Let $s\in(0,1]$. Making $m=1$, $h_1(t)=t^s$ and $h_2(t)=(1-t)^s$ for $t\in[0,1]$ in Theorem 3.1, it follows the desired result. □

Corollary 3.3. *Let $X:I\times\Omega\to\mathbb{R}$ be an mean square integrable on the interval I and s-convex stochastic process in the first sense. Then the following inequalities hold almost everywhere*

$$X\left(\frac{a+b}{2},\cdot\right)\leq\frac{1}{(b-a)}\int_a^b X(u,\cdot)du\leq\frac{X(a,\cdot)+X(b,\cdot)}{2}.$$

Proof. Let $s\in(0,1]$. Making $m=1$, $h_1(t)=t^s$ and $h_2(t)=1-t^s$ for $t\in[0,1]$ in Theorem 3.1, we have the desired result. □

Corollary 3.4. *Let $X:I\times\Omega\to\mathbb{R}$ be an mean square integrable on the interval I and P-convex stochastic process. Then the following inequalities hold almost everywhere*

$$X\left(\frac{a+b}{2},\cdot\right)\leq\frac{2}{b-a}\int_a^b X(u,\cdot)du\leq 2(X(a,\cdot)+X(b,\cdot)).$$

Proof. Letting $m=1$, $h_1(t)=h_2(t)=1$ for $t\in[0,1]$ in Theorem 3.1 we obtain the desired result. □

Corollary 3.5. *Let $X:I\times\Omega\to\mathbb{R}$ be an mean square integrable on the interval I and MT-convex stochastic process. Then the following inequalities hold almost everywhere*

$$X\left(\frac{a+b}{2},\cdot\right)\leq\frac{1}{2(b-a)}\int_a^b X(u,\cdot)du\leq\frac{\pi(X(a,\cdot)+X(b,\cdot))}{4}.$$

Proof. Letting $m=1$, $h_1(t)=\sqrt{t}/2\sqrt{1-t}$ and $h_2(t)=\sqrt{1-t}/2\sqrt{t}$ for $t\in[0,1]$ in Theorem 3.1 we have the desired result. □

Remark 3.2. The inequality found in Corollary 3.1 coincides with that presnted in [8], the result found in Corollary 3.2 coincides with that presented in Theorem 6 in [23].

Theorem 3.2. *Let $X,G:I\times\Omega\to\mathbb{R}$ be a mean square integrable stochastic process on the interval $[0,b/m]$ and (m,h_1,h_2)-convex. If X is dominated by G, then the following inequalities hold almost everywhere*

$$\begin{aligned}&\left|h_1(1/2)\frac{1}{b-a}\int_a^b X(u,\cdot)du+\frac{m^2h_2(1/2)}{b-a}\int_{a/m}^{b/m}X(u,\cdot)-X\left(\frac{a+b}{2},\cdot\right)\right|\\ \leq&h_1(1/2)\frac{1}{b-a}\int_a^b G(u,\cdot)du+\frac{m^2h_2(1/2)}{b-a}\int_{a/m}^{b/m}G(u,\cdot)-G\left(\frac{a+b}{2},\cdot\right)\end{aligned}$$

and

$$\left|\frac{(X(a,\cdot)+X(b,\cdot))}{2}I(h_1)+\frac{m}{2}\left(X\left(\frac{a}{m},\cdot\right)+X\left(\frac{b}{m},\cdot\right)\right)I(h_2)-\frac{1}{b-a}\int_a^b X(u,\cdot)\,du\right|$$
$$\leq\frac{(G(a,\cdot)+G(b,\cdot))}{2}I(h_1)+\frac{m}{2}\left(G\left(\frac{a}{m},\cdot\right)+G\left(\frac{b}{m},\cdot\right)\right)I(h_2)-\frac{1}{b-a}\int_a^b G(u,\cdot)\,du,$$

where

$$I(h_1)=\int_0^1 h_1(t)dt \quad \textit{and} \quad I(h_2)=\int_0^1 h_2(t)dt.$$

Proof. Let $a,b\in I$ and $m\in(0,1]$. Then, for $t\in[0,1]$ we have

$$X\left(\frac{a+b}{2},\cdot\right)=X\left(\frac{ta+(1-t)b+(1-t)a+tb}{2},\cdot\right)$$

and

$$G\left(\frac{a+b}{2},\cdot\right)=G\left(\frac{ta+(1-t)b+(1-t)a+tb}{2},\cdot\right).$$

Using definition of (m,h_1,h_2)-convexity dominated by G we obtain that

$$\left|h_1(t)X(ta+(1-t)b,\cdot)+mh_2(t)X\left((1-t)\frac{a}{m}+t\frac{b}{m},\cdot\right)-X\left(\frac{a+b}{2},\cdot\right)\right|$$
$$\leq h_1(1/2)G(ta+(1-t)b,\cdot)+mh_2(1/2)G\left((1-t)\left(\frac{a}{m}\right)+t\left(\frac{b}{m}\right),\cdot\right)-G\left(\frac{a+b}{2},\cdot\right).$$

Integrating over $t\in[0,1]$ it follows that

$$\left|h_1(1/2)\frac{1}{b-a}\int_a^b X(u,\cdot)du+\frac{m^2h_2(1/2)}{b-a}\int_{a/m}^{b/m}X(u,\cdot)-X\left(\frac{a+b}{2},\cdot\right)\right|$$
$$\leq h_1(1/2)\frac{1}{b-a}\int_a^b G(u,\cdot)du+\frac{m^2h_2(1/2)}{b-a}\int_{a/m}^{b/m}G(u,\cdot)-G\left(\frac{a+b}{2},\cdot\right).$$

So, the first inequality is obtained.

Now, also we have

$$\left|h_1(t)X(a,\cdot)+mh_2(t)X\left(\frac{b}{m},\cdot\right)-X(ta+(1-t)b,\cdot)\right|$$
$$\leq h_1(1/2)G(a,\cdot)+mh_2(1/2)G\left(\frac{b}{m},\cdot\right)-G(ta+(1-t)b,\cdot)$$

and

$$\left|h_1(t)X(b,\cdot)+mh_2(t)X\left(\frac{a}{m},\cdot\right)-X((1-t)a+tb,\cdot)\right|$$
$$\leq h_1(1/2)G(b,\cdot)+mh_2(1/2)G\left(\frac{a}{m},\cdot\right)-G((1-t)a+tb,\cdot).$$

Adding these inequalities, integrating over $t \in [0,1]$ and taking the notation

$$I(h_1) = \int_0^1 h_1(t)dt \quad \text{and} \quad I(h_2) = \int_0^1 h_2(t)dt,$$

we obtain the desired result. □

Corollary 3.6. *Let $X, G : I \times \Omega \to \mathbb{R}$ be two mean square integrable stochastic processes on the interval I and convex. If X is dominated by G, then the following inequalities hold almost everywhere*

$$\left|\frac{1}{b-a}\int_a^b X(u,\cdot)du - X\left(\frac{a+b}{2},\cdot\right)\right| \leq \frac{1}{b-a}\int_a^b G(u,\cdot)du - G\left(\frac{a+b}{2},\cdot\right)$$

and

$$\left|\frac{(X(a,\cdot)+X(b,\cdot))}{2} - \frac{1}{b-a}\int_a^b X(u,\cdot)\,du\right| \leq \frac{G(a,\cdot)+G(b,\cdot)}{2} - \frac{1}{b-a}\int_a^b G(u,\cdot)\,du.$$

Proof. Letting $m = 1$, $h_1(t) = t$ and $h_2(t) = 1-t$, $t \in [0,1]$, in Theorem 3.2 we achieve the desired result. □

Corollary 3.7. *Let $X, G : I \times \Omega \to \mathbb{R}$ be two mean square integrable stochastic processes on the interval I and s-convex in the second sense. If X is dominated by G, then the following inequalities hold almost everywhere*

$$\left|\frac{1}{b-a}\int_a^b X(u,\cdot)du - 2^{s-1}X\left(\frac{a+b}{2},\cdot\right)\right| \leq \frac{1}{b-a}\int_a^b G(u,\cdot)du - 2^{s-1}G\left(\frac{a+b}{2},\cdot\right)$$

and

$$\left|\frac{(X(a,\cdot)+X(b,\cdot))}{s+1} - \frac{1}{b-a}\int_a^b X(u,\cdot)\,du\right| \leq \frac{(G(a,\cdot)+G(b,\cdot))}{s+1} - \frac{1}{b-a}\int_a^b G(u,\cdot)\,du.$$

Proof. Let $s \in (0,1]$. Making $m = 1$, $h_1(t) = t^s$ and $h_2(t) = (1-t)^s$, $t \in [0,1]$, in Theorem 3.2 we have the desired result. □

Corollary 3.8. *Let $X, G : I \times \Omega \to \mathbb{R}$ be two mean square integrable on the interval I and s-convex stochastic process in the first sense. If X is dominated by G, then the following inequalities hold almost everywhere*

$$\left|\frac{1}{2^{s-1}(b-a)}\int_a^b X(u,\cdot) - X\left(\frac{a+b}{2},\cdot\right)\right| \leq \frac{1}{2^{s-1}(b-a)}\int_a^b G(u,\cdot) - G\left(\frac{a+b}{2},\cdot\right)$$

and

$$\left|\frac{(X(a,\cdot)+X(b,\cdot))}{2} - \frac{1}{b-a}\int_a^b X(u,\cdot)\,du\right| \leq \frac{(G(a,\cdot)+G(b,\cdot))}{2} - \frac{1}{b-a}\int_a^b G(u,\cdot)\,du.$$

Proof. Letting $m = 1$, $h_1(t) = t^s$ and $h_2(t) = 1-t^s$, $t \in [0,1]$, in Theorem 3.2 it follows the desired result. □

Corollary 3.9. *Let $X, G : I \times \Omega \to \mathbb{R}$ be two mean square integrable stochastic processes and P-convex. If X is dominated by G, then the following inequalities hold almost everywhere*

$$\left|\frac{2}{b-a}\int_a^b X(u,\cdot)du - X\left(\frac{a+b}{2},\cdot\right)\right| \leq \frac{2}{b-a}\int_a^b G(u,\cdot)du - G\left(\frac{a+b}{2},\cdot\right)$$

and

$$\left|(X(a,\cdot)+X(b,\cdot)) - \frac{1}{b-a}\int_a^b X(u,\cdot)\,du\right| \leq (G(a,\cdot)+G(b,\cdot)) - \frac{1}{b-a}\int_a^b G(u,\cdot)\,du.$$

Proof. Letting $m = 1$, $h_1(t) = h_2(t) = 1$ for all $t \in [0,1]$, in Theorem 3.2 we have the desired result $\square$

Corollary 3.10. *Let $X, G : I \times \Omega \to \mathbb{R}$ be two mean square integrable stochastic processes on the interval I and MT-convex. If X is dominated by G, then the following inequalities hold almost everywhere*

$$\left|\frac{1}{b-a}\int_a^b X(u,\cdot)du - 2X\left(\frac{a+b}{2},\cdot\right)\right| \leq \frac{1}{b-a}\int_a^b G(u,\cdot)du - 2G\left(\frac{a+b}{2},\cdot\right)$$

and

$$\left|\frac{\pi\,(X(a,\cdot)+X(b,\cdot))}{4} - \frac{1}{b-a}\int_a^b X(u,\cdot)\,du\right| \leq \frac{\pi\,(G(a,\cdot)+G(b,\cdot))}{4} - \frac{1}{b-a}\int_a^b G(u,\cdot)\,du.$$

Proof. Letting $m = 1$, $h_1(t) = \sqrt{t}/2\sqrt{1-t}$ and $h_2(t) = \sqrt{1-t}/2\sqrt{t}$ for $t \in [0,1]$ in Theorem 3.2 we obtain the desired result. $\square$

4. Conclusions

In the development of the present work it was introduced the concept of (m,h_1,h_2)-convex stochastic process dominated by another stochastic process of the same type, also some properties associated with them were found (Definition 2.5, Propositions 3.1, 3.2 and 3.3). From the aforementioned definition the Hermite-Hadamard inequality for stochastic processes (Theorem 3.1) was found and some Corollaries that involve the same inequality for classical convex stochastic process and other types of generalized convex stochastic process (Corollaries 3.1–3.5). Also it was studied the absolute value of the difference of the extremes of right and left side of the Hermite-Hadamard inequality for the generalized convex stochastic process under study, similarly some corollaries for other types of convexity were found (Theorem 3.2 and Corollaries 3.6–3.10).

The author hopes that the results presented will stimulate the study of the relationship between generalized convexity and stochastic processes, thus providing a path to possible applications.

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